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by

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Two Choice Optimal Stopping*†

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Abstract

Let $X_n, \ldots, X_1$ be i.i.d. random variables with distribution function F. A statistician, knowing F, observes the X values sequentially and is given two chances to choose X's using stopping rules. The statistician's goal is to stop at a value of X as small as possible. Let V_n^2 equal the expectation of the smaller of the two values chosen by the statistician when proceeding optimally. We obtain the asymptotic behavior of the sequence V_n^2 for a large class of F's belonging to the domain of attraction (for the minimum) $\mathcal{D}(G^\alpha)$, where $G^\alpha(x) = [1 - \exp(-x^\alpha)]\mathbf{I}(x \geq 0)$. The results are compared with those for the asymptotic behavior of the classical one choice value sequence V_n^1, as well as with the "prophet value" sequence $E(\min\{X_n, \ldots, X_1\})$.

1 Introduction

Kennedy and Kertz (1990, 1991) study the asymptotic behavior of the value sequence, as $n \to \infty$, when optimally stopping an n long sequence of i.i.d. random variables with common distribution function F, with the objective being to stop on as large a value as possible. They show that the asymptotic behavior of the value sequence depends upon the domain of attraction, for the maximum, to which F belongs.

Recently Assaf and Samuel-Cahn (2000) and Assaf, Goldstein, and Samuel Cahn (2002) have studied optimal stopping problems where the statistician is given several choices, and his return is the expected value of the maximal element chosen. The goals in these works were the derivation of "prophet inequalities."

In the present paper we study the limiting behavior of the value sequence when the statistician, knowing F, is given two choices. It turns out to be more convenient here to take as objective to stop on as small a value as possible, and therefore to take as the statistician's goal the minimization of the expected value upon stopping.

The two choice problem we consider is more difficult by an order of magnitude than the optimal one-choice problem. To be convinced of this, let $V_n^1(x)$ (which we will also denote by $g_n(x)$) and $V_n^2(x)$ be the value of the optimal one and two choice policy respectively,

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when applied to the i.i.d. sequence $X_n, \ldots, X_1$, when the statistician is already guaranteed the value x. Note that for convenience we are indexing the variables so that the first one observed is X_n and the last is X_1. Then by the dynamic programming principle, for one choice $V_1^1(x) = E[X_1 \wedge x]$ and we have

$$V_{n+1}^1(x) = E[X_{n+1} \wedge V_n^1(x)] \quad \text{for } n \geq 1, \tag{1}$$

whereas with two choices, $V_2^2(x) = E[X_2 \wedge X_1 \wedge x]$ and

$$V_{n+1}^2(x) = E[V_n^1(X_{n+1}) \wedge V_n^2(x)] \quad \text{for } n \geq 2. \tag{2}$$

The first term inside the square brackets in (2) corresponds to choosing the current variable X_{n+1} and being left with only one additional choice among the remaining n observations, while the second term corresponds to passing up the current random variable X_{n+1} and retaining two choices among the remaining n observations. Comparing (1) and (2) we see that for one choice the expectation computed in (1) is with respect to the random variables X_{n+1} with identical distributions, whereas the distribution of the random variable $V_n^1(X_{n+1})$ in (2) depends on the function V_n^1 which changes with n.

Let

$$x_F = \sup\{x : F(x) < 1\}. \tag{3}$$

When nothing is guaranteed, the value for the one and two stop problems will be denoted V_n^1 and V_n^2 respectively, and satisfy $V_n^1 = V_n^1(x_F)$ and $V_n^2 = V_n^2(x_F)$.

As in the one choice problem, the asymptotic behavior of the value sequence depends on which of the three extreme value classes the distribution function F belongs to. In the present paper, we only consider F which belongs to one of these domains of attraction and take up the study of the remaining two classes in subsequent work. Specifically in this paper, by a suitable shift of the origin, we assume that the distribution function F of the i.i.d. random variables belongs to the domain of attraction (for the minimum) $\mathcal{D}(G^\alpha)$, where $\alpha > 0$ and

$$G^\alpha(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp(-x^\alpha) & x \geq 0, \end{cases} \tag{4}$$

and satisfies $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. (This is the Type III of Leadbetter, Lindgren and Rootzén, 1983, and Type Ψ_α of Resnick, 1987.) A necessary and sufficient condition for $F \in \mathcal{D}(G^\alpha)$ is

$$F(x) = x^\alpha L(x)$$

where $L(x)$ is slowly varying at 0, and a sufficient (and close to necessary) condition is

$$\lim_{x \downarrow 0} \frac{xF'(x)}{F(x)} = \alpha,$$

see e.g. de Haan, 1976, Theorem 4.

Let m_n be the minimum of n i.i.d. random variables. The results for the maximum (see e.g. Resnick 1987, Chapter 2.1) and the work of Kennedy and Kertz (1991) translate for the minimum as follows: If $F \in \mathcal{D}(G^\alpha)$, then

$$\begin{aligned} \lim_{n\to\infty} nF(Em_n) &= \Gamma(1 + 1/\alpha)^\alpha \\ \lim_{n\to\infty} nF(V_n^1) &= (1 + 1/\alpha). \end{aligned} \tag{5}$$

Our main result for a statistician with two choices is as follows.

Theorem 1.1 *Let $X_n, \ldots, X_1$ be non-negative integrable i.i.d. random variables with distribution function*

$$F(x) = x^\alpha L(x) \quad \text{where } \lim_{x \downarrow 0} L(x) \text{ exists and equals } \mathcal{L} \in (0, \infty).$$

Then the optimal two choice value V_n^2 satisfies

$$\lim_{n \to \infty} nF(V_n^2) = h^\alpha(b_\alpha) \tag{6}$$

where $b_\alpha > 0$ is the unique solution to

$$\int_0^y h(u)du + (1/\alpha - y)h(y) = 0, \tag{7}$$

and $h(y)$ is the function

$$h(y) = \left(\frac{y}{1 + \alpha y/(\alpha + 1)} \right)^{1/\alpha} \quad \text{for } y \geq 0. \tag{8}$$

The value $h(b_\alpha)$ depends only on α but unfortunately, unlike the values of (5) cannot be given in closed form in terms of α. A short table of the limiting values of (5) and of $h^\alpha(b_\alpha)$ are given in Table 1. The performance improvement between having two choices over having only one is substantial, in that the optimal value becomes much closer to that of the prophet. For example, for a distribution with $\alpha = 1$ such as the uniform, the limiting values (for the minimum) for the statistician with one choice is 2, with two choices it is $1.165\ldots$, while the value for the prophet is 1. More explicitly, with n variables the optimal value for a statistician with one choice is roughly $2/n$, for the prophet it is roughly $1/n$, and for a statistician with two choices it is $1.165\ldots/n$.

Since the limits in (6) are the same for all $F \in \mathcal{D}(G^\alpha)$, we first prove Theorem 1.1 for the case where F is of the form

$$\mathcal{U}^\alpha(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^\alpha & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases} \tag{9}$$

for a fixed value of $\alpha > 0$.

The paper is organized as follows. In Section 2 we derive some fundamental equations for the family (9), and show heuristics which explain the form of the function $h(y)$ of (8). In Section 3 we show that a particular sequence of functions h_n, which determine V_{n+1}^2, converges to h. Section 4 contains some general convergence results. In Section 5 we derive (6) for the special family (9) and some results concerning the finiteness of the limit of the moments of properly scaled randomly selected values. In Section 6 the results are generalized to other distributions belonging to $\mathcal{D}(G^\alpha)$. Section 7 contains numerical results presented in Table 1, along with explanations and several additional remarks.

2 The Fundamental Equations and Heuristics

For X with distribution function F, let

$$g(x) = E[X \wedge x].$$

When $F(0)=0$, writing $g(x)=x-\int_0^x F(u)du$, we see easily that $g(x)$ is positive and strictly increasing on the interval $(0,x_F)$. Hence the same is true for $g_{n+1}(x)=g(g_n(x))$. In the sequel we shall consider $\alpha>0$ as fixed, to avoid the necessity of indexing quantities by α.

For the distribution function $\mathcal{U}^\alpha$ as in (9) we have explicitly on the interval $[0,1]$

$$g(x)=E[X\wedge x]=x-\frac{x^{\alpha+1}}{\alpha+1}, \tag{10}$$

and with $g_1(x)=g(x)$,

$$g_{n+1}(x)=g_n(x)-\frac{g_n(x)^{\alpha+1}}{\alpha+1} \quad n\geq 1. \tag{11}$$

Since a statistician with two choices does at least as well as one with a single choice

$$g_n(0)=0\leq V_n^2\leq V_n^1=g_n(1) \quad n\geq 2.$$

As we are interested in the two choice case, we will henceforth write V_n to denote V_n^2 whenever convenient. Because the function g_n is strictly increasing on $[0,1]$, there exists a unique number $b_n\in[0,1]$ satisfying

$$V_n=g_n(b_n). \tag{12}$$

We call b_n the "threshold value" for the following reason; by (2) the statistician at stage $n+1$ will choose X_{n+1} when $g_n(X_{n+1})<V_n$, that is, when $X_{n+1}<b_n$.

Since $b_n\in[0,1]$, $P(X>b_n)=1-b_n^\alpha$, and the basic equation (2) becomes

$$V_{n+1}=\alpha\int_0^{b_n} g_n(x)x^{\alpha-1}dx+(1-b_n^\alpha)V_n, \quad n\geq 2. \tag{13}$$

Letting U_k be independent $\mathcal{U}[0,1]$ variables, $U_k^{1/\alpha}$ has distribution $\mathcal{U}^\alpha$, and hence we may begin recursion (13) at $V_2=E[U_2^{1/\alpha}\wedge U_1^{1/\alpha}]$. We scale

$$W_n=n^{1/\alpha}V_n, \quad B_n=n^{1/\alpha}b_n \tag{14}$$

and

$$g_n(x)=xf_n(nx^\alpha) \tag{15}$$

with $f_n(0)=1$. Since $g_n(x)$ is defined and positive for $0<x\leq 1$, the function $f_n(x)$ is defined and positive for $0<x\leq n$, and setting $f_n(0)=1$ makes $f_n(x)$ continuous as $x\downarrow 0$, since $g_n'(0)=1$.

Substitute (15) into (13) and make the change of variable $y=nx^\alpha$ to obtain

$$V_{n+1}=n^{-(1+1/\alpha)}\int_0^{B_n^\alpha} y^{1/\alpha}f_n(y)dy+(1-b_n^\alpha)V_n, \quad n\geq 2.$$

Multiply by $n^{1/\alpha}$ and set

$$h_n(y)=y^{1/\alpha}f_n(y) \tag{16}$$

to obtain

$$\left(\frac{n}{n+1}\right)^{1/\alpha}W_{n+1}=\frac{1}{n}\int_0^{B_n^\alpha} h_n(y)dy+(1-b_n^\alpha)W_n, \quad n\geq 2. \tag{17}$$

We can now write (17) as our fundamental equation

$$W_m = c \quad \text{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1} = \frac{1}{n}\int_0^n (h_n(y) \wedge W_n)\, dy \quad \text{for } n \geq m, \tag{18}$$

with $m = 2$ and $c = 2^{1/\alpha} E[U_2^{1/\alpha} \wedge U_1^{1/\alpha}]$, where by (12),(14),(15) and (16),

$$W_n = h_n(B_n^\alpha). \tag{19}$$

Later we allow for arbitrary initial times $m \geq 1$ and any positive starting values c.

