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by

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On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions*

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Abstract

We establish the existence of pure strategy equilibria in monotone bidding functions in first-price auctions with asymmetric bidders, interdependent values and affiliated one-dimensional signals. By extending a monotonicity result due to Milgrom and Weber (1982), we show that single crossing can fail only when ties occur at winning bids or when bids are individually irrational. We avoid these problems by considering limits of ever finer finite bid sets such that no two bidders have a common serious bid, and by recalling that single crossing is needed only at individually rational bids. Two examples suggest that our results cannot be extended to multidimensional signals or to second-price auctions.

Keywords: first-price auction, monotone equilibrium, pure strategy.

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1. Introduction

There is by now a large literature on first-price auctions. While initial efforts centered around the symmetric bidder case (e.g., Milgrom and Weber (1982)), attention has begun to shift toward the even more challenging—and in practice often very relevant—case of asymmetric bidders. A key difference between the two cases is that only the symmetric bidder setting admits closed-form expressions for equilibrium bid functions. Because of this, analysis of equilibrium bidding behavior in asymmetric first-price auctions requires an implicit characterization of equilibrium through first-order necessary conditions for optimal bidding.[1] But if an equilibrium fails to exist, such an analysis is vacuous.

Our objective here is to provide conditions ensuring the existence of a pure-strategy equilibrium in nondecreasing bid functions for asymmetric first-price auctions with affiliated private information and interdependent values. As a by-product, we therefore provide a foundation for the first-order approach to analyzing equilibrium bidding behavior in such auctions.

Recent work on the question of equilibrium existence in first-price auctions can be found in Athey (2001), Bresky (1999), Jackson and Swinkels (2003), Lebrun (1996, 1999), Lizzeri and Persico (2000), Maskin and Riley (2000), and Reny (1999).[2] But there appears to be a common difficulty. The above papers restrict attention either to two bidders, symmetric bidders, independent signals, private values, or common values. That is, the most general case involving three or more asymmetric bidders with affiliated signals and interdependent values is not covered. The reason for this is that standard proof techniques rely on the following *single-crossing condition* (SCC) exploited with great ingenuity in Athey (2001):

If the others employ nondecreasing bid functions and one's payoff from a high bid is no smaller than that from a lower bid, then the high bid remains as good as the lower one when one's signal rises.[3]

However, even when bidders' signals are affiliated, SCC can fail (see Section 3) unless there is but a single other bidder (as in the two-bidder case), or all signals

[1] See, for example, Bajari (1997).

[2] For conditions ensuring *uniqueness* in two-bidder settings, see Lizzeri and Persico (2000), Maskin and Riley (1996) and Rodriguez (2000). Under more restrictive conditions, Maskin and Riley (1996) obtain some uniqueness results for more than two bidders. See also Bajari (1997) and Lebrun (1999).

[3] This is only "half" of the condition. The other half is obtained by reversing the roles of "high(er)" and "low(er)," and replacing "rises" with "falls." See Section 2 for a formal definition.

are symmetric and bidders employ the same bidding function (as in the symmetric case), or signals are independent, or values are either purely private or purely common. It is for this reason that a general result is not yet at hand.

While our main contribution is the existence result, there are two other contributions of note. One of these is our characterization of the precisely two ways that SCC can fail for a bidder. The first failure occurs when the two bids being compared are both individually irrational, each yielding the bidder a negative expected payoff. The second failure occurs when one of the two bids ties for the winning bid with positive probability.

A second contribution is a new method for establishing single-crossing when these two potential failures are ruled out. The new method is based upon an extension of an important result due to Milgrom and Weber (1982) for nondecreasing functions; the extension applies to a class of functions that need not be nondecreasing. Using the new method, we establish the following *individually rational tieless single-crossing condition* (IRT-SCC) for first-price auctions:

If the others employ nondecreasing bid functions and one's payoff from a high bid is nonnegative and no smaller than that from a lower bid, then the high bid remains as good as the lower one when one's signal rises so long as neither bid ties for the winning bid with positive probability.[4]

Once this result is in hand, it is possible to adapt Athey's (2001) finite-action proof techniques to establish the existence of a pure-strategy monotone equilibrium in our first-price auction setting when each bidder is restricted to choose a bid from a finite set, but where the only common bid for distinct bidders is nonserious. The proof is completed by showing that the limit of any sequence of such equilibria, as the bidders' finite grids become dense in their unrestricted bid sets is a pure monotone equilibrium of the model with unrestricted bids.[5] While employing such a limit argument is not novel, the combination of interdependent values and affiliated private information renders a particular property of the uni-

[4] Once again, this is only half of the condition and the other half is obtained as before. See Section 2 for a formal definition.

[5] Jackson and Swinkels (2003) employ Jackson, Swinkels, Simon and Zame (2002) to establish the existence of a *mixed-strategy* equilibrium in a large class of *private-value* auctions. Applying Jackson et. al. (2002) in our setting yields the existence of a *mixed-strategy* equilibrium with an *endogenous tie-breaking rule.* This equilibrium needn't be pure, nondecreasing, or an equilibrium under the standard tie-breaking rule. Consequently, this route does not appear to be fruitful for our purposes.

form tie-breaking rule critical to this part of the argument. By identifying this property, this portion of our proof may also be of some value.

The remainder of the paper is organized as follows. Section 2 describes the class of first-price auctions covered here, provides the assumptions we maintain throughout, and contains our main result. This section also provides a discussion of Athey's (2001) single crossing condition (SCC) and introduces our individually rational tieless single crossing condition (IRT-SCC). Section 3 provides examples of the two ways Athey's (2001) single crossing condition can fail. Section 4 provides both a sketch of the proof of IRT-SCC as well our extension of Milgrom and Weber's (1982) monotonicity theorem, upon which the proof is based. Section 5 provides a private value example suggesting that one-dimensionality of the bidders' signals is essential for the existence of monotone pure-strategy equilibria in the class of first-price auctions studied here, and Section 6 shows that our existence result for first-price auctions can not be extended to second-price auctions. All proofs are contained in the appendix.

2. The Model and Main Result

Consider the following first-price auction game. There is a single object for sale and $N \geq 2$ bidders. Each bidder i receives a private signal $s_i \in [0,1]$. The joint density of the bidders' signals is $f : [0,1]^N \to \mathbb{R}_+$. After receiving their signals, each bidder i submits a sealed bid from $B_i = \{l\} \cup [r_i, \infty) \subseteq \mathbb{R}$, where $l < r_i$ for all i. Either l alone or both l and r_i may be negative. The bidder i submitting the highest bid greater than l wins the object, with ties broken randomly and uniformly.[6] Hence, r_i is bidder i's reserve price and l, being a bid that always loses, corresponds to a decision not to participate. For example, the case of a common reserve price of zero corresponds to $r_i = 0$ for all i and $l < 0$. We shall call bids in $[r_i, \infty)$ *serious bids*, and so l is the only nonserious bid.

If the vector of signals is $s = (s_1, ..., s_N)$ and bidder i wins the object with a bid of b_i, then bidder i's payoff is given by $u_i(b_i, s)$. All other bidders receive a payoff of zero. This specification allows for a variety of attitudes toward risk, as well as a variety of payment rules.

We shall maintain the following assumptions. For all bidders $i = 1, ..., N$:

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[6] Any tie-breaking rule with the property that a high bidder's probability of winning is non-increasing in the others' bids will do. In particular, the probability of winning need not be log-supermodular in the vector of bids as in Athey (2001).

A.1 (i) $u_i : [r_i, \infty) \times [0,1]^N \to \mathbb{R}$ is measurable, $u_i(b_i, s)$ is bounded in $s \in [0,1]^N$ for each $b_i \geq r_i$ and continuous in b_i for each s.

(ii) There exists $\tilde{b} \geq r_i$ such that $u_i(b_i, s) < 0$ for all $b_i > \tilde{b}$ and all $s \in [0,1]^N$.

(iii) For every $b_i \geq r_i$, $u_i(b_i, s)$ is nondecreasing in s_{-i} and strictly increasing in s_i.

(iv) $u_i(\bar{b}_i, s) - u_i(\underline{b}_i, s)$ is nondecreasing in s whenever $\bar{b}_i > \underline{b}_i \geq r_i$.

A.2 (i) $f(s)$ is measurable and strictly positive on $[0,1]^N$.

(ii) $f(s \vee s')f(s \wedge s') \geq f(s)f(s')$ for all $s, s' \in [0,1]^N$, where $\vee$ and $\wedge$ denote component-wise maximum and minimum, respectively.

