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COMPLETE CHARACTERIZATION OF ACCEPTABLE GAME FORMS BY EFFECTIVITY FUNCTIONS

by

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Complete Characterization of Acceptable Game Forms by Effectivity Functions

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Abstract

Acceptable game forms were introduced in Hurwicz and Schmeidler (1978). Dutta (1984) considered effectivity functions of acceptable game forms. This paper unifies and extends the foregoing two papers. We obtain the following characterization of the effectivity functions of acceptable game forms: An effectivity function belongs to some acceptable game form iff (i) it belongs to some Nash consistent game forms; and (ii) it satisfies an extra simple condition (our (3.1) or (4.2)). (Nash consistent game forms have already been characterized by their effectivity functions in Peleg et al. (2001).) As a corollary of our characterization we show that every acceptable game form violates minimal liberalism.

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1 Introduction

In this paper we completely characterize acceptable game forms in terms of their effectivity functions. A game form is acceptable if for every profile of preferences of the players the set of Nash equilibria is non-empty, and every Nash equilibrium outcome is (weakly) Pareto optimal. Acceptable game forms were introduced in Hurwicz and Schmeidler (1978), who obtained, in particular, the following results: For two players only dictatorial game forms are acceptable; for three or more players there exist certain families of acceptable game forms. (The reader should notice that our notion of acceptability is identical to weak acceptability of Hurwicz and Schmeidler (1978). We reserve the term weak acceptability to a weaker notion of stability (see Section 8).)

An effectivity function is (roughly) the coalitional function of a game form. Effectivity functions were introduced in Moulin and Peleg (1982). Dutta (1984) considers acceptable game forms with maximal effectivity function (see Section 2 for the precise definition). He shows that a maximal effectivity function E is an effectivity function of some acceptable game form if, and only if, there is a strong (and proper) simple game G such that E is the effectivity function of G (see, again, Section 2 for the relevant definitions). Our analysis extends Dutta's work in two respects. First, we drop the assumption of maximality and obtain a (still simple) characterization in the general case. Second, in our model the set of alternatives is a (compact Hausdorff) topological space and preferences are continuous. Thus, for finite sets of alternatives we obtain the foregoing result as a corollary. (The reader should notice that our proof is entirely different from that of Dutta, and therefore, it does not yield dominance solvable game forms when the effectivity function is maximal.)

We shall now review our paper. In Section 2 we collected all the necessary definitions. Section 3 contains the proof of a necessary condition on the effectivity function of an acceptable game form (see (3.1) of Lemma 3.1). ((3.1) says that two disjoint coalitions cannot both be strong.) We also reprove Theorem 1 of Hurwicz and Schmeidler (1978) and Theorem 3.5 of Dutta (1984) in a topological framework. Our main result is formulated in Section 4. Theorem 4.1 details our necessary and sufficient conditions. Roughly, it says that an effectivity function belongs to some acceptable game form if, and only if, it belongs to some weakly acceptable game form and satisfies, in addition, our necessary condition. (Here a game form is weakly acceptable if for every profile of continuous preferences it has a Pareto optimal Nash equilibrium).

Fortunately, Peleg et al. (2001) contains sufficient (and almost necessary) conditions for an effectivity function to belong to some weakly acceptable game form. This leads to Theorem 4.2. If the set of alternatives is finite, then Theorem 4.2 yields a complete characterization. Sections 5 and 6 provide the proof of the main result.

Section 7 describes an important application of our result. Let G be a proper simple game (i.e., a committee), and let A be compact Hausdorff space of alternatives. If G contains at most one vetoer, then there exists an acceptable game form (interpreted as a (generalized) voting procedure), that allows G to choose an alternative from A (that is, the effectivity function of the game form is the same as that of G). This result continues and complements Keiding and Peleg (2001a) ((In Keiding and Peleg (2001a) it is proved that every committee *with* vetoers has stable (generalized) voting procedures in economic situations. Clearly, our topological framework includes the Keiding-Peleg framework as a special case. Moreover, our stability requirements are necessarily different.)