For $\mathcal{U}^\alpha$ was in (9), we give a heuristic argument explaining (6) and (7), the appearance of the function h in (8) and of Theorem 1.1. Firstly, $((n+1)/n)^{1/\alpha} = 1 + 1/(\alpha n) + O(n^{-2})$. Thus we have from (17)

$$W_{n+1} - W_n = n^{-1}\int_0^{B_n^\alpha} h_n(y)dy + n^{-1}(1/\alpha - B_n^\alpha)W_n + O(n^{-2}).$$

If for large n the difference $n(W_{n+1} - W_n) = o(1)$, then multiplying by n we have

$$0 = \int_0^{B_n^\alpha} h_n(y)dy + (1/\alpha - B_n^\alpha)W_n + o(1), \tag{20}$$

and if $B_n^\alpha \to b_\alpha, W_n \to d_\alpha$ and $h_n \to h$ as $n \to \infty$ we obtain from (20)

$$0 = \int_0^{b_\alpha} h(y)dy + (1/\alpha - b_\alpha)d_\alpha,$$

where from (19) also

$$d_\alpha = h(b_\alpha),$$

which explains (6) and (7) of Theorem 1.1. By (16) finding the limiting h is equivalent to finding the limiting f, since

$$h(y) = y^{1/\alpha} f(y). \tag{21}$$

Using (11) and (15) and the substitution $y = nx^\alpha$, it follows that

$$f_{n+1}((1 + \frac{1}{n})\, y) = f_n(y) - \frac{y}{(\alpha+1)n} f_n(y)^{\alpha+1}. \tag{22}$$

Subtracting $f_n(y)$ from both sides, dividing by y/n and taking limits as $n \to \infty$ indicates that the limiting function f should satisfy the differential equation

$$f'(y) = -f(y)^{\alpha+1}/(\alpha+1) \tag{23}$$

with the initial condition $f(0) = 1$. Equation (23) has the unique solution

$$f(y) = (1 + \frac{\alpha y}{\alpha+1})^{-1/\alpha}, \tag{24}$$

which together with (21) yields the function h of (8).

3 Preliminary Lemmas

With f_n as in (15) and h_n as in (16), we have the following Lemma.

Lemma 3.1 *The function $f_n(y)$ is strictly decreasing in y for $y \in [0, n]$ and $h_n(y)$ is strictly increasing in y for $y \in [0, n]$.*

Proof: We prove the lemma by induction on n. For $n = 1$ from (10) and (15)

$$f_1(y) = 1 - \frac{y}{\alpha + 1},$$

so the result is immediate for f_1, and for h_1 by (16). Now assume the assertions are true for n. We shall show they are true for $n+1$. Note that for $0 \leq y \leq n$ we have $0 \leq y(n+1)/n \leq n+1$. Differentiating (22), for $0 < y \leq n$,

$$\begin{aligned} \left(\frac{n+1}{n}\right) f'_{n+1}(\frac{n+1}{n}y) &= f'_n(y) - \frac{1}{(\alpha+1)n} f_n(y)^{\alpha+1} - \frac{y}{n} f_n^\alpha(y) f'_n(y) \\ &= f'_n(y)[1 - \frac{y}{n} f_n^\alpha(y)] - \frac{1}{(\alpha+1)n} f_n(y)^{\alpha+1} \\ &< f'_n(y)[1 - \frac{y}{n}] - \frac{1}{(\alpha+1)n} f_n(y)^{\alpha+1} \\ &< 0, \end{aligned}$$

where we have used $f'_n(y) < 0$ and $0 < f_n^\alpha(y) < 1$ for $0 < y \leq n$.

From(22) and (16) we have

$$(\frac{n}{n+1})^{1/\alpha} h_{n+1}(\frac{n+1}{n}y) = h_n(y) - \frac{1}{(\alpha+1)n} h_n(y)^{\alpha+1}.$$

Thus for $0 \leq y < n$ we have

$$(\frac{n}{n+1})^{1/\alpha - 1} h'_{n+1}(\frac{n+1}{n}y) = h'_n(y)[1 - \frac{1}{n} h_n(y)^\alpha] > 0$$

since by the induction hypothesis $h'_n(y) > 0$ and

$$h_n(y)^\alpha < h_n(n)^\alpha = [n^{1/\alpha} f_n(n)]^\alpha < n f_n^\alpha(0) = n. \qquad ♮$$

Let $f(y)$ be given by (24) and define

$$\epsilon_n(y) = f(y) - f_n(y). \tag{25}$$

Lemma 3.2 *With $\epsilon_n(y)$ as in (25),*

$$\epsilon_n(y) > 0 \quad \textit{for } 0 < y \leq n. \tag{26}$$

Proof. We use the following two well known inequalities.

$$\text{For } 0 < \alpha \leq 1 \text{ and } x \geq -1, \quad (1+x)^\alpha \leq 1 + \alpha x, \tag{27}$$

and

$$\text{for } \alpha \geq 1 \text{ and } x \geq -1, \quad 1+\alpha x \leq (1+x)^{\alpha}. \tag{28}$$

We prove the lemma by induction. For $n = 1$ we must show that for $0 < y \leq 1$

$$1 - \frac{y}{\alpha+1} < (1 + \frac{\alpha y}{\alpha+1})^{-1/\alpha}$$

which is equivalent to

$$(1 - \frac{y}{\alpha+1})^{\alpha} < (1 + \frac{\alpha y}{\alpha+1})^{-1}$$

or

$$(1 + \frac{\alpha y}{\alpha+1})(1 - \frac{y}{\alpha+1})^{\alpha} < 1. \tag{29}$$

Now for $0 < \alpha \leq 1$ we have by (27) that the left hand side of (29) is less than or equal to

$$(1 + \frac{\alpha y}{\alpha+1})(1 - \frac{\alpha y}{\alpha+1}) = 1 - (\frac{\alpha y}{\alpha+1})^2 < 1.$$

For $\alpha > 1$ the left hand side of (29) is by (28) less than

$$(1 + \frac{y}{\alpha+1})^{\alpha}(1 - \frac{y}{\alpha+1})^{\alpha} = [1 - (\frac{y}{\alpha+1})^2]^{\alpha} < 1.$$

Thus $\epsilon_1(y) > 0$ for $0 < y \leq 1$.

Now suppose $\epsilon_n(y) > 0$ for $0 < y \leq n$. That $\epsilon_{n+1}(y) > 0$ for $0 < y \leq n+1$, is equivalent to

$$f_{n+1}(y) < (1 + \frac{\alpha y}{\alpha+1})^{-1/\alpha}.$$

By the induction hypothesis

$$f_n(y) < (1 + \frac{\alpha y}{\alpha+1})^{-1/\alpha} \quad \text{for } 0 < y \leq n$$

and thus by (15)

$$g_n(x) < x(1 + \frac{\alpha n x^{\alpha}}{\alpha+1})^{-1/\alpha} \quad \text{for } 0 < x \leq 1,$$

and since $g(\cdot)$ is an increasing function, using (11),

$$g_{n+1}(x) < x(1 + \frac{\alpha n x^{\alpha}}{\alpha+1})^{-1/\alpha}[1 - \frac{x^{\alpha}}{\alpha+1}(1 + \frac{\alpha n x^{\alpha}}{\alpha+1})^{-1}]. \tag{30}$$

Thus, again by (15), it suffices to show that the right hand side of (30) is less than

$$x(1 + \frac{\alpha(n+1)x^{\alpha}}{\alpha+1})^{-1/\alpha}, \quad \text{for } 0 < x \leq 1.$$

Set $y = x^{\alpha}/(\alpha+1)$. Then it suffices to show that

$$(1+\alpha n y)^{-1/\alpha}[1 - \frac{y}{1+\alpha n y}] < (1+\alpha(n+1)y)^{-1/\alpha} \quad \text{for } 0 < y \leq 1,$$

i.e. that

$$[1 + \frac{\alpha y}{1+\alpha n y}]^{1/\alpha}[1 - \frac{y}{1+\alpha n y}] < 1,$$

which is equivalent to

$$[1+\frac{\alpha y}{1+\alpha ny}][1-\frac{y}{1+\alpha ny}]^{\alpha}<1. \tag{31}$$

For $\alpha \leq 1$ use (27) to get that the left hand side of (31) is less than or equal to

$$[1+\frac{\alpha y}{1+\alpha ny}][1-\frac{\alpha y}{1+\alpha ny}]=1-(\frac{\alpha y}{1+\alpha ny})^2<1.$$

For $\alpha > 1$ use (28) to get that the left hand side of (31) is less than

$$[1+\frac{y}{1+\alpha ny}]^{\alpha}[1-\frac{y}{1+\alpha ny}]^{\alpha}=[1-(\frac{y}{1+\alpha ny})^2]^{\alpha}<1.\natural$$

Lemma 3.3 *With $\epsilon_n(y)$ as in (25),*

$$\epsilon_n(y)<\frac{y}{2n} \quad \textit{for } 0<y\leq n. \tag{32}$$

Proof: We prove (32) by induction. For $n=1$ we must show that

$$(1+\frac{\alpha y}{\alpha+1})^{-1/\alpha}<1-y(\frac{1}{\alpha+1}-\frac{1}{2}) \quad \text{for } 0<y\leq 1. \tag{33}$$

For $\alpha \geq 1$, equation (33) is obvious, since the left hand side is less than 1 and the right hand side is greater than 1. For $\alpha < 1$ we have, by (28) that

$$(1+\frac{\alpha y}{\alpha+1})^{1/\alpha}\geq 1+\frac{y}{\alpha+1}.$$

Thus to show (33) it suffices to show

$$\frac{1}{1+y/(\alpha+1)}<1-\frac{y(1-\alpha)}{2(\alpha+1)},$$

i.e. that

$$1<(1+\frac{y}{\alpha+1})(1-\frac{y(1-\alpha)}{2(\alpha+1)})=1+\frac{y}{2}-\frac{y^2(1-\alpha)}{2(\alpha+1)^2}$$

which clearly holds for $0<y\leq 1$.

Now suppose (32) holds for n. Let $0<y\leq n+1$, and $p_n=n/(n+1)$. By (22)

$$\begin{aligned}
\epsilon_{n+1}(y) &= f(y)-f_{n+1}(y)\\
&= f(y)-f_n(p_ny)+\frac{p_ny}{(\alpha+1)n}f_n(p_ny)^{\alpha+1}\\
&= (f(y)-f(p_ny))+(f(p_ny)-f_n(p_ny))+\frac{y}{(\alpha+1)(n+1)}f_n(p_ny)^{\alpha+1}.
\end{aligned}$$

Thus

$$\epsilon_{n+1}(y)=f(y)-f(p_ny)+\epsilon_n(p_ny)+\frac{y}{(\alpha+1)(n+1)}f_n(p_ny)^{\alpha+1}. \tag{34}$$

Note that

$$f'(y)=-f(y)^{\alpha+1}/(\alpha+1)<0 \quad \text{for } y>0 \tag{35}$$

and

$$f''(y) = f(y)^{2\alpha+1}/(\alpha+1) > 0 \quad \text{for } y > 0. \tag{36}$$

Thus if we use the Taylor expansion

$$f(x+\Delta) = f(x) + \Delta f'(x) + \frac{\Delta^2}{2} f''(x+\xi\Delta) \quad \text{for some } 0 < \xi < 1$$

with $x = p_n y$ and $\Delta = y/(n+1)$ so that $x + \Delta = y$, we get, by use of (35) and (36)

$$f(y) - f(p_n y) = -\frac{y}{(\alpha+1)(n+1)} f(p_n y)^{\alpha+1} + \frac{y^2}{2(n+1)^2(\alpha+1)} f(\theta y)^{2\alpha+1} \tag{37}$$

where $p_n < \theta < 1$. Substituting (37) into (34) yields

$$\epsilon_{n+1}(y) = \epsilon_n(p_n y) - \frac{y}{(\alpha+1)(n+1)}[f(p_n y)^{\alpha+1} - f_n(p_n y)^{\alpha+1}] + \frac{y^2}{2(\alpha+1)(n+1)^2} f(\theta y)^{2\alpha+1}. \tag{38}$$