Remark 1. *This model does not cover settings in which bidders must first commit to participate before receiving their signals and then submit bids at or above their reserve price after receiving their signals. While such prior commitment is unusual in practice, our results hold even in this case so long as each bidder i has a feasible bid guaranteeing a nonnegative payoff conditional on winning (e.g., if $u_i(r_i, s) \geq 0$ for all s).*

Remark 2. *According to A.1(iii), it is not necessary that $u_i(b_i, s)$ decrease in b_i, only that it is eventually negative for large enough b_i. Thus, while we require the winner to be the highest bidder, we do not require the winner to pay his bid, nor even an amount that is an increasing function of his bid. It is important, however, that the winner's payment depend only upon his own bid.*

Remark 3. *Note that A.1(iv) is satisfied automatically when bidder i is risk neutral and the winner must pay his bid because in this case $u_i(b_i, s) = w_i(s) - b_i$ and so the difference expressed in A.1(iv) is constant in s. More generally, if $u_i(b_i, s) = U_i(w_i(s) - b_i)$, then A.1(iv) holds when $w_i(s)$ is nondecreasing in s and $U_i'' \leq 0$ (i.e. bidder i is risk averse).*

Remark 4. *Assumption A.2(i) implies that, given any s_i, the support of i's conditional distribution on the others' signals is $[0,1]^{N-1}$. A.2(ii) requires the bidders' signals to be affiliated (see Milgrom and Weber (1982)).*

Given a vector of bids $b = (b_1, ..., b_N) \in \times_i B_i$, let $v_i(b, s)$ denote bidder i's expected payoff when the vector of signals is s. That is,

$$v_i(b,s) = \begin{cases} \frac{1}{m} u_i(b_i, s), & \text{if } m = \#\{j : b_j = b_i = \max_k b_k > l\} \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that this specification implies that a lone bid equal to one's reserve price is a winning bid.

Throughout, the upper case letter, S_i, will denote bidder i's signal as a random variable while the lower case letter, s_i, will denote its realization. A pure strategy for bidder i is a measurable (bid) function $\mathbf{b}_i : [0,1] \to B_i$. Given a vector of pure strategies $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_N)$, let $V_i(\mathbf{b})$ denote bidder i's (ex-ante) expected payoff in the auction. That is,

$$V_i(\mathbf{b}) = E[v_i(\mathbf{b}(S), S)],$$

where $\mathbf{b}(S)$ denotes the random vector $(\mathbf{b}_1(S_1), ..., \mathbf{b}_N(S_N))$ and the expectation is taken with respect to f. It will also be convenient to define bidder i's interim payoff. Accordingly, let $V_i(b_i, \mathbf{b}_{-i} \mid s_i)$ denote bidder i's expected payoff conditional on his signal s_i and given that he bids b_i and the others employ the strategies $\mathbf{b}_{-i}$. That is,[7]

$$V_i(b_i, \mathbf{b}_{-i}|s_i) = E[v_i(b_i, \mathbf{b}_{-i}(S_{-i}), s_i, S_{-i})|S_i = s_i].$$

A pure-strategy equilibrium is an N-tuple of pure strategies $\mathbf{b}^* = (\mathbf{b}_1^*, ..., \mathbf{b}_N^*)$ such that for all bidders i, $V_i(\mathbf{b}^*) \geq V_i(\mathbf{b}_i', \mathbf{b}_{-i}^*)$ for all pure strategies $\mathbf{b}_i'$; or equivalently, for every i and a.e. s_i, $V_i(\mathbf{b}_i^*(s_i), \mathbf{b}_{-i}^* \mid s_i) \geq V_i(b_i, \mathbf{b}_{-i}^* \mid s_i)$ for every $b_i \in B_i$.

Our interest lies in establishing, for any first-price auction game, the existence of a pure-strategy equilibrium in which each bidder's bid function is nondecreasing in his signal. We shall refer to this as a *monotone pure-strategy equilibrium.* This brings us to our main result.

Theorem 2.1. *All first-price auction games satisfying assumptions A.1 and A.2 possess a monotone pure-strategy equilibrium.*

Remark 5. *Suppose that for each pair of distinct bidders, i and j, the strictly increasing function $\phi_j^i(\cdot)$ maps $\mathbb{R}$ onto $\mathbb{R}$, and ϕ_j^i and ϕ_i^j are mutual inverses. With*

[7] All statements involving conditional probabilities are made with respect to the following version of the conditional density: $f(s_{-i}|s_i) = f(s) / \int_{[0,1]^{N-1}} f(s_i, s_{-i}) ds_{-i}$, which, by A.2(i), is well defined.

obvious modifications, the proof of Theorem 2.1 is valid under the more general rule that any serious bid b_i by i is among the highest if $b_i \geq \max_{j \neq i} \phi_j^i(b_j)$ with ties broken uniformly when there is more than one high bidder. Hence, Theorem 2.1 extends, for example, to settings in which the auction rules favor a subset of bidders by increasing their effective bids against those outside the subset.

The proof of Theorem 2.1 is in the appendix and consists of two main steps. The first step establishes that a monotone equilibrium exists when bidders are restricted to finite sets of bids with only the nonserious bid l in common, while the second step shows that the limit of such equilibria, as each bidder i's set of permissible bids becomes dense in B_i, is an equilibrium when, for each bidder i, any bid in B_i is feasible.

The novelty of our approach lies in the first step, where standard techniques have up to now failed. For example, it would be straightforward to establish the existence of a monotone equilibrium with finite bid sets if one could establish the single-crossing condition employed in Athey (2001). One could then simply appeal directly to Athey's Theorem 1.

In the context of our first-price auction, Athey's (2001) single-crossing condition is as follows. For any bidder i, any feasible bids b_i and b_i', and any nondecreasing bid functions $\mathbf{b}_j$ for all bidders $j \neq i$, the following must hold:

SCC. If $V_i(b_i', \mathbf{b}_{-i} \,|\, s_i) \geq V_i(b_i, \mathbf{b}_{-i} \,|\, s_i)$ then this inequality is maintained when s_i rises if $b_i' > b_i$, while it is maintained when s_i falls if $b_i' < b_i$.

Unfortunately, Athey's (2001) result cannot be applied because, for arbitrary finite or infinite bid sets, SCC can fail in two ways (see Section 3). First, SCC can fail when there are ties at winning bids and this is why we must approximate the bidders' bid sets with finite sets whose only common bid is the losing bid, l.

Second, SCC can fail if a bidder employs an individually irrational bid, that is, a bid yielding a negative expected payoff against the others' strategies. But this failure of SCC does not pose a problem for us because the heart of Athey's (2001) proof technique nonetheless applies. To see this, recall that Athey employs SCC only to establish that when the others use monotone strategies, a bidder's best-reply correspondence, as a function of his signal, is increasing in the strong set order.[8,9] Because our finite bid sets include the losing bid l, best replies are necessarily individually rational. Consequently, strong set-order monotonicity can be established as in Athey (2001) *so long as SCC holds for bids that are best replies*. That is, the failure of SCC for individually irrational bids is immaterial for establishing strong set-order monotonicity of the best-reply correspondence whenever each bidder has a bid that guarantees a nonnegative payoff.

Consequently, the existence of a monotone pure-strategy equilibrium can be established in a first-price auction with our particular finite-bid set approximation if SCC can be established whenever ties at winning bids are absent and the two relevant bids are best replies. We in fact establish a single-crossing condition that is stronger than this restricted form of SCC (but of course weaker than SCC itself), which, a fortiori, suffices for our purposes.

The following *individually rational tieless single-crossing condition* (IRT-SCC) requires that, in addition to the absence of ties at winning bids, one of the two relevant bids be individually rational. But neither bid need be a best reply.

Definition 2.2. *A first-price auction satisfies IRT-SCC if for each bidder i, all pairs of bids $b_i, b_i' \in B_i$, and all nondecreasing bid functions $\mathbf{b}_j : [0,1] \to B_j$ of the other bidders such that $\Pr(l < \max_{j \neq i} \mathbf{b}_j(S_j) = b_i \text{ or } b_i') = 0$, the following condition is satisfied:*

If $V_i(b_i', \mathbf{b}_{-i} \mid s_i) \geq 0$ and $V_i(b_i', \mathbf{b}_{-i} \mid s_i) \geq V_i(b_i, \mathbf{b}_{-i} \mid s_i)$, then the second inequality is maintained when s_i rises if $b_i' > b_i$, while it is maintained when s_i falls if $b_i' < b_i$.

The main contribution leading to the proof of Theorem 2.1 is the following proposition. Its proof can be found in the appendix.

Proposition 2.3. *Under assumptions A.1 and A.2, a first-price auction satisfies IRT-SCC.*

[8] That is, in this one-dimensional setting, if a high bid is best at a low signal and a low bid is best at a high signal, then both bids are best at both signals. (Milgrom and Shannon (1994) introduced the strong set order into the economics literature and, using it, established a number of important comparative statics results.)

