

האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

**ON STOCHASTIC COMPARISONS
OF EXCESS TIMES**

by

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Discussion Paper # 302

November 2002

**מרכז לחקר הרציונליות
וההחלטות האינטראקטיביות**

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On stochastic comparisons of excess times

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October 9, 2002

Abstract

A stationary renewal process based on iid random variables X_i is observed at a given time. The excess time, that is, the residual time until the next renewal event, is of course smaller than the total current X which consists of the residual time plus the current age. Nevertheless in certain types of data the distribution of the excess times is stochastically larger than that of X_i 's. We find necessary and sufficient conditions that explain this phenomenon, and related results on stochastic orderings arising from observations on renewal processes.

Keywords and phrases: renewal process, hazard rate, NWU and DFR, distributions, likelihood ratio ordering

MSC : 60K.

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1 Introduction

Consider a renewal process generated by random variables $X_i \geq 0$ with mean $\mu < \infty$ and common distribution function F_X . The X 's may, for example, be a series of lifetimes of a device which is replaced upon failure, or service times (of telephone operators, say), between interruptions (due to illness, say). The process is observed at some given time t .

For the device operating at time t , let Y denote its residual (excess) lifetime (or service time from t to the next interruption), let Z denote its age at time t (or the service time between the last interruption and t), and let $W = Y + Z$ be the total lifetime or service time.

It is well-known that if the process is stationary or when $t \rightarrow \infty$, Y and W have asymptotic distributions given by

$$P[Y \leq x] = \mu^{-1} \int_0^x (1 - F_X(u)) du, \quad P[W \leq x] = \mu^{-1} \int_0^x u F_X(du) \quad x \geq 0.$$

Y and Z are identically distributed ($Y \sim Z$), and, in general, *not* independent. If X is exponential, then $P[Y \leq x] = F_X(x)$, $Y \sim X$, and Y and Z are independent. In general, if X has a density f_X with support $S = [0, s)$ with $0 < s \leq \infty$, then the joint density of (Y, Z) is $g_{(Y,Z)}(y, z) = \mu^{-1} f_X(y + z)$ for $y, z, y + z \in S$. This formula can be used to show that if Y and Z are independent, then they are exponentially distributed, and then so is X .

In this paper we study necessary and sufficient conditions for stochastic ordering comparisons between the distributions of X and Y . The article [6], studies, with motivation similar to ours, a characterization of the class of d.f.'s F_X for which $Y \sim X/q$, for some $q > 0$; alternatively, taking without loss of generality $\mu = 1$, the class of d.f.'s satisfying $F(qx) = \int_0^x (1 - F(u)) du$, $x \geq 0$. Although X represents a complete lifetime and Y a residual one, there are natural cases where X is stochastically smaller than Y . This apparent paradox is explained in part by the fact that W has the F -length-biased distribution which is stochastically larger than F . Moreover, when the density of X , $F'_X = f_X$ exists, then $f_W(t) = t f_X(t) / \mu$, and the likelihood ratio $f_W(t) / f_X(t) = t / \mu$ is increasing, so that W dominates X also in the likelihood ratio sense.

In the next Section we shall recall some basic definitions of stochastic orderings and of aging notions which will be needed for our analysis.

2 Stochastic orderings for excess times

We begin this section by reviewing some basic notions concerning stochastic orderings and aging notions. We refer to [5] for details.

First, recall that a random variable U is said to be *stochastically larger* than a r.v. V (or its distribution function F_U is said to be stochastically larger than F_V) if $\bar{F}_U(t) \geq \bar{F}_V(t)$, $\forall t \in \mathbb{R}$, where $\bar{F} := 1 - F$ is the survival function corresponding to the distribution function F ; in this case, we write $U \geq_{ST} V$, or $\mathcal{L}(U) \geq_{ST} \mathcal{L}(V)$, where $\mathcal{L}(M)$ denotes the law of the random variable M . A stronger order which implies the previous one is the *hazard rate ordering*: we write $U \geq_{HR} V$ if $x \mapsto \bar{F}_U(x)/\bar{F}_V(x)$ is increasing. An even stronger ordering is given by the *likelihood ratio ordering*, well-defined when the random variables involved are absolutely continuous: we write $U \geq_{LR} V$ if the ratio of the densities $f_U(t)/f_V(t)$ is increasing.