Since by (26) $f(p_n y) > f_n(p_n y)$ for $0 < y \leq n+1$, we have

$$f(p_n y)^{\alpha+1} - f_n(p_n y)^{\alpha+1} > f(p_n y)^{\alpha}[f(p_n y) - f_n(p_n y)] = f(p_n y)^{\alpha} \epsilon_n(p_n y). \tag{39}$$

Substituting (39) into (38) yields

$$\epsilon_{n+1}(y) < \epsilon_n(p_n y)[1 - \frac{y}{(\alpha+1)(n+1)} f(p_n y)^{\alpha}] + \frac{y^2}{2(\alpha+1)(n+1)^2} f(\theta y)^{2\alpha+1}. \tag{40}$$

It follows from the induction hypothesis that for $0 < y \leq n+1$ (so that $0 < p_n y \leq n$)

$$\epsilon_n(p_n y) < \frac{p_n y}{2n} = \frac{y}{2(n+1)}.$$

Thus (40) yields

$$\begin{aligned}
\epsilon_{n+1}(y) &< \frac{y}{2(n+1)}[1 - \frac{y}{(\alpha+1)(n+1)} f(p_n y)^{\alpha}] + \frac{y^2}{2(\alpha+1)(n+1)^2} f(\theta y)^{2\alpha+1} \\
&< \frac{y}{2(n+1)}[1 - \frac{y f(p_n y)^{\alpha}\{1 - f(\theta y)^{\alpha+1}\}}{(\alpha+1)(n+1)}] < \frac{y}{2(n+1)},
\end{aligned}$$

where we have used the fact that f is decreasing, $f < 1$ and $\theta > p_n$. ♮

Corollary 3.1

$$\begin{aligned}
f_n(y) \to f(y) &= (1 + \frac{\alpha y}{\alpha+1})^{-1/\alpha} \quad \textit{for all } y > 0, \textit{ as } n \to \infty \\
h_n(y) \to h(y) &= \left(\frac{y}{1 + \alpha y/(\alpha+1)}\right)^{1/\alpha} \quad \textit{for all } y > 0, \textit{ as } n \to \infty.
\end{aligned}$$

Remark 3.1 *Note that by (15),(16) and (1)*

$$h_n(n) = n^{1/\alpha} g_n(1) = n^{1/\alpha} V_n^1$$

and thus, by (5)

$$\lim_{n\to\infty} h_n(n) = [1 + 1/\alpha]^{1/\alpha}.$$

On the other hand, we also have

$$\lim_{y\to\infty} h(y) = [1 + 1/\alpha]^{1/\alpha}$$

Thus, the convergence to h in Corollary 3.1 satisfies

$$\lim_{n\to\infty} h_n(n) = \lim_{y\to\infty} \lim_{n\to\infty} h_n(y).$$

4 Convergence of Recursions

To prove convergence of the sequence W_n determined by the recursion (18), we first study the behavior of a sequence Z_n, whose values are given by the simpler recursion (42) where the function in the integral does not depend on n. For $\alpha > 0$ a fixed value and $q(\cdot)$ a given function, define

$$Q(y) = \int_0^y q(u)du + (1/\alpha - y)q(y). \tag{41}$$

We prove the convergence of Z_n under the following conditions:
(i) $q(0) = 0$
(ii) $q(u)$ for $0 < u < \infty$ is non-decreasing everywhere and strictly increasing and differentiable for $0 < u < A$ where $1/\alpha < A \leq \infty$.
(iii) There exists a unique positive root $b \in (1/\alpha, A)$ to the equation $Q(y) = 0$.
Under (i) and (ii), Lemma 4.1 shows that $Q(A) = \lim_{y\uparrow A} Q(y)$ exists and is in $[-\infty, \infty)$, even when $A = \infty$, and that (iii) is satisfied if $Q(A) < 0$.

Lemma 4.1 *Under conditions (i) and (ii), the function $Q(\cdot)$ is strictly increasing for $0 < y < 1/\alpha$, strictly decreasing for $1/\alpha < y < A$, and non-increasing for $A < y$. Hence $Q(A)$ exists and (iii) holds if $Q(A) < 0$.*

Proof: For $0 \leq y_1 < y_2 < 1/\alpha$ straightforward calculations yield

$$Q(y_2) - Q(y_1) \geq (q(y_2) - q(y_1))(1/\alpha - y_2),$$

and for $1/\alpha < y_1 < y_2$,

$$Q(y_2) - Q(y_1) \leq (q(y_2) - q(y_1))(1/\alpha - y_1).$$

The claims now follow directly. $\natural$

The main result of this Section is

Theorem 4.1 *Let (i), (ii) and (iii) hold, let $m \geq 1$ be any integer and $c \in (0, \infty)$ any constant. If*

$$Z_m = c \quad \textit{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\int_0^n (q(y) \wedge Z_n)dy \quad \textit{for } n \geq m, \tag{42}$$

then the limit of Z_n exists and

$$\lim_{n\to\infty} Z_n = d,$$

where $d = q(b)$, where b is the unique root of $Q(y) = 0$.

Lemma 4.2 is the crux of of the proof of Theorem 4.1.

Lemma 4.2 *Assume that (i), (ii) and (iii) hold. Let $m \geq 1$ be any integer and $c \in (0, \infty)$ any constant, and suppose that Z_n for $n \geq m$ is defined by (42). Then for every $\delta \in (0, \min\{q(A) - d, d - q(1/\alpha)\})$ there there exists $\Delta > 0$ and n_0 such that for all $n \geq n_0$,*

$$\textit{if } Z_n < d - \delta \textit{ then } Z_{n+1} \geq (1 + \Delta/n)Z_n, \tag{43}$$

$$\textit{if } Z_n > d + \delta \textit{ then } Z_{n+1} \leq (1 - \Delta/n)Z_n, \tag{44}$$

$$\textit{if } Z_n < d \textit{ then } Z_{n+1} < d, \textit{ and} \tag{45}$$

$$\textit{if } |Z_n - d| \leq \delta \textit{ then } |Z_{n+1} - d| \leq \delta. \tag{46}$$

Proof: We have

$$(1 + \frac{1}{n})^{1/\alpha} = 1 + \frac{1}{\alpha n} + \frac{1}{\alpha}(\frac{1}{\alpha} - 1)\frac{1}{2n^2} + O_\alpha(n^{-3}),$$

and hence

$$(\frac{n+1}{n})^{1/\alpha}(1 - \frac{1}{n\gamma}) = 1 - \frac{1}{n}(\frac{1}{\gamma} - \frac{1}{\alpha}) + \frac{1}{n^2}\left(\frac{1}{2\alpha}(\frac{1}{\alpha} - 1) - \frac{1}{\alpha\gamma}\right) + O_{\alpha,\gamma}(n^{-3}), \tag{47}$$

where we write $O_\lambda(f_n)$ to indicate a sequence bounded in absolute value by f_n times a constant depending only on λ, a collection of parameters.

Define

$$M(t) = \int_0^{q^{-1}(t)} \left(1 - \frac{q(y)}{t}\right) dy \quad \text{for } 0 \leq t < q(A).$$

From (41), $Q(b) = 0$ and $d = q(b)$, we have

$$M(d) = 1/\alpha.$$

It is not hard to see that $M(t)$ is strictly increasing over its range. Hence, setting $\Delta_1 = (1/\alpha - M(d - \delta))/2$ and $\Delta_2 = (M(d + \delta) - 1/\alpha)/2$ we have $\Delta = \min\{\Delta_1, \Delta_2\} > 0$. Now consider the function

$$r_n(t) = \frac{1}{n}\int_0^n \left(\frac{q(y)}{t} \wedge 1\right) dy = 1 - \frac{1}{n}\int_0^{q^{-1}(t)\wedge n} \left(1 - \frac{q(y)}{t}\right) dy.$$

Since $Z_m > 0$ we have $Z_n > 0$ for all $n \geq m$, and now by (42) we have

$$Z_{n+1}/Z_n = \left(\frac{n+1}{n}\right)^{1/\alpha} r_n(Z_n). \tag{48}$$

By definition

$$r_n(t) = 1 - \frac{1}{n} M(t) \quad \text{for } 0 \leq t < q(n).$$

To prove (43), assume $Z_n < d - \delta$. Since r_n is decreasing, using (48) and (47), we have for all $n > q^{-1}(d - \delta)$,

$$\begin{aligned} Z_{n+1} &\geq Z_n(\frac{n+1}{n})^{1/\alpha} r_n(d-\delta) \\ &= Z_n(\frac{n+1}{n})^{1/\alpha}(1 - \frac{1}{n}M(d-\delta)) \\ &= (1 + \frac{1}{n}(\frac{1}{\alpha} - M(d-\delta)) + O_{\alpha,d-\delta}(n^{-2}))Z_n \\ &\geq (1 + \frac{\Delta_1}{n})Z_n \geq (1 + \frac{\Delta}{n})Z_n \end{aligned}$$

for all n sufficiently large, showing (43).

Next we prove (44). When $Z_n \geq d + \delta$, we have similarly that for $n > q^{-1}(d + \delta)$,

$$\begin{aligned} Z_{n+1} &\leq Z_n(\frac{n+1}{n})^{1/\alpha} r_n(d+\delta) \\ &= Z_n(\frac{n+1}{n})^{1/\alpha}(1 - \frac{1}{n}M(d+\delta)) \\ &= (1 - \frac{1}{n}(M(d+\delta) - \frac{1}{\alpha}) + O_{\alpha,d+\delta}(n^{-2}))Z_n \\ &\leq (1 - \frac{\Delta_2}{n})Z_n \leq (1 - \frac{\Delta}{n})Z_n \end{aligned}$$

for all n sufficiently large.

Turning now to (45) and (46), for $Z_n \leq d + \delta$, since $d + \delta < q(A)$, β_n is well defined by

$$q(\beta_n) = Z_n.$$

Now by (42) and (41)

$$\left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\left(\int_0^{\beta_n} q(y)dy + (n - \beta_n)q(\beta_n)\right) = \frac{1}{n}Q(\beta_n) + (1 - \frac{1}{\alpha n})Z_n;$$

thus

$$Z_{n+1} = \left(1 + \frac{1}{n}\right)^{1/\alpha} \frac{1}{n} Q(\beta_n) + R_n Z_n \tag{49}$$

where

$$R_n = (1 + \frac{1}{n})^{1/\alpha}(1 - \frac{1}{\alpha n}). \tag{50}$$

Consider
$$Q(q^{-1}(u)) = \int_0^{q^{-1}(u)} q(y)dy + (1/\alpha - q^{-1}(u))u.$$
Since $q^{-1}(u)$ is differentiable for $0 < u < q(A)$,
$$\frac{d}{du}Q(q^{-1}(u)) = 1/\alpha - q^{-1}(u).$$
Hence, evaluating $Q(q^{-1}(u))$ by a Taylor expansion around d, and using $Q(b) = Q(q^{-1}(d)) = 0$, we obtain that there exists some ξ_{Z_n} between d and Z_n such that
$$Q(\beta_n) = Q(q^{-1}(Z_n)) = (Z_n - d)(1/\alpha - q^{-1}(\xi_{Z_n})). \tag{51}$$
Subtracting d from both sides of (49) and using (51) we obtain
$$Z_{n+1} - d = \left\{1 - (1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(\xi_{Z_n}) - \frac{1}{\alpha})\right\}(Z_n - d) + [R_n - 1]Z_n. \tag{52}$$
Take n_1 such that for all $n \geq n_1$
$$(1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(d) - 1/\alpha)) < 1.$$
Then for $Z_n < d$ we have $\xi_{Z_n} < d$ and hence $q^{-1}(\xi_{Z_n}) < q^{-1}(d)$, and so
$$0 < \left\{1 - (1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(\xi_{Z_n}) - \frac{1}{\alpha})\right\}.$$
Hence the first term on the right hand side of (52) is strictly negative. Next, there exists $n_2 \geq n_1$ so that for $n \geq n_2$ we have $0 < R_n < 1$, by (50) and (47) with $\gamma = \alpha$. For such n the second term on the right hand side is also negative, and the sum of these two terms is therefore negative. This proves (45).