[9] Athey's (2001) convexity results then apply and existence follows, as Athey shows, from Kakutani's theorem.

Remark 6. *It can in fact be shown that, given individual rationality, the single-crossing inequality is maintained even if ties occur at the* higher *of the two bids* b_i' *and* b_i. *It is only ties at the* lower *of the two bids that cause single-crossing to fail. This tighter characterization might prove useful in understanding when* all *equilibria must be monotone. (See McAdams (2003) for some work in this direction.)*

We next illustrate the two ways that SCC can fail.

3. The Two Failures of Single-Crossing

In each of the two examples below, there are three bidders, each with a zero reserve price, and the joint distribution of their signals is as follows.

Bidders $i = 1, 2$ have signals, s_i, that are i.i.d. uniform on $[0,1]$. These are drawn first. Bidder 3's signal, s_3, is drawn from $[0,1]$ conditional on 1's signal according to the density

$$g(s_3|s_1) = \begin{cases} 1, & \text{if } s_1 \leq 1/2 \\ 2/3, & \text{if } s_1 > 1/2 \text{ and } s_3 \leq 1/2 \\ 4/3 & \text{if } s_1 > 1/2 \text{ and } s_3 > 1/2 \end{cases}$$

Thus, 3's signal is uniform on $[0,1]$ if $s_1 \leq 1/2$. If $s_1 > 1/2$, then 3's signal is twice as likely to be above $1/2$ as below $1/2$, but is otherwise uniformly distributed on each of the two halves of the interval $[0,1]$. So defined, the bidders' signals are affiliated.

The examples will be constructed so that SCC fails for bidder 1. Consequently, bidders 2 and 3 can, for example, be given private values. In each example, Bidder 1's utility will take the quasilinear form $u_1(b,s) = w_1(s) - b$, where, for $v_0 \leq v_1 \leq v_2 \leq v_3$,

$$w_1(s_1, s_2, s_3) = \begin{cases} v_3, & \text{if } (s_2, s_3) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \\ \\ v_2, & \text{if } (s_2, s_3) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}) \\ \\ v_1, & \text{if } (s_2, s_3) \in [0, \frac{1}{2}) \times [\frac{1}{2}, 1] \\ \\ v_0, & \text{if } (s_2, s_3) \in [0, \frac{1}{2}) \times [0, \frac{1}{2}) \end{cases}$$

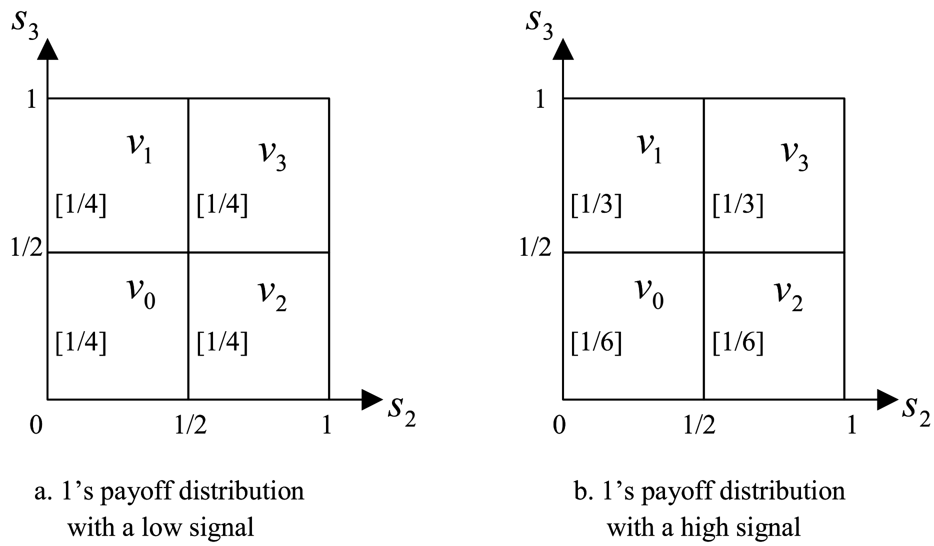

Figure 3.1: 1's Payoff Distribution

Figure 3.1 illustrates both the distribution of the others' signals conditional on 1's signal, and bidder 1's value for the good, $w_1(s_1, s_2, s_3)$, as a function the other bidders' signals.[10] In each panel, the numbers in square brackets are the probabilities of each of the four regions conditional on 1's signal. In panel (a) 1's signal is low (i.e., 1/2 or less), while in panel (b) it is high (i.e., above 1/2). In both panels, the joint density of s_2 and s_3 is uniform within each region.

The two failures of SCC result from two distinct specifications of the values v_0, v_1, v_2, and v_3.

3.1. The First Failure: Individually Irrational Bids

Consider the following values.

$$v_0 = 1, \quad v_1 = 2, \quad v_2 = 8, \quad v_3 = 9$$

Suppose also that bidders 2 and 3 each employ a strictly increasing bidding function that specifies a bid of 5 at the signal 1/2 and a bid of 6 at the signal 1.

[10]Note that $w_1(s)$ is nondecreasing in s, but, contrary to A.1, it is independent of s_1. This is for simplicity only. Adding εs_1 to $w_1(s)$, for $\varepsilon > 0$ small enough, renders 1's utility strictly increasing in his signal and maintains the failure of SCC in both examples.

Now consider two signals, $\underline{s}_1$ and $\bar{s}_1$, for bidder 1 such that $\underline{s}_1 < 1/2 < \bar{s}_1$. Given the bid functions of bidders 2 and 3, bidder 1 is indifferent between bidding 5 and 6 when his signal is $\underline{s}_1$, but that bidding 5 is strictly better than bidding 6 when 1's signal increases to $\bar{s}_1$ (see below). This of course violates SCC.

Furthermore, both bids, 5 and 6, are individually irrational for bidder 1 whether his signal is high or low. As we have already indicated, without ties in bids this is the only way that single-crossing can fail. The requisite calculations follow.

$$\begin{aligned}
V_1(b_1 = 6, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(9+8+2+1) - 6 &&= -1 \\
V_1(b_1 = 5, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(1-5) &&= -1 \\
V_1(b_1 = 6, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= (\tfrac{1}{3}9 + \tfrac{1}{6}8 + \tfrac{1}{3}2 + \tfrac{1}{6}1) - 6 &&= -\tfrac{5}{6} \\
V_1(b_1 = 5, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}(1-5) &&= -\tfrac{2}{3}
\end{aligned}$$

3.2. The Second Failure: Ties at Winning Bids

Consider now the following values.

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = 336, \quad v_3 = 336$$

Suppose this time that bidders 2 and 3 bid zero when their signal is 1/2 or lower and bid 120 when their signal is above 1/2. Consequently, bidders 2 and 3 bid zero and 120 with positive probability each.

Consider again two signals, $\underline{s}_1$ and $\bar{s}_1$, for bidder 1 such that $\underline{s}_1 < 1/2 < \bar{s}_1$. Direct calculations now establish that, given the bidding functions of bidders 2 and 3, bidder 1's unique best reply among the bids $\{0, 120, 167\}$ is 167 when his signal is $\underline{s}_1$, but his unique best reply when his signal increases to $\bar{s}_1$ is 120. Thus SCC is again violated. However, this time the chosen bids, 120 and 167, are individually rational. Note, however, that 120 ties as a winning bid with positive probability. The relevant calculations are as follows.

$$\begin{aligned}
V_1(b_1 = 167, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(0+0+336+336) - 167 &&= 1 \\
V_1(b_1 = 120, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(0-120) + \tfrac{1}{4}(\tfrac{1}{2})(336-120) && \\
&\quad + \tfrac{1}{4}(\tfrac{1}{2})(0-120) + \tfrac{1}{4}(\tfrac{1}{3})(336-120) &&= 0
\end{aligned}$$

$$
\begin{aligned}
V_1(b_1 = 0, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(\tfrac{1}{3})(0-0) &&= 0 \\
V_1(b_1 = 167, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}336 + \tfrac{1}{3}336 - 167 &&= 1 \\
V_1(b_1 = 120, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}(0-120) + \tfrac{1}{6}(\tfrac{1}{2})(336-120) && \\
&\quad + \tfrac{1}{3}(\tfrac{1}{2})(0-120) + \tfrac{1}{3}(\tfrac{1}{3})(336-120) &&= 2 \\
V_1(b_1 = 0, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}(\tfrac{1}{3})(0-0) &&= 0
\end{aligned}
$$

4. IRT-SCC: Proof Sketch

We now provide a sketch of the proof of IRT-SCC. For simplicity, suppose there are three bidders, 1, 2, and 3, each with a reserve price of zero. Consider two bids, $\bar{b}_1 > \underline{b}_1 > 0$ for bidder 1, and suppose that bidders 2 and 3 each employ a strictly increasing bidding function with the property that bidder $j = 2, 3$ bids below $\underline{b}_1$ when $s_j < \underline{s}_j$ and bids between $\underline{b}_1$ and $\bar{b}_1$ when $s_j \in (\underline{s}_j, \bar{s}_j)$. So, consulting Figure 4.1, bidder 1 wins with a bid of $\bar{b}_1$ when (s_2, s_3) is in any one of the cells A_k, and wins with a bid of $\underline{b}_1$ when (s_2, s_3) is in cell A_0.[11] Assume that each A_k has positive probability conditional on any signal for bidder 1.