The Liberal Paradox (Sen (1970)) appears in Section 8: Every acceptable game form violates minimal liberalism. Thus, acceptable game forms cannot represent constitutions that allow for individual rights. (For the investigation of representations of constitutions see Peleg (1998), Keiding and Peleg (2001b), and Peleg et al. (2001).)

2 Definitions and Notations

Throughout this paper, A denotes the set of alternatives. The set A may be finite or infinite; however, if A is finite, then $|A| \geq 2$. (If D is a finite set, then $|D|$ is the number of members of D.) Further, we assume that A is a compact Hausdorff (topological) space. The topology on A is denoted by τ. A preference ordering on A is a complete and transitive binary relation. A preference ordering R is continuous if the sets $\{b \in A|aRb\}$ and $\{b \in A|bRa\}$ are closed (in (A, τ)), for every $a \in A$. We denote by V the set of all continuous preference orderings on A. If $R \in V$ and $a \in A$, then $L(a, R) = \{b \in A|aRb\}$. For a set S, $V^S = \{f|f : S \to V\}$ is the set of mappings from S to V. If D is a set, then $P(D) = \{D^*|D^* \subseteq D\}$, and $P_0(D) = P(D) \setminus \{\emptyset\}$.

Finally,

$$\kappa(A) = \{B \in P_0(A) | B \text{ is closed}\}.$$

Let $N = \{1, \ldots, n\}$ be the set of players and let (A, τ) be the (topological) space of alternatives. An *effectivity function* (EF) is a function $E : P(N) \to P(\kappa(A))$ that satisfies the following conditions:

(i) $E(N) = \kappa(A)$; (ii) $E(\emptyset) = \emptyset$; and (iii) $A \in E(S)$ for every $S \in P_0(N)$. As a general interpretation, $B \in E(S)$ means that the coalition S can force the final alternative to be an element of B. The interpretation of the three conditions is fairly obvious.

An EF E is *superadditive* if it satisfies the following condition: If $S_i \in P_0(N)$ and $B_i \in E(S_i)$ for $i = 1, 2$, and $S_1 \cap S_2 = \emptyset$, then

$$B_1 \cap B_2 \in E(S_1 \cup S_2)$$

The EF E is *monotonic* if

$$[B \in E(S),\ B^* \in \kappa(A),\ B \subseteq B^* \text{ and } \quad S \subseteq S^*] \Rightarrow B^* \in E(S^*)$$

Monotonicity and superadditivity of EF's are natural properties in view of the foregoing interpretation. Moreover, EF's derived form game forms (see below) have these properties.

The *polar* of an EF E is the EF E^* defined by: $E^*(\emptyset) = \emptyset$, and for $S \in P_0(N)$,

$$E^*(S) = \{B \in \kappa(A) | B \cap B' \neq \emptyset \text{ for all } B' \in E(N \setminus S)\}$$

Thus, if $B \in E^*(S)$, then $N \setminus S$ cannot guarantee that the final alternative is not in B. An EF E is *maximal* if E is superadditive and $E = E^*$.

We need also a topology on $\kappa(A)$. We shall use the *upper topology* τ_u. A basis for τ_u is given by

$$\{B \in \kappa(A) | B \subseteq U\}, \quad U \in \tau.$$

If E is an EF, then $E^*(S)$ is closed in $(\kappa(A), \tau_u)$ for every $S \in P(N)$.

We recall the following definitions. A social choice correspondence is a function $H : V^N \to P_0(A)$. A social choice correspondence H is *Maskin monotonic* if for all $Q^N, R^N \in V^N$ and $a \in A$,

$$[a \in H(R^N) \text{ and } L(a, R^i) \subseteq L(a, Q^i) \text{ for all } i \in N] \Rightarrow a \in H(Q^N).$$

We also use some basic properties of simple games. A simple game is a pair (N, W), where $N = \{1, \ldots, n\}$ is a society and $W \subseteq P_0(N)$ is the set of winning coalitions. We always assume *monotonicity*

$$[S \in W \text{ and } S \subseteq T \subseteq N] \Rightarrow T \in W.$$

A simple game $G = (N, W)$ is *proper* if

$$S \in W \Rightarrow N \setminus S \notin W \quad \text{for all} \quad S \in P_0(N).$$

G is *strong* if G is proper and

$$S \notin W \Rightarrow N \setminus S \in W \quad \text{for all} \quad S \in P_0(N).$$

G is *weak* if

$$VET = \cap\{S | S \in W\} \neq \emptyset.$$

VET is the set of *vetoers* of G. G is *dictatorial* if there exists $j \in N$ such that $S \in W$ iff $j \in S$.