To consider (46) suppose that $|Z_n - d| \leq \delta$. Then $|\xi_{Z_n} - d| \leq \delta$, and therefore
$$q^{-1}(d-\delta) \leq q^{-1}(\xi_{Z_n}) \leq q^{-1}(d+\delta).$$
Hence, for all n sufficiently large so that
$$(1+\frac{1}{n})^{1/\alpha}\frac{1}{n}\left(q^{-1}(d+\delta) - 1/\alpha\right) \leq 1,$$
letting $\Delta_3 = q^{-1}(d-\delta) - 1/\alpha > 0$ we have $q^{-1}(\xi_{Z_n}) - 1/\alpha \geq \Delta_3$ and therefore
$$0 \leq \left\{1 - (1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(\xi_{Z_n}) - \frac{1}{\alpha})\right\} \quad \leq \quad 1 - \frac{\Delta_3}{n.} \tag{53}$$
Further, from (50), again using (47) with $\gamma = \alpha$, there exists K_α such that
$$|R_n - 1| \leq \frac{K_\alpha}{n^2}.$$

Then for all n so large that

$$\frac{K_\alpha}{n}(d+\delta) \leq \Delta_3 \delta$$

we have, using (52) and (53),

$$\begin{aligned} |Z_{n+1} - d| &\leq (1 - \frac{\Delta_3}{n})|Z_n - d| + |R_n - 1| Z_n \\ &\leq (1 - \frac{\Delta_3}{n})\delta + \frac{K_\alpha}{n^2}(d+\delta) \\ &\leq \delta. \end{aligned}$$

This proves (46). ♮

Proof of Theorem 4.1: Let $\delta \in (0, \min\{q(A) - d, d - q(1/\alpha)\})$, and $n \geq n_0$.
Case I: $Z_{n_0} > d + \delta$. If $Z_n > d + \delta$ for all $n \geq n_0$ then by (44) we would have

$$Z_{n+1} \leq \prod_{j=n_0}^{n} (1 - \frac{\Delta}{j}) Z_{n_0} \to 0,$$

a contradiction. Hence for some $n_1 \geq n_0$ we have $Z_{n_1} \leq d + \delta$, and we would therefore be in Case II or Case III.

Case II: $Z_{n_1} < d - \delta$ for some $n_1 \geq n_0$. If $Z_n < d - \delta$ for all $n \geq n_1$ we would have by (43) that

$$Z_{n+1} \geq \prod_{j=n_1}^{n} (1 + \frac{\Delta}{j}) Z_{n_1} \to \infty,$$

a contradiction. Hence there exists $n_2 \geq n_1$ such that $Z_{n_2} \geq d - \delta$. By (45), $Z_{n_2} < d$, reducing to Case III.

Case III: $|Z_{n_1} - d| \leq \delta$ for some $n_1 \geq n_0$. In this case $|Z_n - d| \leq \delta$ for all $n \geq n_1$, by (46). Hence $|Z_n - d| \leq \delta$ for all n sufficiently large. Since δ can be taken arbitrarily small, the Theorem is complete. ♮.

The following Lemma may be of general interest, and presumably has been noticed independently by others. We will apply it to obtain asymptotic properties of moments in Section 5.

Lemma 4.3 *A. Let $D_n, n \geq n_0$ be a non-negative sequence satisfying*

$$D_{n+1} \leq \vartheta_n D_n + \gamma_n, \quad n \geq n_0. \tag{54}$$

Suppose that there exist $\vartheta > 0$ and $C \geq 0$ such that

$$0 \leq \vartheta_n \leq (1 - \vartheta/n) \quad \textit{and} \quad 0 \leq \gamma_n \leq \frac{C}{n}.$$

Then

$$\limsup_{n\to\infty} D_n < \infty.$$

B. Let $D_{n_0} > 0$, $n \geq n_0$ satisfy

$$D_{n+1} \geq \vartheta_n D_n + \gamma_n, \quad n \geq n_0. \tag{55}$$

Suppose there exists $\vartheta > 0$ such that

$$\vartheta_n \geq (1 + \vartheta/n), \quad \textit{and} \quad \gamma_n \geq 0.$$

Then

$$\lim_{n\to\infty} D_n = \infty.$$

Proof: Consider A. If (54) holds, then by induction, for all $n \geq n_0$ and $k \geq 0$,

$$D_{n+k+1} \leq \left(\prod_{j=n}^{n+k} \vartheta_j\right) D_n + \sum_{j=n}^{n+k} \left(\prod_{l=j+1}^{n+k} \vartheta_l\right) \gamma_j. \tag{56}$$

Using $\vartheta_n \leq (1 - \vartheta/n)$ and $1 - x \leq e^{-x}$ we have

$$\begin{aligned} \prod_{l=j+1}^{n+k} \vartheta_l &\leq \prod_{l=j+1}^{n+k} e^{-\vartheta/l} \\ &= \exp(-\vartheta \sum_{l=j+1}^{n+k} 1/l) \\ &\leq \exp(-\vartheta(\log(n+k) - \log(j+1))) \\ &= \left(\frac{j+1}{n+k}\right)^{\vartheta}. \end{aligned}$$

Hence, from (56), for all $k \geq 0$,

$$\begin{aligned} D_{n+k+1} &\leq \left(\prod_{j=n}^{n+k} \vartheta_j\right) D_n + \sum_{j=n}^{n+k} \left(\prod_{l=j+1}^{n+k} \vartheta_l\right) \gamma_j \\ &\leq D_n + \sum_{j=n}^{n+k} \left(\frac{j+1}{n+k}\right)^{\vartheta} \frac{C}{j} \\ &\leq D_n + \frac{2^{\vartheta} C}{(n+k)^{\vartheta}} \sum_{j=n}^{n+k} j^{\vartheta-1} \\ &\leq D_n + \frac{2^{\vartheta} C}{\vartheta} \left(\frac{n+k+1}{n+k}\right)^{\vartheta}. \end{aligned}$$

Letting $k \to \infty$ we see that the D_n sequence is bounded.

To prove B, note that for all j sufficiently large

$$\vartheta_j \geq (1 + \vartheta/j) \geq \exp(\vartheta/(2j)),$$

which gives, by (55),

$$D_{n+k+1} \geq \left(\prod_{j=n}^{n+k} \vartheta_j\right) D_n \geq \exp(\frac{\vartheta}{2} \sum_{j=n}^{n+k} \frac{1}{j}) D_n \to \infty \quad \text{as } k \to \infty. \; \natural$$

5 The Family $\mathcal{U}^\alpha$

As in (41), with $h(\cdot)$ defined in (8), let

$$H(y) = \int_0^y h(u)du + (1/\alpha - y)h(y);$$

note that $h(\cdot)$ is strictly increasing for $0 \le y < \infty$.

Lemma 5.1 *There exists a unique value $b_\alpha > 1/\alpha$ such that $H(b_\alpha) = 0$, and*

$$h^\alpha(b_\alpha) < 1 + \frac{1}{\alpha}. \tag{57}$$

Proof: By Lemma 4.1, $H(y)$ is strictly increasing for $0 < y < 1/\alpha$ and strictly decreasing for $1/\alpha < y < \infty$. Hence a root exists in $(1/\alpha, \infty)$ and is unique if H is ever negative. Since

$$H'(y) = (1/\alpha - y)h'(y),$$

for some constant a

$$H(y) = a + \int_{1/\alpha}^y (1/\alpha - u)h'(u)du. \tag{58}$$

Now, since $h(y)$ converges to a finite positive limit at infinity, and

$$h'(y) = \frac{1}{\alpha} h(y)^{1-\alpha} \frac{1}{(1 + \alpha y/(\alpha+1))^2},$$

we have that $y^2 h'(y)$ is bounded away from zero and infinity as $y \to \infty$, and therefore

$$\int_{1/\alpha}^\infty h'(u)du < \infty \quad \text{and} \quad \int_{1/\alpha}^y uh'(u)du \to \infty \quad \text{as } y \to \infty,$$

yielding from (58) that

$$\lim_{y\to\infty} H(y) = -\infty.$$

Inequality (57) follows from $\lim_{y\to\infty} h^\alpha(y) = 1 + 1/\alpha$.♮

For $f(y)$ as given in (24), setting

$$f_j^*(y) = f(y) - y/2j \tag{59}$$

we have

$$\frac{d}{dy}\left(y^{1/\alpha} f_j^*(y)\right) = y^{1/\alpha - 1}\left(\frac{f(y)}{\alpha} - \frac{yf(y)^{\alpha+1}}{\alpha+1} - \frac{y}{2j}(1/\alpha + 1)\right). \tag{60}$$

Since $yf(y)^\alpha$ is strictly increasing with limit $(\alpha+1)/\alpha$ at infinity, $f(y)/\alpha > yf(y)^{\alpha+1}/(\alpha+1)$ for all $y \ge 0$. Hence, for any $A > b_\alpha$ we have

$$\inf_{0 \le y \le A}\left(\frac{f(y)}{\alpha} - \frac{yf(y)^{\alpha+1}}{\alpha+1}\right) > 0.$$

It follows that there exists $j_0 = j_0(A)$ such that the derivative in (60) is positive for all $0 < y \le A$ and all $j > j_0$. For these j, set

$$k_j(y) = \begin{cases} y^{1/\alpha} f_j^*(y) & \text{for } 0 \le y < A \\ A^{1/\alpha} f_j^*(A) & \text{for } A \le y < \infty \end{cases} \tag{61}$$

and

$$K_j(y) = \int_0^y k_j(u)du + (1/\alpha - y)k_j(y). \tag{62}$$

Lemma 5.2 *There exists j_1 such that for all $j > j_1$ there are unique values $b_{j,\alpha} > 1/\alpha$ such that $K_j(b_{j,\alpha}) = 0$. Setting $d_{j,\alpha} = k_j(b_{j,\alpha})$ we have*

$$b_{j,\alpha} \to b_\alpha \quad \textit{and} \quad d_{j,\alpha} \to d_\alpha \quad \textit{as } j \to \infty, \textit{ where } d_\alpha = h(b_\alpha)\ . \tag{63}$$

Proof: We apply Lemma 4.1. The functions $k_j(\cdot)$ satisfy $k_j(0) = 0$, are non-decreasing everywhere and are strictly increasing and differentiable for $0 < y < A$. Further, $k_j(y)$ converges uniformly to $h(y)$ in $[0, A]$, yielding the uniform convergence of $K_j(y)$ to $H(y)$ in $[0, A]$. Since H is strictly decreasing in $(1/\alpha, \infty)$, it follows that $H(A) < H(b_\alpha) = 0$. Hence, since $K_j(A) \to H(A)$ as $j \to \infty$, for all j sufficiently large $K_j(A) < 0$. For such j Lemma 4.1 now yields the existence of a unique root $b_{j,\alpha} > 1/\alpha$ satisfying $K_j(b_{j,\alpha}) = 0$.

The uniform convergence of K_j to H implies $H(b_{j,\alpha}) \to 0$ as $j \to \infty$, from which the convergence of $b_{j,\alpha}$ to b_α follows. That $d_{j,\alpha}$ converges to d_α follows from the uniform convergence of k_j to h in $[0, A]$. $\natural$.

It will become convenient to consider value and scaled value sequences arising from stopping on the independent variables $U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, X_{m-1}, \ldots, X_1$. The scaled value sequence for this problem satisfies (18) with $c = m^{1/\alpha} V_m(X_m, \ldots, X_1)$. Note that for any m and c there exists $X_m, \ldots, X_1$ such that $c = m^{1/\alpha} V_m(X_m, \ldots, X_1)$; the simplest construction is obtained by letting $X_j = cm^{-1/\alpha}$ for $1 \le j \le m$. Our suppression of the dependence of W_n on m and c is justified by Theorem 5.1, which states that the limiting value of W_n is the same for all such sequences.