Consider two signals, $\underline{s}_1 < \bar{s}_1$, for bidder 1. Our objective is to illustrate the role played by individual rationality and the absence of ties in obtaining single crossing for bidder 1 with respect to his bids, $\underline{b}_1 < \bar{b}_1$, and his signals, $\underline{s}_1 < \bar{s}_1$. Single crossing requires, in part, that if $\bar{b}_1$ is at least as good as $\underline{b}_1$ when 1's signal is $\underline{s}_1$, then the same is true when his signal increases to $\bar{s}_1$.

Note that ties have already been ruled out by supposing that bidders 2 and 3 employ strictly increasing bidding functions. Indeed, this has been incorporated into Figure 4.1; there are no cells in which bidder 1 wins with probability other than zero or one. Consequently, single crossing cannot fail because of ties. To ensure that single crossing cannot fail because bids are individually irrational, it is enough to suppose that bidder 1's low bid, $\underline{b}_1$, is individually rational at his high signal, $\bar{s}_1$.

We wish now to demonstrate that single crossing holds for bider 1. Consult Figure 4.1. The ex-post difference in 1's payoff from bidding $\bar{b}_1$ versus $\underline{b}_1$, which

[11] Ties occur with probability zero given the strictly increasing functions of bidders 2 and 3.

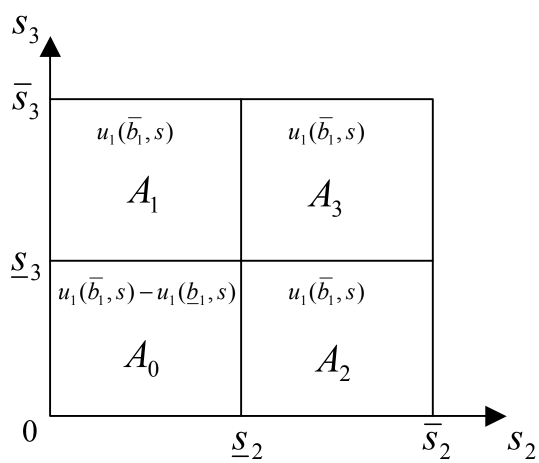

Figure 4.1: Payoff difference from $b_1 = \bar{b}_1$ vs. $b_1 = \underline{b}_1$

we shall denote by $\Delta(s)$, is $u_1(\bar{b}_1, s) - u_1(\underline{b}_1, s)$ when the signals of bidders 2 and 3 are in cell A_0, while it is simply $u_1(\bar{b}_1, s)$ in the other three cells because $\underline{b}_1$ loses in those cells.

Let A denote the union of the four cells, i.e., the event that $\bar{b}_1$ is a winning bid. Consequently, when his signal s_1, the difference in 1's payoff from bidding $\bar{b}_1$ versus $\underline{b}_1$ is $\Pr(A|s_1)E(\Delta(S)|A, s_1)$. Single crossing requires that

$$\Pr(A|\underline{s}_1)E(\Delta(S)|A, \underline{s}_1) \geq 0 \implies \Pr(A|\bar{s}_1)E(\Delta(S)|A, \bar{s}_1) \geq 0.$$

But because $\bar{b}_1$ wins with positive probability regardless of 1's signal, both $\Pr(A|\bar{s}_1)$ and $\Pr(A|\underline{s}_1)$ are positive. Hence, single crossing holds if

$$E(\Delta(S)|A, \underline{s}_1) \leq E(\Delta(S)|A, \bar{s}_1). \tag{4.1}$$

Consequently, we may restrict attention to signals of bidders 2 and 3 that lie in A.

Now, because $\Delta(s)$ is nondecreasing in s_1, if it were also nondecreasing in (s_2, s_3) on A, then (4.1) would follow from the following monotonicity result due to Milgrom and Weber (1982; Theorem 5): If $X_1, ..., X_n$ are affiliated and $\phi(x_1, ..., x_n)$ is a nondecreasing real-valued function, then the expectation of ϕ, conditional on any number of events of the form $X_k \in [a_k, b_k]$, is nondecreasing in all the a_k and b_k, where $a_k = b_k$ is permitted.

However, $\Delta(s)$ can fail to be nondecreasing in (s_2, s_3). For example, if at some point on the border between A_0 and A_1, $u_1(\underline{b}_1, s) < 0$, then $\Delta(s)$ strictly decreases as (s_2, s_3) increases through that point from just below the border to just above it. Hence, Milgrom and Weber's result does not apply.

Fortunately, an extension of their result that requires only an *average* form of monotonicity on the cells A_k does apply (see Lemma 7.1 in the appendix). In the present example, this average form of monotonicity is that for $k = 1, 2$,

$$E(u_1(\bar{b}_1, s) - u_1(\underline{b}_1, s)|A_0, \bar{s}_1) \leq E(u_1(\bar{b}_1, s)|A_k, \bar{s}_1) \leq E(u_1(\bar{b}_1, s)|A_3, \bar{s}_1). \quad (4.2)$$

Hence, to establish (4.1) we need only show that (4.2) holds.

The inequality on the right in (4.2) follows from Milgrom and Weber's monotonicity result. To obtain the inequality on the left we use the individual rationality of the bid $\underline{b}_1$ at $\bar{s}_1$. Note that a bid of $\underline{b}_1$ wins precisely when the others' signals are in region A_0, and this occurs with positive probability. Hence, individual rationality implies

$$E(u_1(\underline{b}_1, S)|A_0, \bar{s}_1) \geq 0. \quad (4.3)$$

The inequality on the left in (4.2) then follows since, for $k = 1, 2$,

$$\begin{aligned} E(u_1(\bar{b}_1, S)|A_k, \bar{s}_1) &\geq E(u_1(\bar{b}_1, S)|A_0, \bar{s}_1) \\ &\geq E(u_1(\bar{b}_1, S) - u_1(\underline{b}_1, S)|A_0, \bar{s}_1), \end{aligned}$$

where the first inequality follows from Milgrom and Weber's monotonicity result and the second follows from (4.3). Hence, (4.1), and therefore also single crossing, holds.

5. Multi-Dimensional Signals: A Private Value Counterexample

We now provide a private value example suggesting that Theorem 2.1 fails if the bidders' signals are not one-dimensional. The example possesses a unique equilibrium, which is pure and nonmonotone.

The example is only suggestive because, while it satisfies the multi-dimensional signal analogue of assumption A.1, it involves several extreme distributional specifications. For example, some signals are discrete random variables rather than continuous ones, and some signals are perfectly correlated with others. Consequently, the joint signal distribution has no density function and so, formally, A.2 fails. However, the signals in our example are affiliated, and we conjecture that no smoothed nearby example satisfying A.1 and A.2 will possess a monotone pure-strategy equilibrium either.[12] But this remains an open question.

There are three bidders, 1,2,3, each with a zero reserve price, and so the losing bid l is strictly less than zero. All bidders observe a public signal, Y, which affects only the values of bidders 2 and 3. In addition, bidder 1 receives a private signal, X, which determines his value, and bidders 2 and 3 observe a common signal Z which affects their value.

One interpretation is that these are three firms bidding on a contract to upgrade their computer systems to enhance production. Firm 1 is a monopolist in market A and $-X$ is a signal about the cost of his inputs. Firms 2 and 3 compete in market B and Y is a public signal about demand in market B, while $-Z$ is a signal about the common cost of inputs to firms 2 and 3 in market B. Another interpretation, suggested by a referee, focuses on information leakage. Specifically, bidder 1 receives signal X, bidder 2 receives Y and bidder 3 receives Z. In addition, it is common knowledge that bidder 1 spies on 2 and finds out Y, bidder 2 spies on 3 and finds out Z, and bidder 3 spies on 2 and finds out Y. This interpretation is particularly interesting because bidders will typically have an incentive to find out their opponents' signals, if only to better predict their bids. Hence, whether the opportunity to find out others' signals is present can be an important issue.

So, altogether, bidder 1 receives the two-dimensional signal $S_1 = (X, Y)$, while bidders 2 and 3 each receive the common two-dimensional signal $S_2 = S_3 = (Y, Z)$. Let us suppose that X and Y are i.i.d. random variables taking on the values 0 and 1 with probability 1/2 each, and Z is independently and uniformly distributed on $[0, 1] \cup [2, 3]$. Consequently, the six real random variables $(S_1, S_2, S_3) = (X, Y, Y, Z, Y, Z)$ are affiliated.