Let $G = (N, W)$ be a simple game. We associate with G an $EF, E(G)$, by

$$E(G)(S) = \begin{cases} \kappa(A), & S \in W, \\ \{A\}, & S \notin W, \quad S \neq \emptyset, \\ \emptyset, & S = \emptyset. \end{cases}$$

An EF E is *dictatorial* if there exists a dictatorial simple game G such that $E = E(G)$.

We now turn to define some basic properties of game forms. A *game form* (GF) is an $(n+2)$-tuple $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ where (i) Σ^i is the (non-empty) set of strategies of player $i \in N$; and (ii) $\pi : \Sigma^1 \times \cdots \times \Sigma^n \to A$ is the outcome function. For $S \in P_0(N)$ we denote $\Sigma^S = \times_{i \in S} \Sigma^i$. Also, we denote $\Sigma = \Sigma^N$. Let $R^N \in V^N$. The pair (Γ, R^N) defines, in an obvious way, a game in strategic form. A strategy combination $\sigma \in \Sigma$ is a Nash equilibrium (NE) if

$$[\tau^i \in \Sigma^i, i \in N] \Rightarrow \pi(\sigma) R^i \pi(\sigma^{N \setminus \{i\}}, \tau^i)$$

(Here $\sigma^{(N \setminus \{i\}}$ is the restriction of σ to $N \setminus \{i\}$.) The set of all NE's of (Γ, R^N) is denoted by $NE(\Gamma, R^N)$. A GF Γ is *Nash consistent* if $NE(\Gamma, R^N) \neq \emptyset$ for all $R^N \in V^N$. Γ is *acceptable* if (i) Γ is Nash consistent; and (ii) if $\sigma \in$

$NE(\Gamma, R^N)$, then $\pi(\sigma)$ is (weakly) Pareto optimal with respect to R^N ($x \in A$ is (weakly) Pareto optimal with respect to R^N if for every $y \in A$ there exists $i \in N$ such that xR^iy).

Let $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be a GF and assume that π is surjective. The EF E^Γ, associated with Γ, is defined in the following way. For $S \in P_0(N)$ and $B \in \kappa(A)$, S is effective for B if there exists $\sigma^S \in \Sigma^S$ such that $\pi(\sigma^S, \tau^{N\setminus S}) \in B$ for all $\tau^{N\setminus S} \in \Sigma^{N\setminus S}$. Now E^Γ is defined by $E^\Gamma(\emptyset) = \emptyset$ and $E^\Gamma(S) = \{B \in \kappa(A) | S \text{ is effective for } B\}$, for $S \in P_0(N)$. Clearly, E^Γ is superadditive and monotonic. Let $E : P(N) \to P(\kappa(A))$ be an EF. A GF Γ is a *representation* of E if $E(S) = E^\Gamma(S)$ for every $S \in P_0(N)$. Basically, this means that the GF distributes the same power among the players as the EF does.

Notational convention: Instead of $E(\{i\})$ and $E^*(\{i\})$ we usually write $E(i)$ and $E^*(i)$.

3 Some properties of acceptable game forms

Let (A, τ) be a compact Hausdorff space. If Γ is an acceptable GF on V^N, then the EF of Γ, E^Γ, satisfies the following restriction.

Lemma 3.1. *Let $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be an acceptable GF on V^N. Then*
(3.1)

$$[S, T \in P_0(N), B \in E^\Gamma(S), C \in E^\Gamma(T), \text{ and } S \cap T = \emptyset] \Rightarrow B \cup C = A$$

Proof. Assume, on the contrary, that there exist $S, T \in P_0(N), B \in E^\Gamma(S)$, and $C \in E^\Gamma(T)$ such that $S \cap T = \emptyset$ and $B \cup C \neq A$. Let $x \in A \setminus (B \cup C)$. (A, τ) is normal, because it is compact and Hausdorff. Hence there exists a continuous function $u : A \to [0, 1]$ such that $u(x) = 1$ and $u(y) = 0$ for all $y \in B \cup C$. Define now a profile $R^N \in V^N$ by yR^iz iff $u(y) \geq u(z)$, for all $y, z \in A$ and $i \in N$. We shall construct a Pareto-dominated NE with respect to R^N.