Lemma 5.3 *Let $m \ge 1$ be any integer and $c \in (0, \infty)$ be any constant. For $n > m$ let W_n be determined by the recursion (18) with starting value $W_m = c$, and let*

$$Z_m^+ = c \quad \textit{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1}^+ = \frac{1}{n} \int_0^n \left(h(y) \wedge Z_n^+\right) dy \quad \textit{for } n \ge m. \tag{64}$$

With j_1 as in Lemma 5.2, for all $j > j_1$ let $m_j^* = \max\{m, j\}$. Now define sequences $Z_{j,n}^-$ for $n \ge m_j^*$, by

$$Z_{j,m_j^*}^- = W_{m_j^*} \quad \textit{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} Z_{j,n+1}^- = \frac{1}{n} \int_0^n \left(k_j(y) \wedge Z_{j,n}^-\right) dy \quad \textit{for } n \ge m_j^*. \tag{65}$$

Then for all $n \ge m_j^*$,

$$Z_{j,n}^- \le W_n \le Z_n^+ \tag{66}$$

and

$$\lim_{n\to\infty} Z_{j,n}^- = d_{j,\alpha} \quad \textit{and} \quad \lim_{n\to\infty} Z_n^+ = d_\alpha. \tag{67}$$

Proof: With $j > j_1$ and f_j^* defined in (59), Lemmas 3.3, 3.2 and monotonicity of f_n give

$$f_j^*(y) < f_n(y) < f(y) \quad \text{for all } n \geq j \text{ and } 0 < y \leq n.$$

Therefore, by (61), (16) and (21),

$$k_j(y) < h_n(y) < h(y) \quad \text{for all } n \geq j \text{ and } 0 < y \leq n.$$

Equation (66) now follows by a comparison of (65), (18) and (64), and (67) follows directly from Theorem 4.1. ♮

Theorem 5.1 *Let $m \geq 2$ be any integer and suppose the variables $U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, \ldots, X_1$ are independent. Let*

$$V_{n,m} = V_n(U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, \ldots, X_1), \tag{68}$$

be the optimal two choice value, and suppose $V_m(X_m, \ldots, X_1) = c \in (0, \infty)$. Then

$$W_n = n^{1/\alpha} V_{n,m} \quad \textit{for } n > m,$$

satisfies

$$\lim_{n \to \infty} W_n = h(b_\alpha), \tag{69}$$

where b_α is the unique solution to (7).

In particular, the optimal two stop value V_n for a sequence of i.i.d. variables with distribution function $\mathcal{U}^\alpha(x) = x^\alpha$ for $0 \leq x \leq 1$ and $\alpha > 0$ satisfies

$$\lim_{n \to \infty} n\mathcal{U}^\alpha(V_n) = h^\alpha(b_\alpha); \tag{70}$$

that is, the conclusion of Theorem 1.1 holds for the $\mathcal{U}^\alpha$ family of distributions.

Proof: We apply Lemma 5.3. Letting $n \to \infty$ in (66) and using (67),

$$d_{j,\alpha} \leq \liminf_{n \to \infty} W_n \leq \limsup_{n \to \infty} W_n \leq d_\alpha \quad \text{for all } j > j_1.$$

Now letting $j \to \infty$ and using (63) gives (69). The W_n values for the i.i.d. sequence with distribution function $\mathcal{U}^\alpha$ are generated by recursion (18) for the particular case $m = 2$ and $c = 2^{1/\alpha} E[U_2^{1/\alpha} \wedge U_1^{1/\alpha}]$, thus proving (70) ♮.

We conclude this section with some results on the existence of moments for both the one and two-stop problems.

Theorem 5.2 *Let $U_n^{1/\alpha}, \ldots, U_1^{1/\alpha}$ be an i.i.d. sequence with distribution function $\mathcal{U}^\alpha$, a_n a sequence of constants in $[0, 1]$ with $a_0 = 1$, and*

$$T_n = \max\{1 \leq k \leq n : U_k^{1/\alpha} \leq a_{k-1}\}. \tag{71}$$

When $A_n = n^{1/\alpha} a_n$ satisfies

$$0 < \underline{\kappa} = \liminf_{n \to \infty} A_n \leq \limsup_{n \to \infty} A_n = \overline{\kappa} < \infty$$

we have
$$\limsup_{n\to\infty} E(n^{1/\alpha}U_{T_n}^{1/\alpha})^r < \infty \quad \textit{for all } r < \alpha\underline{\kappa}^\alpha. \tag{72}$$
If
$$\overline{\kappa} < \infty,$$
we have
$$\lim_{n\to\infty} E(n^{1/\alpha}U_{T_n}^{1/\alpha})^r = \infty \quad \textit{for all } r > \alpha\overline{\kappa}^\alpha. \tag{73}$$

Proof: Let
$$M_n(r) = E(U_{T_n}^{r/\alpha})$$
be the r^{th} moment of the a_k stopped sequence. The sequence $M_n(r)$ satisfies the recursion
$$M_{n+1}(r) \;=\; \int_0^{a_n} x^r \alpha x^{\alpha-1}dx + (1-a_n^\alpha)M_n(r), \quad n \geq 1.$$
Substituting $y = nx^\alpha$,
$$M_{n+1}(r) \;=\; \frac{1}{n^{1+r/\alpha}}\int_0^{A_n^\alpha} y^{r/\alpha}dy + (1-a_n^\alpha)M_n(r).$$
Multiplying by $n^{r/\alpha}$, and letting $n^{r/\alpha}M_n(r) = S_n(r)$,
$$\begin{aligned}\left(\frac{n}{n+1}\right)^{r/\alpha} S_{n+1}(r) &= \frac{1}{n}\int_0^{A_n^\alpha} y^{r/\alpha}dy + (1-\frac{A_n^\alpha}{n})S_n(r)\\ &= \frac{A_n^{\alpha+r}}{n(1+r/\alpha)} + (1-\frac{A_n^\alpha}{n})S_n(r).\end{aligned}$$
To show (72), first note that
$$\left(\frac{n+1}{n}\right)^{r/\alpha} = 1 + \frac{r}{\alpha n} + O_{r/\alpha}(n^{-2}).$$
Now multiply by $((n+1)/n)^{r/\alpha}$ and use the boundedness of the sequence A_n and $r < \alpha\underline{\kappa}^\alpha$ to obtain, for all n sufficiently large,
$$S_{n+1}(r) \leq \frac{2^{r/\alpha}A_n^{\alpha+r}}{n(1+r/\alpha)} + (1 - \frac{(A_n^\alpha - r/\alpha)}{2n})S_n(r);$$
(72) now follows from Lemma 4.3 A.

To show (73) we note that for all n sufficiently large, recalling that $r > \alpha\overline{\kappa}^\alpha$,
$$\begin{aligned} S_{n+1}(r) &\geq (\frac{n+1}{n})^{r/\alpha}(1-\frac{A_n^\alpha}{n})S_n(r)\\ &= (1+\frac{r}{\alpha n} + O_{r/\alpha}(n^{-2}))(1-\frac{A_n^\alpha}{n})S_n(r)\\ &= \left(1 + \frac{(r/\alpha - A_n^\alpha)}{n} + O_{r/\alpha,\overline{\kappa}}(n^{-2})\right)S_n(r)\\ &\geq (1+\frac{(r/\alpha - A_n^\alpha)}{2n})S_n(r).\end{aligned}$$
Now apply Lemma 4.3 B.

Corollary 5.1 *Let $\mathbf{1}_n^{U^{1/\alpha}}$ and $\mathbf{2}_n^{U^{1/\alpha}}$ be the one and two choice random values obtained from optimally stopping an independent sequence of variables having distribution $\mathcal{U}^\alpha$. In the one choice case,*

$$\text{if } r < 1+\alpha, \quad \limsup_{n\to\infty} E(n^{1/\alpha}\mathbf{1}_n^{U^{1/\alpha}})^r < \infty, \tag{74}$$
$$\text{while if } r > 1+\alpha, \quad \limsup_{n\to\infty} E(n^{1/\alpha}\mathbf{1}_n^{U^{1/\alpha}})^r = \infty.$$

In the two choice case,

$$\text{if } r < 1+\alpha, \quad \limsup_{n\to\infty} E(n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}})^r < \infty. \tag{75}$$

Proof: For one choice, apply Theorem 5.2 with $a_n = V_n^1$, and therefore $T_n = \mathbf{1}_n^{U^{1/\alpha}}$. By (5),

$$\lim_{n\to\infty} n^{1/\alpha}V_n^1 = \lim_{n\to\infty}(n\mathcal{U}^\alpha(V_n^1))^{1/\alpha} = (1+1/\alpha)^{1/\alpha}.$$

The one choice results now follow from (72) and (73) of Theorem 5.2 with $\overline{\kappa} = \underline{\kappa} = (1+1/\alpha)^{1/\alpha}$.

For two choices, let T_n be defined as in (71) with b_n, the first choice thresholds given in (12), replacing a_n, and let $B_n = n^{1/\alpha}b_n$. Then as $\mathbf{2}_n^{U^{1/\alpha}} \le U_{T_n}^{1/\alpha}$, it clearly suffices to show that for $r < 1+\alpha$,

$$\limsup_{n\to\infty} E(n^{1/\alpha}U_{T_n}^{1/\alpha})^r < \infty.$$

Reiterating (19), $W_n = h_n(B_n^\alpha)$, and by Theorem 5.1

$$\lim_{n\to\infty} W_n = d_\alpha = h(b_\alpha).$$

We show $\lim_{n\to\infty} B_n^\alpha = b_\alpha$. Suppose $\limsup_{n\to\infty} B_n^\alpha = B^\alpha > b_\alpha$. Then there exists $\epsilon > 0$ such that $B^\alpha - \epsilon > b_\alpha$. But then $\limsup_{n\to\infty} h_n(B_n^\alpha) \ge \limsup_{n\to\infty} h_n(B^\alpha-\epsilon) = h(B^\alpha-\epsilon) > h(b_\alpha)$, a contradiction. Similarly if $\liminf_{n\to\infty} B_n^\alpha < b_\alpha$. Thus the limit of B_n exists and

$$n^{1/\alpha}b_n = B_n \to b_\alpha^{1/\alpha}.$$

By (72) it suffices to show that $b_\alpha > 1+1/\alpha$, which, by Lemmas 4.1 and 5.1, would follow from $H(1+1/\alpha) > 0$. Now

$$\begin{aligned} H(1+1/\alpha) &= (1+\alpha)^{1/\alpha}\left[\int_0^{1+1/\alpha}\left(\frac{y}{\alpha+1+\alpha y}\right)^{1/\alpha} dy - \left(\frac{1+1/\alpha}{2(1+\alpha)}\right)^{1/\alpha}\right] \\ &> (1+\alpha)^{1/\alpha}\left[\left(\frac{1}{2(1+\alpha)}\right)^{1/\alpha}\int_0^{1+1/\alpha} y^{1/\alpha}dy - \left(\frac{1+1/\alpha}{2(1+\alpha)}\right)^{1/\alpha}\right] \\ &= 0, \end{aligned}$$

completing the proof. ♮

Remark 5.1 *Kennedy and Kertz (1991, Theorem 1.4) obtain the limiting distribution of the scaled optimal one stop random variable $n^{1/\alpha}\mathbf{1}_n^{U^{1/\alpha}}$. It is easily checked that this limiting distribution has a finite r^{th} moment if and only if $r < 1+\alpha$, which is not surprising, when compared with (74) in Corollary 5.1.*

Remark 5.2 *From the proof that $b_\alpha > 1+1/\alpha$ in Corollary 5.1, it follows that the limiting thresholds b_n for the first choice in the optimal two-choice problem are larger than the corresponding values V_n^1 for the optimal one choice problem, for all $\alpha > 0$. This is reasonable, as with two choices one 'can afford' to make the first of the two choices in the two stop problem earlier than the only choice in the one stop problem.*

Another interpretation of the inequality $b_n > V_n^1$ is gained by applying $V_n^1()$ to both sides, to obtain $V_n > V_{2n}^1$ i.e. one is better off having one choice among $2n$ variables than having two choices among n variables.