The bidders have private values and quasilinear utilities. Bidder 1's value is

$$v_1(x, y) = 6x,$$

while bidders 2 and 3 have identical values $v_2(y, z) = v_3(y, z) = v(y, z)$, where

$$v(y, z) = \begin{cases} 7 & \text{if } y = 1, \; z \in [2, 3] \\ \\ z & \text{otherwise.} \end{cases}$$

[12] Arbitrarily nearby examples satisfying A.1 and A.2 exist.

Proposition 5.1. *The above first-price auction example possesses a unique equilibrium outcome. Every equilibrium is (up to ex ante probability zero events) pure, nonmonotone and of the following form: For* $y = 0, 1$ *and a.e.* z, $b_1(0, y) \in \{l, 0\}$, $b_1(1, 0) = 3$, $b_1(1, 1) = 1$; $b_j(y, z) = v(y, z)$ $j = 2, 3$. *Thus, the only nonuniqueness concerns bidder 1's strategy and whether, when* $X = 0$, *he bids* l *or* 0, *both of which lose with probability one.*

The proof can be found in Reny and Zamir (2002), but the argument is straightforward. Because bidders 2 and 3 have identical values, and because their signals are also identical, their identical values are common knowledge between them. Consequently, a standard Bertrand competition argument establishes that bidders 2 and 3 must each bid their value in equilibrium. Hence, bidders 2 and 3 employ monotone pure strategies. It remains only to find bidder 1's best reply.

When 1's signal $S_1 = (x, y) = (0, y)$, a best reply for bidder 1 requires bidding so that he loses with probability one, because his value is $v_1 = 6x = 0$. Hence, he must bid either l or 0. The interesting case is when $x = 1$.

When $S_1 = (x, y) = (1, 0)$, bidder 1's value is $v_1 = 6$ and he knows that $Y = 0$. Consequently, he knows that the common bid of bidders 2 and 3 is $v(0, Z) = Z$, which is uniformly distributed on $[0, 1] \cup [2, 3]$, and a straightforward calculation establishes that 1's unique best reply is to bid 3.

However, when $S_1 = (x, y) = (1, 1)$, bidder 1's value is again $v_1 = 6$, but he now knows that $Y = 1$. Consequently, he knows that bidders 2 and 3 each bid $v(1, Z) = 7$ if $Z \in [2, 3]$ while they each bid $v(1, Z) = Z$ if $Z \in [0, 1]$. Clearly, it would be suboptimal for bidder 1 to bid 7 or more since his value is only 6. Consequently, bidder 1 will bid less than 7 and so can condition on the event that 2 and 3 bid less than 7 as well. But, conditional on bidding less than 7, bidders 2 and 3 submit a common bid that is uniformly distributed on $[0, 1]$. Another straightforward calculation establishes that bidder 1's unique best reply now is to bid 1.

Hence, every best reply for bidder 1 is nonmonotone, with his bid falling from 3 to 1 when his signal increases from $(1, 0)$ to $(1, 1)$.

One reason for the failure of monotonicity here is the failure of affiliation to be inherited by monotone functions of multi-dimensional affiliated random variables. Specifically, even though the random variables Y and Z are independent and hence affiliated, the random variables Y and $v(Y, Z)$ are *not* affiliated, despite the fact that $v(Y, Z)$ is nondecreasing. To see this, observe that

$$\frac{\Pr(v(Y, Z) \in [2, 3] | Y = 0)}{\Pr(v(Y, Z) \in [0, 1] | Y = 0)} = 1 > 0 = \frac{\Pr(v(Y, Z) \in [2, 3] | Y = 1)}{\Pr(v(Y, Z) \in [0, 1] | Y = 1)}.$$

Consequently, in the example, one of the dimensions of bidder 1's signal, Y, is not affiliated with the equilibrium bids of the other bidders.

6. Second-Price Auctions: A Counterexample

Returning to the one-dimensional signal setting, one might wonder whether second-price auctions admit monotone pure-strategy equilibria under assumptions A.1 and A.2.[13] The answer is trivially 'yes' if dominated strategies are permitted and if for some bidder i, $E(u_i(r_i, S)|s_i) \geq 0$ for all $s_i \in [0, 1]$. In this case there is always an equilibrium in which, regardless his signal, bidder i bids $\tilde{b} > r_i$ such that for all $j \neq i$, $u_j(\tilde{b}, s) < 0$ for all s, and all others drop out by bidding l. Hence, the substantive question is: *Under assumptions A.1 and A.2 does a second-price auction necessarily possess a monotone pure-strategy equilibrium in undominated strategies?* The answer is 'no,' and we now provide an example. For the sake of brevity, the signals in the example are discrete and some are perfectly correlated. However, the example can easily be smoothed so that it satisfies both A.1 and A.2.

There are three bidders. Bidder i receives signal s_i. Bidder 1's value is $v_1 = s_1 + s_2$, while bidders 2 and 3 have private values; $v_2 = s_2$ and $v_3 = s_3$. Each bidder receives either a high or a low signal. Specifically, $s_1 \in \{1, 2\}$, $s_2 \in \{0, 4\}$, and $s_3 \in \{0, 3\}$. Bidders 1 and 3 have perfectly correlated signals, both being low with probability 1/2 or both being high with probability 1/2. Bidder 2's signal is independent of the others', being low with probability p and high with probability $1 - p$, where $p \in (2/3, 1)$. The bidders' signals are therefore affiliated.

Because bidders 2 and 3 have private values, their only undominated strategy is to bid their value. Bidders 2 and 3 therefore each employ a pure nondecreasing bid function. It remains only to compute an undominated best reply for bidder 1.

When $s_1 = 1$, bidder 1 knows that $\mathbf{b}_3(s_3) = s_3 = 0$. Since 1's value is $v_1 = 1 + s_2$, his value is 1 if $\mathbf{b}_2(s_2) = s_2 = 0$, while it is 5 if $\mathbf{b}_2(s_2) = s_2 = 4$. Consequently, 1's value is always strictly above the high bid of the others and so it is optimal for him to bid so that he wins with probability one. Because $p < 1$, this requires a bid above 4. However, when $s_1 = 1$ his value is never more than 5, and so in any undominated equilibrium we must have $\mathbf{b}_1(1) \in (4, 5]$.

When $s_1 = 2$, bidder 1 knows that $\mathbf{b}_3(s_3) = s_3 = 3$. Consequently, a bid above 4 still wins with probability one, but bidder 1 now pays 3 when $\mathbf{b}_2(s_2) = s_2 = 0$

[13]We thank a referee for asking precisely this question.

and $v_1 = 2$, and pays 4 when $\mathbf{b}_2(s_2) = s_2 = 4$ and $v_1 = 6$. Hence, a bid above 4 yields bidder 1 a payoff of $p(2-3)+(1-p)(6-4)$, which is negative because $p > 2/3$. Similarly, any bid weakly between 3 and 4 yields a negative payoff. Consequently, we must have $\mathbf{b}_1(2) < 3$. Hence, an undominated pure-strategy equilibrium exists, but in no such equilibrium is bidder 1's strategy nondecreasing.

7. APPENDIX

We begin with our extension of Theorem 5 of Milgrom and Weber (1982). In $\mathbb{R}^k$, call the product of k real intervals — each of which can be closed, open or half-open — a *cell.* If C and C' are cells in $\mathbb{R}^k$, then we will write $C \leq C'$ if the lower (upper) endpoint of each interval in the product defining C is no greater than the lower (upper) endpoint of the corresponding interval in the product defining C'.[14]

A function $h : [0,1]^k \to \mathbb{R}$ is *cellwise nondecreasing with respect to the density* $g : [0,1]^k \to \mathbb{R}_+$ if there is a finite partition, $\{C^i\}$, of $[0,1]^k$ into cells such that, (i) the restriction of h to each cell is nondecreasing, and (ii) $E_g(h|C^i) \leq E_g(h|C^j)$ whenever $C^i \leq C^j$.

Evidently, by considering the partition consisting of the single cell $[0,1]^k$, every nondecreasing function on $[0,1]^k$ is cellwise nondecreasing for every density. On the other hand, a function can be cellwise nondecreasing with respect to a given density without being nondecreasing on its domain. Our extension of Milgrom and Weber's monotonicity result is as follows.