Next, we recall some basic notions of univariate aging. A random variable $U > 0$ is said to be *New Worse than Used in Expectation (NWUE)* if $\mathbb{E}(U - t|U > t) \geq \mathbb{E}(U) \quad \forall t \geq 0$. A stronger notion which implies NWUE is that of *New Worse than Used (NWU)* which states that $P(U - t > x|U > t) \geq P(U > x)$, $\forall x, t \geq 0$, or equivalently $P(U > t + x) \geq P(U > t)P(U > x)$. A further stronger notion is that of *Decreasing Failure Rate (DFR)*. Recall that if U has a probability density f then U DFR means that the failure or hazard rate $h(x) = f(x)/\bar{F}(x)$ is decreasing in $x \geq 0$. The corresponding notions of positive aging (NBUE, NBU, IFR) are defined analogously.

A further notion is that of *Increasing Mean Residual Lifetime (IMRL)*: We say that a positive r.v. U is IMRL (DMRL) if the *mean residual life* of U , namely, the function

$$\mu_{F_U}(t) := \mathbb{E}(U - t|U > t) = \int_0^\infty \frac{\bar{F}_U(t+u)}{\bar{F}_U(t)} du = \frac{\int_t^\infty \bar{F}_U(u) du}{\bar{F}_U(t)}, \quad (1)$$

is increasing (decreasing).

We make two preliminary observations concerning IMRL:

1. The notion of DFR implies that of IMRL, which in turn implies NWUE. However, IMRL neither implies nor is implied by NWU.
2. If $\mathbb{E}(U) = 1$, the numerator in the last term of (1) is the survival function of the excess distribution corresponding to U . Thus, we may rephrase the definition by saying that U is IMRL if the ratio excess-survival/survival is increasing.

The phenomenon that the distribution of observed partial lifetimes Y and Z often dominates in some stochastic sense the distribution of the full time X , is elucidated by the following proposition.

Proposition 1. *Let $X \sim F_X$ be a positive random variable and let $Y \sim F_Y$ be the corresponding excess variable. Then*

1. *The following statements are equivalent:*

- (a) X is NWUE;
- (b) $Y \geq_{ST} X$.
- (c) The failure rate h_Y of Y satisfies the relation: $h_Y(t) \leq h_Y(0), \forall t \geq 0$.

2. *The following statements are equivalent:*

- (a) X is IMRL;
- (b) $Y \geq_{HR} X$;
- (c) $\mathcal{L}(Y - t|Y > t) \geq_{ST} \mathcal{L}(X - t|X > t), \forall t \geq 0$.
- (d) Y is DFR

3. *Assume that \bar{F}_X is strictly positive and differentiable. The following statements are equivalent:*

- (a) X is DFR
- (b) $Y \geq_{LR} X$,

Similar statements hold for the positive aging notions NBU, DMRL, IFR.

Proof. The equivalence of 1a and 1b is given as Theorem 1.8.6. in [4], and we repeat it for completeness. We have

$$\begin{aligned}
 \mathbb{E}(X - t|X > t) &= \int_0^\infty P(X - t > u|X > t)du \\
 &= \int_0^\infty \frac{\bar{F}_X(t+u)}{\bar{F}_X(t)}du = \frac{1}{\bar{F}_X(t)} \int_t^\infty \bar{F}_X(z)dz \quad (2) \\
 &= \frac{\mu}{\bar{F}_X(t)} \bar{F}_Y(t) = \mathbb{E}(X) \frac{\bar{F}_Y(t)}{\bar{F}_X(t)},
 \end{aligned}$$

so that 1a and 1b are equivalent. The previous equalities yield also:

$$h_Y(t) = \frac{f_Y(t)}{\bar{F}_Y(t)} = \frac{1}{\mu} \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} = \frac{1}{\mathbb{E}(X - t | X > t)}, \quad (3)$$

and the equivalence between 1a and 1c follows.