Remark 5.3 *Whereas it follows from Resnick, (1987, Proposition 2.1) that all scaled moments of the minimum exist, it is of interest to note that no moment with $r > 1+\alpha$ exists for the optimal scaled one-choice value.*

6 Extension to General Distributions

In Theorem 5.1 we considered the special case where the variables had distribution function $\mathcal{U}^\alpha(x)$ as in (9). In this section we prove Theorem 1.1, thus extending our results to a much wider class.

For $X_n, \ldots, X_1$ an i.i.d. sequence of random variables with distribution function F_X, we will let V_n^X denote its optimal two stop value. The proof of Theorem 1.1 will be given at the end of this section. First note, however, that without loss of generality, we may take $\lim_{x\downarrow 0} L(x) = 1$, since if $F_X(x) = x^\alpha L_{\mathcal{L}}(x)$ with $\lim_{x\downarrow 0} L_{\mathcal{L}}(x) = \mathcal{L} \in (0,\infty)$, then $Z = \mathcal{L}^{1/\alpha}X$ has distribution function $F_Z(z) = z^\alpha(1/\mathcal{L})L_{\mathcal{L}}(\mathcal{L}^{-1/\alpha}z)$ with $\lim_{z\downarrow 0}(1/\mathcal{L})L_{\mathcal{L}}(\mathcal{L}^{-1/\alpha}z) = 1$. Since $V_n^Z = \mathcal{L}^{1/\alpha}V_n^X$, we have

$$F_Z(V_n^Z) = F_X(V_n^X),$$

and hence we can assume that X has distribution function F such that

$$F(x) = x^\alpha L(x) \quad \lim_{x\downarrow 0} L(x) = 1. \tag{76}$$

To prove Theorem 1.1 the two stop problem is considered for $X_n, \ldots, X_1$, independent but not necessarily identically distributed random variables; it is direct to see that the dynamic programming equations given in the introduction for an i.i.d. sequence hold under the assumption of independence alone. In particular, the functions $V_n^1(x)$ and the two stop value V_n are again given through (1) and (2) respectively. We begin by giving conditions such that the threshold sequences are uniquely defined.

Lemma 6.1 *Let $X_n, \ldots, X_1$ be non-negative independent random variables with distribution functions $F_n, \ldots, F_1$ respectively, and with x_F given in (3), let*

$$x_{\bar{F}_k} = \min\{x_{F_1}, \ldots, x_{F_k}\}, \quad 1 \le k \le n.$$

Then the function $V_k^1(x)$ given by (1) is continuous and strictly monotone increasing in $x \in [0, x_{\bar{F}_k}]$. Furthermore, assuming $E(X_2 \wedge X_1) < \infty$, the indifference numbers $b_k, 2 \le k \le n-1$ given by

$$V_k = V_k^1(b_k)$$

exist and are unique in $[0, x_{\bar{F}_k}]$.

Proof: The function $V_1^1(x) = E[x \wedge X_1] = x - \int_0^x F_1(y)dy$ is continuous and strictly increasing for x in $[0, x_{F_1}]$, and $0 \le V_1^1(x) \le x$. Now assume $V_{k-1}^1(x)$ is continuous and strictly increasing in $[0, x_{\bar{F}_{k-1}}]$ and that $0 \le V_{k-1}^1(x) \le x$. That $V_k^1(x)$ is continuous follows directly from the dominated convergence theorem. To prove strict monotonicity, take $0 \le x < y \le x_{\bar{F}_k}$. Since $x_{\bar{F}_k} \le x_{\bar{F}_{k-1}}$ we have $0 \le V_k^1(x) = E[V_{k-1}^1(x) \wedge X_k] \le V_{k-1}^1(x) \le x$. Using the induction hypotheses, $0 \le V_{k-1}^1(x) < V_{k-1}^1(y) \le y \le x_{\bar{F}_k}$. Therefore $P(X_k > V_{k-1}^1(x)) > 0$, and hence

$$V_k^1(x) = E[V_{k-1}^1(x) \wedge X_k] < E[V_{k-1}^1(y) \wedge X_k] = V_k^1(y).$$

Since

$$V_k^1(b_k) = V_k \le V_k^1 = V_k^1(x_{\bar{F}_k}),$$

and $V_k^1(x)$ is continuous and strictly monotone increasing in $[0, x_{\bar{F}_k}]$, the value $b_k \le x_{\bar{F}_k}$ is determined uniquely in this interval.♮

Lemma 6.2 *For any sequence of nonnegative independent random variables $X_n, \ldots, X_1$ with $E[X_2 \wedge X_1] < \infty$ the b_n sequence is monotone non-increasing.*

Proof: We first show that

$$V_{n+1}^2 \le E[X_{n+1} \wedge V_n^2].$$

The right hand side is the value of the one choice problem for the two variables X_{n+1}, V_n^2. But this value can be achieved in the two choice problem by the suboptimal rule of choosing X_{n+1} and forgetting about any second choice if X_{n+1} is less than V_n^2, and retaining two choices otherwise. Therefore

$$V_{n+1}^1(b_{n+1}) = V_{n+1}^2 \le E[X_{n+1} \wedge V_n^2] = E[X_{n+1} \wedge V_n^1(b_n)] = V_{n+1}^1(b_n).$$

Since the functions $V_n^1(x)$ are strictly monotone increasing in $[0, x_{\bar{F}_n}]$ for all n, the Lemma is shown. ♮.

Lemma 6.3 *Let $X_n, \ldots, X_1$ and $Y_n, \ldots, Y_1$ be sequences of independent non-negative random variables satisfying*

$$E[Y_2 \wedge Y_1] \le E[X_2 \wedge X_1] < \infty, \tag{77}$$

and having two choice value and threshold sequences V_j^X, V_j^Y and b_j^X, b_j^Y respectively.

If for some $m \ge 2$,

$$Y_j \le_d X_j, \quad j = 3, \ldots, m \tag{78}$$

and there exists τ such that

$$\tau \ge \max\{b_m^X, b_m^Y\} \quad and \quad \tau \wedge Y_{j+1} \le_d \tau \wedge X_{j+1} \quad for\ m \le j < n, \tag{79}$$

then

$$V_j^Y \le V_j^X \quad for\ j = 2, \ldots, n.$$

Hence, if the inequalities in (77), (78) and (79) are replaced by equalities, then $V_j^Y = V_j^X, j = 2, 3, \ldots, n$, and so in particular V_n^Y is unchanged upon replacing any Y_{j+1} by $\tau \wedge Y_{j+1}, 2 \le j < n$, for any $\tau \ge b_j^Y$.

Proof: Clearly $V_j^Y \le V_j^X$ for $2 \le j \le m$. Let $V_n^{X,1}(x)$ and $V_n^{Y,1}(x)$ denote the optimal one choice value functions for the X and Y sequences respectively, with guaranteed value x, as in (1). Clearly, for $j = m$ we have $V_j^Y \le V_j^X$, and $V_j^{Y,1}(x) \le V_j^{X,1}(x)$ for all $x \le \tau$. Assuming these statement are true for some $m \le j < n$, then for $x \le \tau$ we have $V_j^{X,1}(x) \le V_j^{X,1}(\tau) \le \tau$ and so

$$Y_{j+1} \wedge V_j^{Y,1}(x) \le Y_{j+1} \wedge V_j^{X,1}(x) \le_d X_{j+1} \wedge V_j^{X,1}(x),$$

giving

$$V_{j+1}^{Y,1}(x) = E[Y_{j+1} \wedge V_j^{Y,1}(x)] \le E[X_{j+1} \wedge V_j^{X,1}(x)] = V_{j+1}^{X,1}(x).$$

Therefore

$$\begin{aligned}
V_{j+1}^Y &= E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^Y] = E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^{Y,1}(b_j^Y) \wedge V_j^Y] = E[V_j^{Y,1}(Y_{j+1} \wedge b_j^Y) \wedge V_j^Y] \\
&\le E[V_j^{Y,1}(X_{j+1} \wedge b_j^Y) \wedge V_j^Y] \le E[V_j^{X,1}(X_{j+1} \wedge b_j^Y) \wedge V_j^Y] \\
&= E[V_j^{X,1}(X_{j+1}) \wedge V_j^{X,1}(b_j^Y) \wedge V_j^{Y,1}(b_j^Y)] = E[V_j^{X,1}(X_{j+1}) \wedge V_j^{Y,1}(b_j^Y)] \\
&= E[V_j^{X,1}(X_{j+1}) \wedge V_j^Y] \le E[V_j^{X,1}(X_{j+1}) \wedge V_j^X] \\
&= V_{j+1}^X. \natural
\end{aligned}$$

Corollary 6.1 *Let $X_n, \ldots, X_1$ be a sequence of i.i.d. non-negative random variables with $E[X_2 \wedge X_1] < \infty$ and distribution function satisfying (76). Then there exists an i.i.d. sequence $Y_n, \ldots, Y_1$ of bounded non-negative random variables with distribution function satisfying (76) and $V_n^Y = V_n^X$ for all $n \ge 2$.*

Proof: Assume $x_F = \infty$, else there is nothing to prove. For all $x > 0$ sufficiently small, using the non-degeneracy of the distribution on $[0, x]$, Jensen's inequality applied to the concave function $\psi(u) = u \wedge x$ yields

$$E[x \wedge X_1] \le x \wedge EX_1, \quad \text{with strict inequality for all } x \text{ sufficiently small.}$$

Thus

$$E[X_2 \wedge X_1 | X_2] \le X_2 \wedge EX_1, \quad \text{with strict inequality having positive probability}$$

and therefore, since $V_1^1(\infty) = EX_1$ (which may be infinite),

$$V_2 = E[X_2 \wedge X_1] < E[X_2 \wedge EX_1] = V_2^1(\infty).$$

Using $x_F = \infty$ and Lemma 6.1, $V_2^1(x)$ is continuous and strictly monotone increasing on $(0, \infty)$, hence the solution b_2 to

$$V_2 = V_2^1(x)$$

exists, is unique, and satisfies $b_2 < \infty$.

For $j = 1, \ldots, n$ let

$$Y_j = \begin{cases} X_j & \text{for } X_j \le b_2 \\ K & \text{for } X_j > b_2. \end{cases}$$

Since the distribution of X_j is unbounded, $P(X_j > b_2) > 0$, which guarantees that K can be chosen to yield $E[Y_2 \wedge Y_1] = E[X_2 \wedge X_1]$. Now apply Lemma 6.3 with $m = 2$ and $\tau = b_2$. $\natural$.

We have the following Lemma.

Lemma 6.4 *Let X have distribution function $F(x) = P(X \le x)$, and set*

$$F^{-1}(u) = \sup\{x : F(x) < u\} \quad \textit{for } 0 < u < 1.$$

Then

$$F(x) \ge u \quad \textit{if and only if} \quad x \ge F^{-1}(u), \tag{80}$$

and with $U \sim \mathcal{U}(0,1)$ we have

$$X =_d F^{-1}(U). \tag{81}$$

In addition, if

$$F(x) = x^\alpha L_F(x), \quad \textit{for all } x \ge 0, \textit{ with } \lim_{x\downarrow 0} L_F(x) = 1,$$

then there exists a function L^* such that

$$F^{-1}(u) = u^{1/\alpha} L_{F^{-1}}(u) = u^{1/\alpha} L^*(u^{1/\alpha}), \quad \textit{with} \quad \lim_{u\downarrow 0} L^*(u) = 1, \tag{82}$$

so that by (81)

$$X =_d U^{1/\alpha} L^*(U^{1/\alpha}). \tag{83}$$

Proof: Let $A_u = \{x : F(x) < u\}$. If $F(x) \ge u$ then $x \notin A_u$ and therefore $F^{-1}(u) \le x$. If $F(x) < u$ then by right continuity there exists $\epsilon > 0$ such that $F(x+\epsilon) < u$. Thus $x + \epsilon \in A_u$, which gives that $F^{-1}(u) \ge x + \epsilon > x$. This demonstrates (80). Now replacing u by a random variable U having the $\mathcal{U}[0,1]$ distribution we obtain (81), by $P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$.