Lemma 7.1. *Suppose that* $h : [0,1]^{n+m} \to \mathbb{R}$, *that* $X_1, ..., X_n, Y_1, ..., Y_m$ *are affiliated random variables with joint strictly positive density* $g : [0,1]^{n+m} \to \mathbb{R}_{++}$, *and that* $\bar{x} \geq \underline{x}$ *are vectors in* $[0,1]^n$. *If* $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_m)$, *and*

(i) *for every* $y \in [0,1]^m$, $h(x,y)$ *is nondecreasing in* x *on* $[0,1]^n$, *and*

(ii) $h(x,\cdot)$ *is cellwise nondecreasing w.r.t* $g(\cdot|X = x)$ *when* $x = \bar{x}$ *or* $\underline{x}$,

then $E[h(X,Y)|X = \underline{x}] \leq E[h(X,Y)|X = \bar{x}]$.

Proof. We provide the proof for the case in which $h(\bar{x},\cdot)$ is cellwise nondecreasing w.r.t $g(\cdot|X = \bar{x})$, the other case being similar. Let $\{C^i\}$ denote the associated

[14] So, in $\mathbb{R}^2$ for example, $(a_1, b_1] \times [a_2, b_2) \leq (a_1', b_1') \times (a_2', b_2')$ if $a_i \leq a_i'$ and $b_i \leq b_i'$, for $i = 1, 2$.

partition of $[0,1]^m$. Hence, the step function, $\phi : [0,1]^m \to \mathbb{R}$ defined by $\phi(y) = E[h(X,Y)|X = \bar{x}, Y \in C^i]$ if $y \in C^i$, is nondecreasing. Therefore,

$$
\begin{aligned}
E[h(X,Y)|X = \underline{x}] &= \textstyle\sum_i \Pr(C^i|X = \underline{x}) E[h(X,Y)|X = \underline{x}, Y \in C^i] \\
&\leq \textstyle\sum_i \Pr(C^i|X = \underline{x}) E[h(X,Y)|X = \bar{x}, Y \in C^i] \\
&= E[\phi(Y)|X = \underline{x}] \\
&\leq E[\phi(Y)|X = \bar{x}] \\
&= \textstyle\sum_i \Pr(C^i|X = \bar{x}) E[h(X,Y)|X = \bar{x}, Y \in C^i] \\
&= E[h(X,Y)|X = \bar{x}],
\end{aligned}
$$

where the two inequalities follow from Theorem 5 of Milgrom-Weber (1982), henceforth MW. ■

Proof of Proposition 2.3. To establish IRT-SCC, fix $\bar{b}_i > \underline{b}_i$ and, for all $j \neq i$, fix nondecreasing bid functions, $\mathbf{b}_j$, so that $\Pr(l < \max_{j \neq i} \mathbf{b}_j(S_j) = \bar{b}_i \text{ or } \underline{b}_i) = 0$. In what follows, all statements about i's payoff, etc., correspond to these fixed bid functions of the others.

Clearly, if $b_i = \bar{b}_i$ or $\underline{b}_i$, then the probability that b_i is among the highest serious bids is equal to the probability that b_i is the *unique* highest bid (and hence serious). Moreover, because $f > 0$ and the event that b_i is uniquely highest depends only upon the others' signals, this event (and any other with this property) has positive probability if and only if it has positive probability conditional on every $s_i \in [0,1]$. These facts will be used repeatedly in what follows.

Let A denote the event that $\bar{b}_i$ is a winning bid. That is,

$$A = \{s \in [0,1]^N : \max_{j \neq i} \mathbf{b}_j(s_j) < \bar{b}_i\}.$$

If $\underline{b}_i$ wins with probability zero, then $V_i(\underline{b}_i, \mathbf{b}_{-i}|s_i) = 0$ for every s_i. Consequently, IRT-SCC holds because, by MW Theorem 5, $E(u_i(\bar{b}_i, S)|s_i, A)$ is nondecreasing in s_i whenever A has positive probability. Consequently, we can assume that $\underline{b}_i$ wins with positive probability. A fortiori, $\bar{b}_i$ wins with positive probability and so A has positive probability.

Partition A as follows. For every subset J of $\{1, ..., N\} \backslash \{i\}$, define

$$A(J) = A \cap \{s \in [0,1]^N : \forall j \neq i,\ \mathbf{b}_j(s_j) \geq \underline{b}_i \text{ iff } j \in J\}.$$

Hence, ignoring ties, $A(J)$ is the event that $\underline{b}_i$ loses against precisely those bidders in J. Because $A(J)$ is contained in A, $\bar{b}_i$ wins against every $j \neq i$ in each event $A(J)$. Note that $A(\emptyset)$, being the event that $\underline{b}_i$ loses against no one, is the event that $\underline{b}_i$ wins the auction and so has positive probability.

Because both $\bar{b}_i$ and $\underline{b}_i$ win with positive probability and tie the maximum of the others' bids with probability zero, IRT-SCC reduces to the following: For $\bar{s}_i > \underline{s}_i$,

$$E[u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \underline{s}_i] \geq 0 \Longrightarrow E[u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \bar{s}_i] \geq 0 \tag{7.1}$$

when $E(u_i(\bar{b}_i, S) \,|\, A, \underline{s}_i) \geq 0$, and

$$E[u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \bar{s}_i] \leq 0 \Longrightarrow E[(u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \underline{s}_i] \leq 0 \tag{7.2}$$

when $E[u_i(\underline{b}_i, S) | A(\emptyset), \bar{s}_i] \geq 0$.

Now, by MW Theorem 5, $E[u_i(\bar{b}_i, S) | A, \underline{s}_i] \geq 0$ implies $E[u_i(\bar{b}_i, S) | A, \bar{s}_i] \geq 0$. Consequently, if in addition $E[u_i(\underline{b}_i, S) I_{A(\emptyset)} | \, A, \bar{s}_i] < 0$, then (7.1) holds simply because the second difference is positive, being the difference between a nonnegative and a negative number. Hence, it suffices to establish (7.1) and (7.2) when $E[u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \bar{s}_i] \geq 0$, or equivalently, when $E[u_i(\underline{b}_i, S) | A(\emptyset), \bar{s}_i] \geq 0$. We shall in fact show more than this, namely, that if $E[u_i(\underline{b}_i, S) | A(\emptyset), \bar{s}_i] \geq 0$, then

$$E[u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \underline{s}_i] \leq E[u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) I_{A(\emptyset)} | A, \bar{s}_i]. \tag{7.3}$$

To see this, let $\Delta_i(s) = u_i(\bar{b}_i, s) - u_i(\underline{b}_i, s) I_{A(\emptyset)}$ and note that $\Delta_i(s_i, s_{-i})$ is nondecreasing in s_i on $[0, 1]$. Hence, by Lemma 7.1, it suffices to show that $\Delta_i(\bar{s}_i, \cdot) : A \to \mathbb{R}$ is cellwise nondecreasing with respect to $f(s_{-i} | A, \bar{s}_i)$. Consider the above finite partition, $\{A(J)\}$, of A into cells. We may restrict attention to those subsets J such that $A(J)$ is nonempty.

If $A(J') \geq A(J)$, then

$$E[u_i(\bar{b}_i, S) | A(J'), \bar{s}_i] \geq E[u_i(\bar{b}_i, S) | A(J), \bar{s}_i],$$

follows from MW Theorem 5. Furthermore, because every $A(J) \geq A(\emptyset)$,

$$E[u_i(\bar{b}_i, S) | A(J), \bar{s}_i] \geq E[u_i(\bar{b}_i, S) | A(\emptyset), \bar{s}_i] \geq E[u_i(\bar{b}_i, S) - u_i(\underline{b}_i, S) | A(\emptyset), \bar{s}_i],$$

where the first inequality follows from MW Theorem 5, and the second follows because $E[u_i(\underline{b}_i, S) | A(\emptyset), \bar{s}_i] \geq 0$. Hence, $\Delta_i(\bar{s}_i, \cdot)$ is cellwise nondecreasing. ■

Proof of Theorem 2.1. PART 1. In this part of the proof we consider a modified first-price auction game. There are two modifications. First, we restrict the bidders to finite sets of bids. Each finite set contains the losing bid, l, but no two sets have any serious bid in common. This means that ties can only occur at bids of l. Second, we restrict the bidders' strategies so that each bidder must bid l when his signal is in $[0, \varepsilon)$, where $\varepsilon \in (0, 1)$ is fixed.[15] Therefore, because the joint density of signals is strictly positive on $[0, 1]^N$, any serious bid submitted by a bidder wins with strictly positive probability regardless of his signal. The import of this will be explained in the second part of the proof. We now wish to show the following.

Under the two modifications above, a monotone pure-strategy equilibrium exists.

To establish this, it would suffice to verify the single-crossing condition (SCC) employed in Athey (2001). We could then appeal to Athey's Theorem 1. However, as we have seen, SCC does not hold in our setting. Fortunately, Athey's *finite-action* existence proof goes through if the following, more permissive, *best-reply* single-crossing condition holds in our modified auction.