Let $\sigma^S \in \Sigma^S$ satisfy $\pi(\sigma^S, \tau^{N\setminus S}) \in B$ for all $\tau^{N\setminus S} \in \Sigma^{N\setminus S}$, and let $\sigma^T \in \Sigma^T$ satisfy $\pi(\sigma^T, \tau^{N\setminus T}) \in C$ for all $\tau^{N\setminus T} \in \Sigma^{N\setminus T}$. Further let $\sigma^{N\setminus(S\cup T)}$ be a member of $\Sigma^{N\setminus(S\cup T)}$. Then $\sigma = (\sigma^S, \sigma^T, \sigma^{N\setminus(S\cup T)})$ is an NE of (Γ, R^N) whose outcome is Pareto-dominated by x. Thus the desired contradiction has been obtained. □

Lemma 3.1 has the following interpretation. If S and T are disjoint coalitions, then $E^{\Gamma}(S)$ and $E^{\Gamma}(T)$ cannot both be large. For example, if $\{x\} \in E^{\Gamma}(S)$ for some $x \in A$, then T is powerless, that is, $E^{\Gamma}(T) = \{A\}$.

We shall now characterize all maximal EF's that satisfy (3.1).

Theorem 3.1. *Let $E\colon P(N) \to P(\kappa(A))$ be a monotonic and maximal EF. E satisfies*

(3.2) $[S_i \in P_0(N),\ B_i \in E(S_i),\ i = 1, 2, \text{ and } S_1 \cap S_2 = \emptyset] \Rightarrow B_1 \cup B_2 = A,$

iff there exists a strong simple game G such that $E = E(G)$.

Proof. Necessity. Let $E : P(N) \to P(\kappa(A))$ be a monotonic and maximal EF that satisfies (3.2). Further let $S \in P_0(N), S \neq N$, and let $x \in A$. We claim that

$$\{x\} \in E(S) \cup E(N \setminus S). \tag{3.3}$$

Assume, on the contrary, that $\{x\} \notin E(S) \cup E(N \setminus S)$. Call $D \in \kappa(A)$ *thick* if there exists an open set U such that $x \in U$ and $cl(U) = D$. Let

$$\delta = \{D \in \kappa(A) | D \text{ is thick}\}.$$

Then $(\delta, \supseteq)$ is a net. Indeed, if $D_i = cl(U_i)$ is in $\delta,\ i = 1, 2$, then

$$D_1 \cap D_2 \supseteq cl(U_1 \cap U_2) \text{ and } cl(U_1 \cap U_2) \in \delta.$$

The net $(\delta, \supseteq)$ converges in $(\kappa(A), \tau_u)$ to $\{x\}$. Indeed, let $U \in \tau$ and $x \in U$. Then there exists $Q \in \tau$ such that $x \in Q$ and $D = cl(Q) \subseteq U$. Hence, $D \in \delta$ and if $D^* \in \delta$ and $D^* \subseteq D$, then $D^* \subseteq U$.

$E(T)$ is closed in $(\kappa(A), \tau_u)$ for every $T \in P_0(N)$, because E is maximal. Hence, there exists $D \in \delta$ such that $D \notin E(S) \cup E(N \setminus S)$. There exists $U \in \tau$ such that $x \in U$ and $D = cl(U)$. Therefore, by the monotonocity of E,

$$B = A \setminus U \in E^*(S) \cap E^*(N \setminus S) = E(S) \cap E(N \setminus S). \tag{3.4}$$

As $B \neq A$,(3.4) contradicts (3.2). Thus, (3.3) is proved.