The claim in (82) is equivalent to

$$\lim_{u\downarrow 0} \frac{(F^{-1}(u))^\alpha}{u} = 1. \tag{84}$$

Using that $F(x) = x^\alpha L_F(x)$,

$$F^{-1}(u) = \sup\{x : x^\alpha L_F(x) < u\},$$

and hence, setting $L_\alpha(y) = L_F(y^{1/\alpha})$,

$$(F^{-1}(u))^\alpha = \sup\{x^\alpha : x^\alpha L_F(x) < u\} = \sup\{y : y L_F(y^{1/\alpha}) < u\} = \sup\{y : y L_\alpha(y) < u\}. \tag{85}$$

Note $y L_\alpha(y) = F(y^{1/\alpha})$ is non-decreasing. Let $\epsilon \in (0,1)$ be given. Since $\lim_{y\downarrow 0} L_\alpha(y) = 1$, there exists $\delta > 0$ such that

$$1 - \epsilon < L_\alpha(y) < 1 + \epsilon \quad \text{for all } 0 < y < \delta. \tag{86}$$

Let $0 < u < \delta(1-\epsilon)$. Then if $0 < y < u/(1+\epsilon)$ we have $y < \delta$ and so

$$y L_\alpha(y) < y(1+\epsilon) < u,$$

so

$$\{y : 0 < y < u/(1+\epsilon)\} \subset \{y : y L_\alpha(y) < u\}.$$

Thus

$$u/(1+\epsilon) \le (F^{-1}(u))^\alpha \quad \text{for all } 0 < u < \delta(1-\epsilon).$$

By (86)

$$u < (1-\epsilon)y < yL_\alpha(y) \tag{87}$$

holds for any $y \in (u/(1-\epsilon), \delta)$; it follows by (85) that

$$(F^{-1}(u))^\alpha \le u/(1-\epsilon).$$

Hence,

$$1/(1+\epsilon) \le \frac{(F^{-1}(u))^\alpha}{u} \le 1/(1-\epsilon) \quad \text{for } 0 < u < \delta(1-\epsilon),$$

and (84) is shown. ♮.

Lemma 6.5 *Let $\chi_n, n = 1, 2, \ldots$ be a uniformly integrable non-negative sequence of random variables, and $0 \le L_n \le L$, L a constant, with $L_n \to_p 1$ as $n \to \infty$. Then*

$$\limsup_{n\to\infty} E\chi_n L_n \quad = \quad \limsup_{n\to\infty} E\chi_n$$

so that in particular, if $\lim_{n\to\infty} E\chi_n$ exists,

$$\limsup_{n\to\infty} E\chi_n L_n = \lim_{n\to\infty} E\chi_n.$$

Proof: Let $\epsilon > 0$ be given. Since χ_n is uniformly integrable, there exists $\delta > 0$ such that

$$E\chi_n \mathbf{1}_A \le \epsilon \quad \text{whenever} \quad P(A) \le \delta. \tag{88}$$

Since $L_n \to_p 1$ as $n \to \infty$, there exists n_0 such that for all $n \ge n_0$

$$\Omega_n = \{|L_n - 1| \le \epsilon\} \quad \text{satisfies} \quad P(\Omega_n) \ge 1 - \delta.$$

Hence, for $n \ge n_0$, using (88) and that $\chi_n \ge 0$, with $A = \Omega_n^c$,

$$(1-\epsilon)E\chi_n \mathbf{1}_{\Omega_n} \le E\chi_n L_n \le (1+\epsilon)E\chi_n \mathbf{1}_{\Omega_n} + L\epsilon$$

and

$$E\chi_n - \epsilon \le E\chi_n \mathbf{1}_{\Omega_n} \le E\chi_n,$$

so that for $n \ge n_0$ we have

$$(1-\epsilon)(E\chi_n - \epsilon) \le E\chi_n L_n \le (1+\epsilon)E\chi_n + L\epsilon.$$

Taking lim sup and recalling $\epsilon > 0$ was arbitrary completes the proof. ♮

Lemma 6.6 *Let $X_n, \ldots, X_1$ be an integrable i.i.d. sequence with distribution function $F(x)$ satisfying (76). Let $W_n^X = n^{1/\alpha} V_n^X$ and $W_n^{U^{1/\alpha}} = n^{1/\alpha} V_n^{U^{1/\alpha}}$. Then*

$$\limsup_{n\to\infty} W_n^X \le \lim_{n\to\infty} W_n^{U^{1/\alpha}}.$$

Proof: Using Lemma 6.4, we construct i.i.d. pairs (U_i, X_i) with $U_i \sim \mathcal{U}, X_i \sim F$, and

$$X_i = U_i^{1/\alpha} L^*(U_i^{1/\alpha}).$$

By Corollary 6.1, without loss of generality we can take the X variables to be bounded, and since $L^*(u) \to 1$ as $u \downarrow 0$, it follows that L^* is bounded.

Let $\mathbf{2}_n^{U^{1/\alpha}}$ and $\mathbf{2}_n^X$ be the optimal random n-variable two-stop value for the $U_n^{1/\alpha}, \ldots, U_1^{1/\alpha}$ and $X_n, \ldots, X_1$ sequences respectively. Since $En^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}} = n^{1/\alpha}V_n^{U^{1/\alpha}} = W_n^{U^{1/\alpha}}$ converges (to $h(b_\alpha)$), we have

$$P(\mathbf{2}_n^{U^{1/\alpha}} > \epsilon) = P(n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}} > n^{1/\alpha}\epsilon) \le \frac{W_n^{U^{1/\alpha}}}{n^{1/\alpha}\epsilon} \to 0 \quad \text{as } n \to \infty.$$

Hence $\mathbf{2}_n^{U^{1/\alpha}} \to_p 0$, and therefore $L^*(\mathbf{2}_n^{U^{1/\alpha}}) \to_p 1$. Furthermore, by Corollary 5.1, the collection $n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}}$ has a bounded r^{th} moment for some $r > 1$ and hence is uniformly integrable.

Let $\mathbf{2}_n^{X,U^{1/\alpha}}$ denote the X sequence stopped on the optimal rules for the $U^{1/\alpha}$ sequence. Then $\mathbf{2}_n^{X,U^{1/\alpha}} = \mathbf{2}_n^{U^{1/\alpha}} L^*(\mathbf{2}_n^{U^{1/\alpha}})$, and since these rules may not be optimal for the X sequence we have

$$En^{1/\alpha}\mathbf{2}_n^X \le En^{1/\alpha}\mathbf{2}_n^{X,U^{1/\alpha}} = En^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}} L^*(\mathbf{2}_n^{U^{1/\alpha}}).$$

Taking limsup and using that $n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}}$ is uniformly integrable and L^* is bounded and $L^*(\mathbf{2}_n^{U^{1/\alpha}}) \to_p 1$, the result follows from Lemma 6.5 and the fact that $W_n^{U^{1/\alpha}}$ converges. ♮

Lemma 6.7 *Let $X_n, \ldots, X_1$ be i.i.d. random variables with distribution function F satisfying (76). Then the indifference values b_n for X satisfy*

$$\lim_{n\to\infty} b_n = 0.$$

Proof: Since b_n is monotone non-increasing by Lemma 6.2, $b_n \downarrow b \ge 0$. We have

$$V_n^1(X_n \wedge b, \ldots, X_1 \wedge b) = g_n(b) \le g_n(b_n) = V_n^X. \tag{89}$$

Hence the two choice value from $X_n, \ldots, X_1$ is greater (worse) than the optimal one choice value of a sequence of i.i.d. random variables $b \wedge X_n, \ldots, b \wedge X_1$. If $b > 0$, by (5), the limit of the scaled optimal one choice value, say, $W_n^{X\wedge b,1}$ of this sequence is the same as the limit of $W_n^{X,1}$, the scaled optimal one choice value for $X_n, \ldots, X_1$. But then, using (89) in the first inequality, Lemma 6.6 for the second inequality, Theorem 5.1 for the equality, (57) for the strict inequality and the results of Kennedy and Kertz (1991) for the last two equalities we have

$$\lim_{n\to\infty} W_n^{X,1} \le \limsup_{n\to\infty} W_n^X \le \lim_{n\to\infty} W_n^{U^{1/\alpha}} = h(b_\alpha) < (1+1/\alpha)^{1/\alpha} = \lim_{n\to\infty} W_n^{U^{1/\alpha},1} = \lim_{n\to\infty} W_n^{X,1},$$

a contradiction. ♮

Lemma 6.8 *Let (U_i, X_i), $i = n, \ldots, 1$ be independent pairs of random variables with U_i uniform on $[0,1]$ and X_i having distribution function F satisfying (76). Let $V_{n,m}$ be defined as in (68), giving in particular $V_{n,n} = V_n^X$. Then for every $\epsilon \in (0,1)$, there exists m such that*

$$\frac{1}{1+\epsilon} \le \liminf_{n\to\infty} \frac{V_{n,m}}{V_{n,n}} \le \limsup_{n\to\infty} \frac{V_{n,m}}{V_{n,n}} \le \frac{1}{1-\epsilon}. \tag{90}$$

Proof: Using (83) of Lemma 6.4, we can construct the i.i.d. X sequence using an i.i.d. sequence $U^{1/\alpha}$ with distribution $\mathcal{U}^\alpha$ by defining X_i as

$$X_i = U_i^{1/\alpha} L^*(U_i^{1/\alpha}) \quad \text{a.s.} \tag{91}$$

where $\lim_{u \downarrow 0} L^*(u) = 1$. Hence, for the given $\epsilon \in (0,1)$ there exists $\delta > 0$ such that

$$1 - \epsilon \le L^*(u^{1/\alpha}) \le 1 + \epsilon \quad \text{for } 0 < u \le \delta, \tag{92}$$

and so by (91) and (92) we have

$$(1+\epsilon)^{-1} X_i \le U_i^{1/\alpha} \le (1-\epsilon)^{-1} X_i \quad \text{when } U_i \le \delta.$$

By condition (76), F is continuous at 0 and satisfies $F(0) = 0$, and therefore there exists $\rho > 0$ with $0 < F(\rho) \le \delta$. But by (80), since

$$U_i \le F(\rho) \quad \text{if and only if} \quad X_i \le \rho,$$

we have

$$\text{if} \quad X_i \le \rho \quad \text{then} \quad U_i \le \delta.$$

Let $\tau = \min\{\delta, \rho\}$, and b_n^X and $b_n^{U^{1/\alpha}}$ be the indifference values for the X and $U^{1/\alpha}$ variables, respectively, which by Lemma 6.7 converge monotonically to zero. Hence there exists m with

$$\max\{b_m^{U^{1/\alpha}}, b_m^X\} \le \tau,$$

and for all $n \ge m$, by Lemma 6.3,

$$\begin{aligned}
& (1+\epsilon)^{-1} V_n(X_n, \ldots, X_1) \\
= \; & (1+\epsilon)^{-1} V_n(X_n \wedge \tau, \ldots, X_{m+1} \wedge \tau, X_m, \ldots, X_1) \\
= \; & V_n((1+\epsilon)^{-1}(X_n \wedge \tau), \ldots, (1+\epsilon)^{-1}(X_{m+1} \wedge \tau), (1+\epsilon)^{-1} X_m, \ldots, (1+\epsilon)^{-1} X_1) \\
\le \; & V_n(U_n^{1/\alpha} \wedge \tau, \ldots, U_{m+1}^{1/\alpha} \wedge \tau, X_m, \ldots, X_1) \\
= \; & V_n(U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, \ldots, X_1) \\
= \; & V_n(U_n^{1/\alpha} \wedge \tau, \ldots, U_{m+1}^{1/\alpha} \wedge \tau, X_m, \ldots, X_1) \\
\le \; & V_n((1-\epsilon)^{-1}(X_n \wedge \tau), \ldots, (1-\epsilon)^{-1}(X_{m+1} \wedge \tau), (1-\epsilon)^{-1} X_m, \ldots, (1-\epsilon)^{-1} X_1) \\
\le \; & (1-\epsilon)^{-1} V_n(X_n \wedge \tau, \ldots, X_{m+1} \wedge \tau, X_m, \ldots, X_1) \\
= \; & (1-\epsilon)^{-1} V_n(X_n, \ldots, X_1).
\end{aligned}$$