Let us consider now case 2. From (2) we have:

$$X \text{ IMRL} \iff \frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \leq \frac{\bar{F}_Y(t+h)}{\bar{F}_X(t+h)} \iff \frac{\bar{F}_X(t+h)}{\bar{F}_X(t)} \leq \frac{\bar{F}_Y(t+h)}{\bar{F}_Y(t)}, \quad \forall t, h \geq 0.$$

The first equivalence relation shows equivalence of 2a and 2b and the second establishes the desired equivalence with 2c.

The result that 2a and 2d are equivalent was stated in [2]. It follows directly from (3).

Finally, let us turn our attention to case 3. We need to consider monotonicity of the ratio of the densities. Simply note that $f_Y(t) = \bar{F}/\mu$ and so $\frac{f_Y}{f_X} = \bar{F}/f\mu$, which is increasing if and only if f/\bar{F} is decreasing, that is, if and only if X is DFR. \square

As we mentioned before, the concept of NWUE is weaker than NWU. For the analysis of the case when the full lifetime X is NWU, we need to introduce a further notion of aging.

Definition 2. *A nonnegative random variable U is said to be Stochastically Larger at Any Age (SLA) than V ($U \geq_{\text{SLA}} V$) if $P(U > t + x | U > t) \geq P(V > x) \quad \forall x, t \geq 0$. Similarly, U is said to be Stochastically Smaller at Any Age (SSA) than V if $P(U > t + x | U > t) \leq P(V > x) \quad \forall x, t \geq 0$.*

Note that under the above definition of SLA, U will survive another x units of time at any age with higher probability than V at birth (that is, when entirely new). Observe that if U is SLA than V , then it is not true in general that V is SSA than U .

With this (perhaps new) notion we have

Proposition 3. *Let $X \sim F_X$ be a positive random variable and let $Y \sim F_Y$ be the corresponding excess variable. If X is NWU then $Y \geq_{\text{SLA}} X$.*

Proof.

$$\begin{aligned}
\bar{F}_Y(t+x) &= \mu^{-1} \int_{t+x}^{\infty} \bar{F}_X(u) du \\
&= \mu^{-1} \int_t^{\infty} \bar{F}_X(u+x) du \geq \bar{F}_X(x) \mu^{-1} \int_t^{\infty} \bar{F}_X(u) du \\
&= \bar{F}_X(x) \bar{F}_Y(t).
\end{aligned}$$

and the conclusion follows. \square

Continuing in this direction, another question that can be raised is to find conditions under which X is either SLA or SSA than the excess time Y . In other words, we are asking for a comparison between $\bar{F}_X(t+x)/\bar{F}_X(t) = P(X > t+x | X > t)$ and $\bar{F}_Y(x) = P(Y > x)$. However, in the Appendix we show that either of the resulting conditions

$$\bar{F}_X(t+x)/\bar{F}_X(t) \leq (\geq) \bar{F}_Y(x)$$

can hold for all t and x only for the exponential distribution. These conditions seem to correspond to the notion of NRBU (NRWU) in [2]. Hence, as already pointed out in [3], for either the NRBU or the NRWU condition to hold, X must be exponentially distributed. We shall deal with this issue in the Appendix, where some identities which may be of independent interest are presented.

3 Negative Aging in renewal processes

Positive aging as in NBU, NBUE, IFR and DMRL distributions is associated with lifetimes of devices which wear out in use. On the other hand, negatively aging distributions (NWU, NWUE, DFR and IMRL) may provide appropriate models for hospitalization periods, service times of operators between absence, say due to illness, and sometimes service times in certain queueing systems. The results here formulate the common observation that waiting time for the end of service for the person currently served when one arrives into a bank at a random time t , seems longer than the average service time, in spite of the fact that part of his service was done before time t .

Consider, for example, data on work periods of employees between interruptions due to illness, say. Each interruption can be identified as an arrival

time in a renewal process (one process for each employee). Such data often exhibit the above mentioned phenomenon of excess times being stochastically larger than the X 's generating the renewal process.

It seems natural to assume that

- Each employee has a distribution for the work periods between absences which is a mixture of exponentials:

$$\mathbb{P}(X > t) = \int_0^\infty \exp^{-\theta t} \pi(d\theta).$$

Notice that a mixture of exponentials is DFR, and hence the conditions for the occurrence of the phenomenon that $Y \geq_{\text{ST}} X$, and stronger comparisons, are satisfied.