By (3.2) and (3.3) for every $S \in P_0(N), S \neq N$, either $E(S) = \kappa(A)$ or $E(N \setminus S) = \kappa(A)$. As E is monotonic, E is, indeed, the EF of a strong simple game.

The sufficiency part is obvious. □

The following corollary extends Theorem 3.5 of Dutta (1984) to topological EF's.

Corollary 3.2. *Let Γ be an acceptable GF on V^N. If E^Γ is maximal, then E^Γ is the EF of a strong simple game.*

We remark that, unlike Dutta, we are restricted to continuous preferences. Therefore, our result applies only to topological EF's.

The final result in this section extends Theorem 1 of Hurwicz and Schmeidler (1978) to topological EF's.

Theorem 3.3. *Let $\Gamma = (\Sigma^1, \Sigma^2; \pi; A)$ be an acceptable GF on V^N. If $E^\Gamma(i)$ is closed in $(\kappa(A), \tau_u)$ for $i = 1, 2$, then E^Γ is dictatorial.*

Proof. Every two-person strong simple game is dictatorial. Hence it is sufficient to prove that E^Γ is maximal. Thus, we have to prove that $(E^\Gamma(i))^* = E^\Gamma(i)$ for $i = 1, 2$. Clearly, $(E^\Gamma(1))^* \supseteq E^\Gamma(1)$. Also by Theorem 4.1 of Peleg et al. (2001), $(E^\Gamma(1))^* \subseteq (E^\Gamma(1))^{**}$. Finally, by Proposition 5.2 in Abdou and Keiding (1991, p. 46), $(E^\Gamma(1))^{**} = cl(E^\Gamma(1)) = E^\Gamma(1)$. Thus, $E^\Gamma(1) = (E^\Gamma(1))^*$. Similarly, $E^\Gamma(2) = (E^\Gamma(2))^*$. □

4 Formulation of the main result

In this section we shall formulate the main result of this paper and some of its corollaries. Let (A, τ) be a compact Hausdorff space and let $N = \{1, 2, \ldots, n\}$, $n \geq 3$, be a society.

Theorem 4.1. *Let $E : P(N) \to P(\kappa(A))$ be an EF. E has an acceptable representation on V^N iff the following conditions are satisfied:*

$$E \text{ is monotonic and superadditive.} \tag{4.1}$$

$$[S_i \in P_0(N),\ B_i \in E(S_i),\ i = 1, 2, \text{ and } S_1 \cap S_2 = \emptyset] \Rightarrow B_1 \cup B_2 = A. \tag{4.2}$$

$$\begin{array}{l} \text{For every } R^N \in V^N \text{ there exists a Pareto optimal alternative} \\ x \in A \text{ such that } L(x, R^i) \in E(N \setminus \{i\}), \text{ for all } i \in N. \end{array} \tag{4.3}$$

As the reader may easily verify, (4.1) and (4.3) are necessary conditions for the existence of an acceptable representation. By Lemma 3.1, (4.2) is also necessary. The sufficiency part of Theorem 4.1 will be proved in Sections 5 and 6.

Peleg et al. (2001) contains sufficient conditions for the validity of (4.3) (see Theorem A.1, Corollary 4.6, and Theorem 4.7 therein). These conditions lead to the following result.

Theorem 4.2. *Let $E : P(N) \to P(\kappa(A))$ be an EF. E has an acceptable representation on V^N if (4.1), (4.2), and the following two additional assumptions are satisfied.*

$$[B^i \in E^*(i) \text{ for all } i \in N] \Rightarrow \bigcap_{i \in N} B^i \neq \emptyset. \tag{4.4}$$

$$E(N \setminus \{i\}) \text{ is closed in } (\kappa(A), \tau_u) \text{ for every } i \in N. \tag{4.5}$$

By Peleg et al. (2001) (4.4) is a necessary condition for Nash consistency. Also, (4.5) is indispensable.

When A is finite (4.5) is satisfied. Hence we obtain a complete characterization of the family of EF's which have an acceptable representation, when $2 \leq |A| < \infty$.