Now dividing by $V_{n,n}$ we see that for all $n \ge m$,

$$\frac{1}{1+\epsilon} \le \frac{V_{n,m}}{V_{n,n}} \le \frac{1}{1-\epsilon},$$

completing the proof. ♮

Proof of Theorem 1.1: Clearly, for all $0 \le m \le n$,

$$\frac{V_n(U_n^{1/\alpha}, \ldots, U_1^{1/\alpha})}{V_n(X_n, \ldots, X_n)} = \frac{V_{n,0}}{V_{n,n}} = \frac{V_{n,0}}{V_{n,m}} \frac{V_{n,m}}{V_{n,n}}.$$

Given $\epsilon \in (0,1)$, let m be such that (90) holds. But for any fixed m we have by Theorem 5.1 that

$$\lim_{n\to\infty} \frac{V_{n,0}}{V_{n,m}} = 1.$$

Hence by Lemma 6.8,

$$\frac{1}{1+\epsilon} \le \liminf_{n\to\infty} \frac{V_{n,0}}{V_{n,n}} \le \limsup_{n\to\infty} \frac{V_{n,0}}{V_{n,n}} \le \frac{1}{1-\epsilon},$$

and therefore the limit of the ratio exists and equals one. Applying Theorem 5.1 to the sequence $n^{1/\alpha}V_{n,0}$ completes the proof of Theorem 1.1. ♮

7 Numerical Results and Additional Remarks

In Table 1, for the $\alpha = 0.1, 0.2, \ldots 1, 2, \ldots 10$ values in column (1), we tabulate the following quantities in the columns indicated:

(2) b_α

(3) $\lim_{n\to\infty} nF(V_n^1) = (1 + 1/\alpha)$

(4) $\lim_{n\to\infty} nF(V_n^2) = h^\alpha(b_\alpha) = d_\alpha^\alpha$ and

(5) $\lim_{n\to\infty} nF(Em_n) = \Gamma(1 + 1/\alpha)^\alpha$,

for $F(x) = x^\alpha L(x)$ and $\lim_{x\to 0} L(x) = \mathcal{L}$ existing in $(0,\infty)$. In columns (6),(7), and (8), we tablulate the ratios (3)/(4), (4)/(5) and (3)/(5). Note that another natural comparison would be among the values listed raised to the power $1/\alpha$, as this would yield a comparison of the actual limiting values of $V_n^1/V_n^2, V_n^2/E(m_n)$ and $V_n^1/E(m_n)$ respectively. The reason that Table 1 lists the values in the way it does is to display them in a comparable order of magnitude to make numerical comparisons easier. The final column of Table 1 presents the relative improvement attained by using two stops rather than one, as compared to the reference value of the prophet,

$$\lim_{n\to\infty} (V_n^1 - V_n^2)/(V_n^1 - Em_n). \tag{93}$$

As evident from the table, the improvement is highly significant for all values of α.

The following asymptotic results can be shown to hold:

(i) For $\alpha \to \infty$,

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} nF(V_n^1) = 1$$

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} nF(V_n^2) = 1 - 1/e$$

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} nF(Em_n) = e^{-\gamma}$$

where $\gamma = .5772\ldots$ is Euler's constant. The limiting value for the relative improvement (93) given in the last column is

$$[1 - \log(e-1)]/\gamma = 0.7946\ldots$$

(ii) For $\alpha \to 0$,

The quantities in columns (3), (4) and (5) all tend to infinity, but the ratios in columns (6),(7),(8) and (9) tend to a finite limit, and are respectively

$$\lim_{\alpha \to 0} \lim_{n\to\infty} \frac{nF(V_n^1)}{nF(V_n^2)} = 2$$

$$\lim_{\alpha \to 0} \lim_{n\to\infty} \frac{nF(V_n^2)}{nF(Em_n)} = e/2 = 1.3591\ldots$$

$$\lim_{\alpha \to 0} \lim_{n\to\infty} \frac{nF(V_n^1)}{nF(Em_n)} = e = 2.7182\ldots$$

The relative improvement (93) given in the last column can be shown to tend to 1.

Remark 7.1 *Though we have proven Theorem 1.1 for the case where $F(x) = x^\alpha L(x)$, $\alpha > 0$ and $L(x)$ having finite positive limit as $x \downarrow 0$, we believe it holds true for all $F \in \mathcal{D}(G^\alpha)$ of (4), that is, whenever $L(x)$ is slowly varying as $x \downarrow 0$.*

Remark 7.2 *The approach in the present paper can easily be applied to obtain the asymptotic behavior of the one-choice value (obtained in Kennedy and Kertz (1991) by a different method), when $F(x) = x^\alpha L(x)$ and $\lim_{x\downarrow 0} L(x) = \mathcal{L} \in (0,\infty)$. First assume that $X \sim \mathcal{U}^\alpha(x)$ as in (9). Then for the one choice value V_n^1, we have*

$$V_{n+1}^1 = E[X \wedge V_n^1] = \alpha \int_0^{V_n^1} x^\alpha dx + (1 - (V_n^1)^\alpha) V_n^1.$$

Set $W_n^1 = n^{1/\alpha} V_n^1$, and make the change of variable $y = nx^\alpha$, as in Section 2. Now multiply by $n^{1/\alpha}$ to obtain

$$\begin{aligned} \left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1}^1 &= \frac{1}{n}\int_0^{(W_n^1)^\alpha} y^{1/\alpha} dy + (1 - (V_n^1)^\alpha) W_n^1 \\ &= \frac{1}{n}\int_0^n (W_n^1 \wedge y^{1/\alpha}) dy. \end{aligned}$$

Thus W_n^1 satisfies (42) with $q(y) = y^{1/\alpha}$, and now Theorem 4.1 can be applied to yield that $W_n^1 \to q(b)$ where b is the unique root of

$$\int_0^y u^{1/\alpha} du + (1/\alpha - y) y^{1/\alpha} = 0,$$

Table 1: Limiting Values of $nF(V_n^1), nF(V_n^2), nF(Em_n)$, and their ratios.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
α	b_α	$\lim nF(V_n^1)$	$\lim nF(V_n^2)$	$\lim nF(Em_n)$	(3)/(4)	(4)/(5)	(3)/(5)	((3)-(4))/((3)-(5))
0.1	11.9312	11.0000	5.72334	4.52873	1.92195	1.26379	2.42894	.99868
0.2	6.8927	6.0000	3.20772	2.60517	1.87049	1.23129	2.30311	.97131
0.3	5.2004	4.3333	2.36372	1.94980	1.83327	1.21229	2.22245	.93248
0.4	4.3485	3.5000	1.93919	1.61670	1.80488	1.19947	2.16490	.90235
0.5	3.8342	3.0000	1.68310	1.41421	1.78242	1.19013	2.12132	.88102
0.6	3.4896	2.6667	1.51157	1.27776	1.76417	1.18298	2.08699	.86571
0.7	3.2423	2.4286	1.38853	1.17940	1.74902	1.17732	2.05916	.85460
0.8	3.0561	2.2500	1.29590	1.10506	1.73624	1.17270	2.03610	.84614
0.9	2.9107	2.1111	1.22362	1.04684	1.72530	1.16887	2.01665	.83958
1.0	2.7940	2.0000	1.16562	1.00000	1.71583	1.16562	2.00000	.83438
2.0	2.2634	1.5000	0.90214	0.78540	1.66270	1.14864	1.90984	.81217
3.0	2.0839	1.3333	0.81309	0.71207	1.63983	1.14186	1.87245	.80556
4.0	1.9934	1.2500	0.76825	0.67497	1.62707	1.13820	1.85193	.80252
5.0	1.9388	1.2000	0.74123	0.65255	1.61895	1.13590	1.83897	.80078
6.0	1.9023	1.1666	0.72316	0.63753	1.61324	1.13432	1.82994	.79967
7.0	1.8762	1.1429	0.71023	0.62677	1.60914	1.13317	1.82343	.79892
8.0	1.8566	1.1250	0.70052	0.61867	1.60592	1.13230	1.81839	.79831
9.0	1.8412	1.1112	0.69296	0.61236	1.60350	1.13162	1.81455	.79789
10.0	1.8291	1.1000	0.68689	0.60731	1.60147	1.13105	1.81134	.79756

giving $b = 1 + 1/\alpha$. Hence, $\lim_{n\to\infty} W_n^1 = (1 + 1/\alpha)^{1/\alpha}$, or, $\lim_{n\to\infty} nF(V_n^1) = (1 + 1/\alpha)$. The general result for the wider class of distribution functions mentioned now follows in a manner similar to, but simpler than, the calculation for two choices.

Remark 7.3 *A similar approach can also be used to obtain the limiting value for more than 2 choices. For three choices one must first obtain the function $h^{(3)}(y)$ which replaces the function $h^{(2)}(y) = h(y)$ of (8). (Note that by Remark 7.2, $h^{(1)}(y) = y^{1/\alpha}$).*

Remark 7.4 *Our results translate easily to the case where the statistician is given two choices and his goal is to pick as large a value as possible, his payoff being the expectation of the larger of the two values chosen. Denote the optimal two-choice value based on n i.i.d. observations by $\tilde{V}_n^2$. Then for $X \sim F(x)$, where $x_F < \infty$, and*

$$F_X(x) = 1 - (x_F - x)^\alpha L(x_F - x)$$

where $L(\cdot)$ satisfies $\lim_{y\downarrow 0} L(y) = \mathcal{L}$ and $0 < \mathcal{L} < \infty$, we have

$$\lim_{n\to\infty} [1 - F(\tilde{V}_n^2)] = h^\alpha(b_\alpha).$$

Bibliography

1. Assaf, D., and Samuel-Cahn, E. (2000) Simple ratio prophet inequalities for a mortal with multiple choices. *J. Appl. Prob.*, **37**, pp. 1084-1091.

2. Assaf, D., Goldstein, L. and Samuel-Cahn, E. (2002). Ratio prophet inequalities when the mortal has several choices. *Ann. Appl. Prob.*, **12**, pp. 972-984.

3. de Haan, L. (1976). Sample extremes: an elementary introduction. *Statist. Neerlandica*, **30**, pp. 161-172.

4. Kennedy, D.P. and Kertz, R.P. (1990). Limit theorems for threshold-stopped random variables with applications to optimal stopping. *Adv. Appl. Prob.* **22**, pp. 396-411.

5. Kennedy, D.P. and Kertz, R.P. (1991). The asymptotic behavior of the reward sequence in the optimal stopping of i.i.d random variables. *Ann. Prob.* **19**, pp. 329-341.

6. Leadbetter, M.R., Lindgren, Georg and Rootzén, Holger.(1983). *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag, New York, Heidelberg, Berlin

7. Resnick, Sidney I. (1987). *Extreme Values, Regular variation, and Point Processes.* Springer. N.Y