BR-SCC. If b_i' is a best reply for s_i against $\mathbf{b}_{-i}$, and b_i is feasible for s_i, then the inequality $V_i(b_i', \mathbf{b}_{-i} \,|\, s_i') \geq V_i(b_i, \mathbf{b}_{-i} \,|\, s_i')$ holds for all $s_i' > s_i$ if $b_i' > b_i$, while it holds for all $s_i' < s_i$ if $b_i' < b_i$.

Our modified auction therefore possesses a monotone pure-strategy equilibrium if BR-SCC holds, which we now establish. Suppose that b_i' is a best reply for s_i against the others' monotone strategy $\mathbf{b}_{-i}$. Because the bids l and b_i are feasible for s_i, we must have $V_i(b_i', \mathbf{b}_{-i} \,|\, s_i) \geq \max(V_i(b_i, \mathbf{b}_{-i} \,|\, s_i), V_i(l, \mathbf{b}_{-i} \,|\, s_i)) \geq 0$. Hence, in addition to being as good as b_i, b_i' is individually rational for i at s_i. Also, because the only common bid available to distinct bidders is l, $\Pr(l < \max_{j \neq i} \mathbf{b}_j(S_j) = b_i' \text{ or } b_i) = 0$. Hence, by Proposition 2.3, $V_i(b_i', \mathbf{b}_{-i} \,|\, s_i') \geq V_i(b_i, \mathbf{b}_{-i} \,|\, s_i')$ holds for all $s_i' > s_i$ if $b_i' > b_i$, while it holds for all $s_i' < s_i$ if $b_i' < b_i$. This establishes BR-SCC for the modified auction and so it possesses a monotone pure-strategy equilibrium.[16]

PART 2. In this part of the proof, we consider a sequence of monotone equilibria of the modified auctions from Part 1. For $n = 1, 2, ...$, let G^n denote the

[15] Athey (2001) also employs this device.

[16] Note that our example of the second failure of SCC in Section 5 demonstrates that BR-SCC fails if one allows the bidders' finite bid sets to have serious bids in common.

modified auction in which $\varepsilon = 1/n$ and bidder i's finite set of bids is denoted by B_i^n. Note then that $l \in B_i^n$ for each n. Further, suppose that $B_i^n \supseteq B_i^{n-1}$ and that $B_i^\infty = \cup_n B_i^n$ is dense in B_i. Let $\mathbf{b}^n$ denote a monotone pure-strategy equilibrium of G^n. By A.1(ii), for every i there exists $\tilde{b} \geq r_i$ such that $u_i(b, s) < 0$ for all $b > \tilde{b}$ and all $s \in [0, 1]^N$. Consequently, because all serious bids for i win with positive probability, and because the bid l is available in G^n, each $\mathbf{b}_i^n$ is bounded above by $\tilde{b}$; and of course $\mathbf{b}_i^n$ is bounded below by l. Hence, by Helley's Theorem, we may assume without loss that for every i, $\mathbf{b}_i^n(s) \to \hat{\mathbf{b}}_i(s)$ for a.e. $s_i \in [0, 1]$, where each $\hat{\mathbf{b}}_i$ is nondecreasing on $[0, 1]$. We shall argue that $\hat{\mathbf{b}}$ is a monotone equilibrium of the first-price auction game.

The argument would be trivial were it not for the possibility of ties, which lead to discontinuities in payoffs. There are two points at which ties must be carefully considered. The first arises when considering a bidder i's payoff from any bid against the limit strategies, $\hat{\mathbf{b}}_{-i}$, of the others. It must be shown that, given his signal, bidder i can either approximate arbitrarily well or improve upon his payoff by employing a slightly higher bid that, with probability one, does not tie the others' bids. This is demonstrated in (7.5), a result that depends upon the tie-break property that a high bidder's probability of winning does not increase when additional bidders tie for the winning bid.

Ties must also be handled carefully when considering the limit of a bidder's payoff as the grid of bids becomes finer and finer. While the a priori possibility of ties at the limit can render the limiting payoff distinct from the payoff at the limit, the limiting payoff, it is shown, can always be obtained by employing the limit strategies, $\hat{\mathbf{b}}$, together with some surrogate tie-breaking rule that is a function of the vector of signals alone. Demonstrating that ties in fact occur with probability zero under $\hat{\mathbf{b}}$ requires establishing that, even against such surrogate tie-breaking rules, a bidder can approximate or improve upon his payoff by increasing his bid slightly. As before, the details of the surrogate tie-breaking rule are critical for establishing this result.

Throughout the remainder of the proof it will be convenient to maintain two conventions regarding the bid $b = l$. First, define $u_i(l, s) = 0$ for all i and all s, and second, when $b = l$ define $b' \downarrow b$ to mean $b' = b$, since l is isolated.

Because $l \in B_i^n$, $V_i(\mathbf{b}^n) \geq 0$ for all i and n. Hence, for every i and a.e. s_i such that $\hat{\mathbf{b}}_i(s_i) > l$ and $\Pr(\max_{j \neq i} \hat{\mathbf{b}}_j(S_j) \leq \hat{\mathbf{b}}_i(s_i)|s_i) > 0$, the following holds for n large enough.

$$
\begin{aligned}
0 &\leq E[u_i(\mathbf{b}_i^n(s_i), S)|s_i, \max_{j\neq i} \mathbf{b}_j^n(S_j) \leq \mathbf{b}_i^n(s_i)] \\
&\leq E[u_i(\mathbf{b}_i^n(s_i), S)|s_i, \max_{j\neq i} \mathbf{b}_j^n(S_j) \leq \hat{\mathbf{b}}_i(s_i) + \delta] \\
&\to E[u_i(\hat{\mathbf{b}}_i(s_i), S)|s_i, \max_{j\neq i} \hat{\mathbf{b}}_j(S_j) \leq \hat{\mathbf{b}}_i(s_i)], \qquad (7.4)
\end{aligned}
$$

where the first line follows because $\hat{\mathbf{b}}_i(s_i) > l$ implies $\mathbf{b}_i^n(s_i)$ is serious for n large enough, implying both that it wins with positive probability (given our strategy restriction), and that the right-hand side is s_i's payoff because ties in G^n cannot occur at serious bids; the second line follows for any $\delta > 0$ by MW Theorem 5; and the third line follows taking the limit first as $n \to \infty$ and then as $\delta \downarrow 0$ along a sequence such that $\hat{\mathbf{b}}_i(s_i) + \delta$ is never one of the at most countably many mass points of $\max_{j\neq i} \hat{\mathbf{b}}_j(S_j)$, ensuring that the first limit, in n, exists for each such δ.[17]

Let $\lambda_i(b, b_{-i})$ denote the probability that i wins when the vector of bids is (b, b_{-i}). Suppose that $V_i(b, \hat{\mathbf{b}}_{-i}|s_i) \geq 0$ for some bid b and some signal s_i of bidder i. Then, letting $H_i = \{s_{-i} : \max_{j\neq i} \hat{\mathbf{b}}_j(s_j) \leq b\}$ and defining $E(\cdot|s_i, H_i) = 0$ if $\Pr(H_i|s_i) = 0$, we have the following:

$$
\begin{aligned}
&0 \leq V_i(b, \hat{\mathbf{b}}_{-i}|s_i) \\
&= \Pr(H_i|s_i)E[u_i(b, S)\lambda_i(b, \hat{\mathbf{b}}_{-i}(S_{-i}))|s_i, H_i] \\
&\leq \Pr(H_i|s_i)E[u_i(b, S)|s_i, H_i]E[\lambda_i(b, \hat{\mathbf{b}}_{-i}(S_{-i}))|s_i, H_i] \qquad (7.5) \\
&\leq \Pr(H_i|s_i)E[u_i(b, S)|s_i, H_i] \\
&= \lim_{b'\downarrow b} V_i(b', \hat{\mathbf{b}}_{-i}|s_i),
\end{aligned}
$$

where the inequality on the third line follows from MW Theorem 23, because $u_i(b, s_i, S_{-i})$ and $1 - \lambda_i(b, \hat{\mathbf{b}}_{-i}(S_{-i}))$, given uniform tie-breaking, are nondecreasing in S_{-i} and so (because the conditioning event is a sublattice) the conditional expectation of their product is no smaller than the product of their conditional expectations, and the inequality on the fourth line follows because, under uniform

[17] Obtaining nonnegativity of the limit in (7.4) is precisely where the restriction on strategies in G^n is required.

tie-breaking, $\Pr(H_i|s_i) > 0$ implies $0 < E[\lambda_i(b, \hat{\mathbf{b}}_{-i}(S_{-i}))|s_i, H_i] \leq 1$.[18]

Because the probability that i wins, $\lambda_i(b_i, b_{-i})$, is nondecreasing in b_i and nonincreasing in b_{-i}, $\lambda_i(\mathbf{b}^n(s))$ is a sequence of functions each of which is monotone in each of its arguments, $s_1, ..., s_N$, being nondecreasing in s_i and nonincreasing in s_{-i}. Hence, by Helley's theorem, there exists $\alpha_i : [0,1]^N \to [0,1]$, nondecreasing in s_i and nonincreasing in s_{-i}, such that (extracting a subsequence if necessary) $\lambda_i(\mathbf{b}^n(s)) \to_n \alpha_i(s)$ for a.e. $s \in [0,1]^N$. Consequently, $\sum_i \alpha_i(s) \leq 1$ a.e. $s \in [0,1]^N$, and for every i and a.e. s_i, $V_i(\mathbf{b}_i^n(s_i), \mathbf{b}_{-i}^n|s_i) = E[u_i(\mathbf{b}_i^n(s_i), S)\lambda_i(\mathbf{b}^n(S))|s_i]$ converges to $E[u_i(\hat{\mathbf{b}}_i(s_i), S)\alpha_i(S)|s_i]$, by the dominated convergence theorem. Hence, one can think of $\alpha_i(\cdot)$ as a surrogate tie-breaking rule that, were it actually employed at the limit, would yield continuity of payoffs there.