Corollary 4.3. *Let $2 \leq |A| < \infty$ and let $E : P(N) \to P(\kappa(A))$ be an EF. Then E has an acceptable representation iff (4.1), (4.2), and (4.4) are satisfied.*

5 A representation with only Pareto optimal NE's

Let (A, τ) be a compact Hausdorff space and let $N = \{1, 2, \ldots, n\}$ be a society. Further, let the EF $E : P(N) \to P(\kappa(A))$ satisfy (4.1) and (4.2). We shall construct a GF Γ with the following two properties: (i) Γ is a representation of E; (ii) for every $R^N \in V^N$ each NE of (Γ, R^N) is Pareto optimal. (Notice that the game (Γ, R^N) may have no NE.)

We now describe the construction. For $i \in N$ let $N^i = \{S \subseteq N | i \in S\}$ and let

$$M^i = \{m : N^i \to N^i \times \kappa(A) | m_1(S) \subseteq S \text{ for all } S \in N^i, \text{ and } m_2(S) \in E(m_1(S))\},$$

where $m(S)_{=}(m_1(S), m_2(S)), S \in N^i$. A selection from $\kappa(A)$ is a function $\varphi : \kappa(A) \to A$ that satisfies $\varphi(B) \in B$ for every $B \in \kappa(A)$. Denote by Φ the set of all selections from $\kappa(A)$. We define a GF $\Gamma = (\Sigma^1, \dots \Sigma^n; \pi; A)$ as follows. The set of strategies of $i \in N$ is $\Sigma^i = M^i \times \Phi \times N \times \{0, 1\}$. Let $\sigma = (\sigma^1, \dots, \sigma^n) \in \Sigma^1 \times \dots \times \Sigma^n$, where $\sigma^i = (m^i, \varphi^i, t^i, q^i)$ for $i \in N$. In order to define $\pi(\sigma)$ we introduce the following sequence of partitions of N. First, for $S \in P_0(N)$, we define an equivalence relation $\sim_\sigma$ on S by

$$i \sim_\sigma j \Leftrightarrow m^i(S) = m^j(S), \text{ all } \quad i, j \in S.$$

Denote by $D(S, \sigma)$ the partition of S with respect to $\sim_\sigma$. Now let the first partition of N be $H_0(\sigma) = \{N\}$, and define inductively the following partitions. If $H_k(\sigma) = \{S_{k,1}, \dots, S_{k,\ell}\}$ is the k-th partition, where $k \geq 0$, then we define

$$H_{k+1}(\sigma) = \bigcup_{j=1}^{\ell} D(S_{k,j}, \sigma).$$

Clearly, there exists a minimal r such that $H_r(\sigma) = H_k(\sigma)$ for all $k \geq r$. Let $H_r(\sigma) = \{S_1, \dots, S_\ell\}$. The coalitions $S_1, \dots, S_\ell$ are called *final* . For each final coalition $S_j, j = 1, \dots, \ell$, there exists $B_j \in E(S_j)$ such that $m^i(S_j) = (S_j, B_j)$, for all $i \in S_j$. Further, a final coalition S is called *decided* if $q^i = 0$ for all $i \in S$. In the definition of $\pi(\sigma)$ we distinguish the following possibilities.

(5.1) $\qquad S_j$ is decided for $\quad j = 1, \dots, \ell$.

Let $k \equiv \sum_{h=1}^{n} t^h(\mod n)$ and define $\pi(\sigma) = \varphi^k(\cap_{j=1}^{\ell} B_j)$. (We notice that $\cap_{j=1}^{\ell} B_j \neq \emptyset$, because E is superadditive).

(5.2) $S_1, \dots, S_h, 1 \leq h \leq \ell$, are undecided, and $S_{h+1}, \dots, S_\ell$ are decided.

In order to simplify notations assume that $\cup_{j=1}^{h} S_j = \{1, \dots, s\}$, where $1 \leq s \leq n$. Let $k \equiv \sum_{j=1}^{s} t^j(\mod s), 1 \leq k \leq s$. Then $\pi(\sigma) = \varphi^k(\cap_{j=h+1}^{\ell} B_j)$. (Thus, if $h = \ell$, then $\pi(\sigma) = \varphi^k(A)$.) This completes the definition of π.

We shall now verify the aforementioned properties of Γ.

Claim 5.1. Γ *is a representation of* E.