- The *type* of employee is modeled by his/her mixing distribution π , that is, we may think of π as being fixed for a given employee, thus characterizing the employee. After each interruption the employee chooses a value of θ from π and then works for an amount of time distributed exponentially with parameter θ . We may also consider a hierarchical model, where π is a random probability measure chosen from a suitable prior by each employee at each interruption, or once for all interruptions.

Such models explain the observation that that in absenteeism-from-work data, for example, where a “lifetime” is taken as an uninterrupted show-up-at-work period, residual lifetimes in the observational windows are demonstrably larger in distribution than regular lifetimes. (See [9] for an example and a discussion that motivated the investigation in [6], and the statistical methodologies in [7] and [8]). Although this may seem paradoxical at first, intuitively this may be not be very surprising. A worker who has not been absent due to illness for a long time, is less at risk of contracting illness than a worker who has recently been ill. Similar data are typical, for example, in public health studies, where patients’ hospitalization duration is recorded over a random time window, and similar biases occur.

Appendix

Let, as before, X be a positive r.v. and let Y be the corresponding excess time. We show here that the relations $\bar{F}_X(t+x)/\bar{F}_X(t) \leq (\geq) \bar{F}_Y(x)$ may

hold only with equality sign, with X and Y exponential r.v.'s.

In fact, setting without loss of generality $\mu = 1$, we have

$$\begin{aligned}\bar{F}_Y(x) - \bar{F}_X(t+x)/\bar{F}_X(t) &= \int_0^\infty \bar{F}_X(u+x)du - \bar{F}_X(t+x)/\bar{F}_X(t) \\ &= \int_0^\infty [\bar{F}_X(u+x)/\bar{F}_X(u)]\bar{F}_X(u)du - \bar{F}_X(t+x)/\bar{F}_X(t) \\ &= \int_0^\infty G_x(u)\bar{F}_X(u)du - G_x(t),\end{aligned}$$

where $G_x(u) = \bar{F}_X(u+x)/\bar{F}_X(u)$. An inequality of the type $\int_0^\infty G_x(u)\bar{F}_X(u)du - G_x(t) \geq (\leq) 0 \forall t \geq 0$ seems possible only if $G_x(u)$ is constant, since if the function is not constant we can find a value $G_x(t)$ which is greater (smaller) than the average with respect to the density $\bar{F}_X(u)du$. This also implies that for either the NRBU or the NRWU conditions of [2] to hold, X must be exponentially distributed.

Another, useful way of presenting this issue is the following: let X be a lifetime random variable with an absolutely continuous cdf $F = F_X$ having a density $f = f_X$ with support $S = [0, s)$ with $0 < s \leq \infty$, and a finite mean μ . Let (Y, Z) be a pair of lifetime rv's with a joint density $g(y, z) = g_{(Y,Z)}(y, z) = \mu^{-1}f(y+z)$, for y and $z \in S$ and $y+z \in S$. Note that (Y, Z) are distributed as the excess and age variables above, but here we consider it as a purely distributional issue, without referring to renewal.

We have the following interesting identity:

$$P(X > x+t | X > x) = P(Y > t | Z = x). \quad (4)$$

Thus, the relation $\bar{F}_X(t+x)/\bar{F}_X(t) \leq \bar{F}_Y(x)$, or equivalently definition (2.1) in [2] of NRBU can be written as:

f is NRBU provided the following holds,

$$P(Y > t | Z = x) \leq P(Y > t) \quad \forall x, t \geq 0.$$

The latter inequality holding for all x with probability one, combined with the relation $P(Y > t) = \mathbb{E}[P(Y > t | Z)]$ implies $P(Y > t | Z) = P(Y > t)$ with probability one, and hence independence of Y and Z . It follows that Y and Z are exponentially distributed, and therefore so is X .

Note that equation (4) can also be expressed as follows: Let $X \sim F$ and $W \sim tF(dt)/\mu_F$, that is, W has the F size biased distribution, and let U be

a uniform $(0, 1)$ random variable, all independent. Then

$$P(X > x + t | X > x) = P(Y > t | Z = x) = P(W(1 - U) > t | WU = x).$$

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