Because each $\hat{\mathbf{b}}_j(S_j)$ has at most countably many mass points and B_i^n becomes dense in B_i, for every $b \in B_i$, every $\varepsilon > 0$, and a.e. s_i, there exists $\bar{n} \geq 1$ and $\bar{b} \in B_i^{\bar{n}}$ such that

$$\begin{aligned}
\lim_{b' \downarrow b} V_i(b', \hat{\mathbf{b}}_{-i}|s_i) &\leq V_i(\bar{b}, \hat{\mathbf{b}}_{-i}|s_i) + \varepsilon, \\
&\leq V_i(\bar{b}, \mathbf{b}_{-i}^n|s_i) + 2\varepsilon, \text{ for } n \geq \bar{n} \\
&\leq V_i(\mathbf{b}_i^n(s_i), \mathbf{b}_{-i}^n|s_i) + 2\varepsilon, \text{ for } n \geq \bar{n},
\end{aligned}$$

where the first and second lines follow because $\bar{b}$ can be chosen so that the probability that any $\hat{\mathbf{b}}_j(S_j)$ equals $\bar{b}$ is arbitrarily small, and the third line follows because $\bar{b} \in B_i^{\bar{n}}$ is feasible in G^n for every $n \geq \bar{n}$ and $\mathbf{b}^n$ is an equilibrium. Because $\varepsilon > 0$ is arbitrary, as is b in (7.5), we obtain, because $l \in B_i$ yields a payoff of zero, the following for all i and a.e. s_i,

$$\sup_{b_i \in B_i} V_i(b, \hat{\mathbf{b}}_{-i}|s_i) \leq \lim_n V_i(\mathbf{b}_i^n(s_i), \mathbf{b}_{-i}^n|s_i) = E[u_i(\hat{\mathbf{b}}_i(s_i), S)\alpha_i(S)|s_i]. \quad (7.6)$$

[18] In particular, (7.5) shows that, under the uniform tie-breaking rule, payoffs from bids that tie for the winning bid with positive probability can be approximated or improved upon by slightly higher bids that tie with probability zero. This fact is not obvious because, unlike the private value setting, winner's curse effects can make it costly to always 'win the tie' by bidding slightly higher. A slightly higher bid implies that a bidder wins for sure even in the "bad news" events in which he wins and only a small number (one say) of the other bidders would have tied with him. Conditional on such events, the higher bid can make him worse off. But under uniform tie-breaking, the benefit of ensuring that he also wins in the "good news" events in which he wins and *many* bidders would have tied with him, outweighs this cost. This is not true for all tie-breaking rules.

For a.e. s_i such that $\hat{\mathbf{b}}_i(s_i) > l$, letting $H_i = \{S_{-i} : \max_{j\neq i} \hat{\mathbf{b}}_j(S_j) \leq \hat{\mathbf{b}}_i(s_i)\}$, we have

$$
\begin{aligned}
0 &\leq E[u_i(\hat{\mathbf{b}}_i(s_i), S)\alpha_i(S)|s_i] \\
&= \Pr(H_i|s_i)E[u_i(\hat{\mathbf{b}}_i(s_i), S)\alpha_i(S)|s_i, H_i] \\
&\leq \Pr(H_i|s_i)E[u_i(\hat{\mathbf{b}}_i(s_i), S)|s_i, H_i]E[\alpha_i(S)|s_i, H_i] \\
&\leq \Pr(H_i|s_i)E[u_i(\hat{\mathbf{b}}_i(s_i), S)|s_i, H_i] \\
&= \lim_{\varepsilon\downarrow 0} V_i(\hat{\mathbf{b}}_i(s_i) + \varepsilon, \hat{\mathbf{b}}_{-i}|s_i) \\
&\leq \sup_{b_i\in B_i} V_i(b_i, \hat{\mathbf{b}}_{-i}|s_i),
\end{aligned}
$$

where the first line follows from (7.6), the second line follows because $\alpha_i(s_i, s_{-i}) = 0$ a.e. $s_{-i} \notin H_i$, the third line follows from MW Theorem 23 (as in (7.5)), and the fourth line follows because $\alpha_i(\cdot) \in [0, 1]$ and, by (7.4), $\Pr(H_i|s_i) > 0$ implies $E[u_i(\hat{\mathbf{b}}_i(s_i), S)|s_i, H_i] \geq 0$.[19] Hence, by the inequality on line six and (7.6), the inequalities on lines three, four, and six must be equalities. In particular, if $\Pr(H_i|s_i) > 0$, then

$$0 \leq E[u_i(\hat{\mathbf{b}}_i(s_i), S)|s_i, H_i]E[\alpha_i(S)|s_i, H_i] = E[u_i(\hat{\mathbf{b}}_i(s_i), S)|s_i, H_i], \tag{7.7}$$

and, because $u_i(b, s)$ is strictly increasing in s_i and H_i increases appropriately (as a set) with s_i, MW Theorem 5 implies that the inequality in (7.7) must be strict, and therefore $E[\alpha_i(S)|s_i, H_i] = 1$, for a.e. s_i such that $\Pr(H_i|s_i) > 0$.

[19] Note the importance of $\alpha_i(s_i, s_{-i})$ being nonincreasing in s_{-i}. The third line can fail otherwise. For example, consider two risk-neutral bidders with uniform [0,1] i.i.d. signals, and where bidder i's value is s_j, the other bidder's signal. If the surrogate tie-breaking rule, $\alpha_i(\cdot)$, is such that upon a tie the bidder with the highest value wins, then $\alpha_i(\cdot)$ is *increasing* in the opponent's signal. The third line now fails if $\hat{\mathbf{b}}_i(\cdot) = 1/2$ for $i = 1, 2$, since bidding above or below 1/2 is then *strictly* suboptimal whenever $s_i \in (0, 1)$. (Hence, $\hat{\mathbf{b}}_i(\cdot) = 1/2$ for $i = 1, 2$ is an equilibrium under this surrogate tie-breaking rule and both players earn positive ex-ante payoffs. Thus, while Maskin and Riley's (2000) Proposition 4 is valid for the tie-breaking rule they actually employ, this example demonstrates that it is not valid for *all* tie-breaking rules, contrary to their footnote 9.)

Consequently, given $\emptyset \neq I \subseteq \{1, ..., N\}$ and letting $T_I = \{s : \hat{\mathbf{b}}_i(s_i) = \max_j \hat{\mathbf{b}}_j(s_j) > l, \forall i \in I\}$, if $\Pr(T_I) > 0$ then for every $i \in I$, $\alpha_i(s) = 1$ for a.e. $s \in T_I$. But $\sum_{i \in I} \alpha_i(s) \leq 1$ a.e. $s \in [0,1]^N$ then implies that $\#I = 1$. Hence, the probability that, under $\hat{\mathbf{b}}$, two or more bidders simultaneously submit the highest bid above l, is zero. But then for every i and a.e. s_i, $V_i(\cdot|s_i)$ is continuous at $(\hat{\mathbf{b}}_i(s_i), \hat{\mathbf{b}}_{-i})$, being continuous there whenever $\hat{\mathbf{b}}_i(s_i) = l$ because l is isolated. Therefore, $\lim_n V_i(\mathbf{b}_i^n(s_i), \mathbf{b}_{-i}^n|s_i) = V_i(\hat{\mathbf{b}}_i(s_i), \hat{\mathbf{b}}_{-i}|\mathbf{s}_i)$ a.e. s_i, and so (7.6) implies that $\hat{\mathbf{b}}$ is an equilibrium. ■

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