Proof. Let $S \in P_0(N)$ and $B \in E(S)$. Let $\bar{m}^i(T) = (S, B)$ for all $T \supseteq S$ and $i \in S$. If $\sigma^i = (\bar{m}^i, \varphi^i, t^i, 0)$ for all $i \in S$, then for every $\tau^{N\setminus S} \in \Sigma^{N\setminus S}$ S is a decided coalition with respect to $(\sigma^S, \tau^{N\setminus S})$. Hence, by (5.1) and (5.2), $\pi(\sigma^S, r^{N\setminus S}) \in B$ for all $\tau^{N\setminus S} \in \Sigma^{N\setminus S}$. Thus $B \in E^\Gamma(S)$ and we have proved that $E^\Gamma(S) \supseteq E(S)$ for all $S \in P_0(N)$.

Now let $S \in P_0(N)$ and $C \in \kappa(A) \setminus E(S)$. Then $S \neq N$ because $E(N) = \kappa(A)$. Also, $B \setminus C \neq \emptyset$ for every $B \in E(S)$, because E is monotonic. Let $\sigma^S \in \Sigma^S$ be fixed. We shall choose strategies $\bar{\sigma}^i = (\bar{m}^i, \bar{\varphi}^i, \bar{t}^i, \bar{q}^i)$, $i \in N \setminus S$, such that $\pi(\sigma^S, \bar{\sigma}^{N\setminus S}) \notin C$. Indeed, let $\bar{m}^i(T) = (N \setminus S, A)$ for all $T \supseteq N \setminus S$ and $i \in N\setminus S$. Further, let $\bar{q}^i = 1$ for all $i \in N\setminus S$. Then $N\setminus S$ is an undecided final coalition with respect to $\sigma^* = (\sigma^S, \bar{\sigma}^{N\setminus S})$. If $S_1, \ldots, S_h$, $h \geq 0$, are the decided coalitions in the final partition $H_r(\sigma^*)$, and $m_2^i(S_j) = B_j, i \in S_j, j = 1, \ldots, h$, where $\sigma^i = (m^i, \varphi^i, t^i, q^i)$, $i \in S$, then $B = \cap_{j=1}^h B_j \in E(S)$. Let $k \notin S$. $N \setminus S$ can choose $\bar{t}^i$ and $\bar{\varphi}^i$, $i \notin S$, such that $\pi(\sigma^*) = \bar{\varphi}^k(B) \notin C$. Thus, $C \notin E^\Gamma(S)$. $\square$

We now prove that all NE's of Γ are Pareto-optimal.

Claim 5.2. *Let* $R^N \in V^N$. *Then every* NE *of* (Γ, R^N) *is Pareto optimal.*

Proof. Let $\sigma = (\sigma^1, \ldots, \sigma^n)$ be an NE of (Γ, R^N) and let $\{S_1, \ldots, S_\ell\}$ the partition of N into final coalitions which is associated with σ. We distinguish the following possibilities.

(i) No final coalition is decided.

As σ is an NE, $\pi(\sigma) R^i x$ for all $x \in A$ and $i \in N$. Thus $\pi(\sigma)$ is Pareto optimal.

(ii) Exactly one final coalition is decided.

Let S_j be the decided coalition. Then $\pi(\sigma) R^i x$ for all $x \in A$ and $i \in S_j$. Hence $\pi(\sigma)$ is Pareto optimal.

(iii) There exist at least two decided coalitions.

In order to simplify notations assume that $S_1, \ldots, S_h$, $h \geq 2$, are decided. Let $\sigma^i = (m^i, \varphi^i, t^i, q^i)$, $i \in N$, and let $B_j = m_2^i(S_j)$, for $i \in S_j$ and $j = 1, \ldots, h$. Denote $C_j = \cap\{B_k | k = 1, \ldots, j-1, j+1, \ldots, h\}$ for $j = 1, \ldots, h$. By (5.2) $\pi(\sigma) R^i x$ for all $i \in S_j, x \in C_j$, and $j = 1, \ldots, h$. By (4.2) $\cup_{j=1}^h C_j = A$. Hence $\pi(\sigma)$ is Pareto optimal. $\square$

We point out again that Γ may not be Nash consistent.

6 Proof of Theorem 4.1

Let (A, τ) be a compact Hausdorff space and let $N = \{1, \ldots, n\}, n \geq 3$ be a society. Further let $E : P(N) \to P(\kappa(A))$ be an EF that satisfies (4.1)-(4.3). We shall construct an acceptable representation of E.

Let $H : V^N \to P_0(A)$ be defined by

$$H(R^N) = \{a \in A | L(a, R^i) \in E(N\backslash\{i\}) \text{ for all } i \in N \text{ and } a \text{ is Pareto optimal}\}.$$

Clearly, H is well-defined by (4.3), and is Maskin monotonic. We denote

$$\text{graph } (H) = \{(R^N, a) | R^N \in V^N \text{ and } a \in H(R^N)\},$$

and proceed to define a GF Γ_0 with the desirable properties. Let $\Gamma_0 = (\Sigma_0^1, \ldots, \Sigma_0^n; \pi_0; A)$ where

$$\Sigma_0^i = \text{graph } (H) \times \{0,1\} \times \{0,1\} \times N \times E(i) \times \Phi \times \Sigma^i, \quad i \in N.$$

(Here Φ and Σ^i have already been defined in Section 5.) It remains to define π_0. Let $\eta^i = (R_i^N, a^i, q_1^i, q_2^i, t_0^i, B_0^i, \varphi_0^i, \sigma^i)$, $i \in N$, be an n-tuple of strategies. We distinguish the following possibilities.

(6.1) $$\eta^i = (R^N, a, 0, 0, t_0^i, B_0^i, \varphi_0^i, \sigma^i) \text{ for all } i \in N.$$

In this case $\pi_0(\eta^1, \ldots, \eta^n) = a$.

(6.2) There exists $j \in N$ such that $\eta^i = (R^N, a, 0, 0, t_0^i, B_0^i, \varphi_0^i, \sigma^i)$ for all $i \in N \setminus \{j\}$, $(R_j^N, a^j, q_1^j, q_2^j) \neq (R^N, a, 0, 0)$, and $q_1^j = 0$.

Define $\pi_0(\eta^1, \ldots, \eta^n) = \varphi_0^j(L(a, R^j))$, where R^j is the j-th component of R^N. (π_0 is well defined, because $n \geq 3$.)

(6.3) There exists $j \in N$ such that $\eta^i = (R^N, a, 0, 0, t_0^i, B_0^i, \varphi_0^i, \sigma^i)$ for all $i \in N \setminus \{j\}$ and $q_1^j = 1$.

Define $\pi_0(\eta^1,\ldots,\eta^n) = \varphi_0^j(L(a,R^j)\cap B_0^j)$. (Notice that $L(a,R^j) \in E(N\setminus \{j\})$. Hence $L(a,R^j)\cap B_0^j \neq \emptyset$, because E is superadditive.)

(6.4)

There exists $j,h \in N, j\neq h,$ such that $\eta^i = (R^N,a,0,0,t_0^i,B_0^i,\varphi_0^i,\sigma^i)$ for $i \in N\setminus\{h,j\}$, $\eta^h = (R^N,a,0,1,t_0^h,B_0^h,\varphi_0^h,\sigma^h)$, and $q_1^j = 1$.

Define $\pi_0(\eta^1,\ldots,\eta^n) = \varphi_0^h(B_0^j)$.

(6.5)

There exist $j,h\in N,\ j\neq h,$ such that $\eta^i = (R^N,a,0,0,t_0^i,B_0^i,\varphi_0^i,\sigma^i)$, for all $i\in N\setminus\{h,j\}, \eta^h = (R^N,a,0,1,t_0^h,B_0^h,\varphi_0^h,\sigma^h), q_1^j = 0,$ and $(R_j^N,a^j,q_1^j,q_2^j)\neq(R^N,a,0,0)$.

Let $\sum_{i=1}^{n} t_0^i(\mod n)\equiv k,\ \ 1\leq k\leq n$. Define $\pi_0(\eta^1,\ldots,\eta^n) = \varphi_0^k(A)$. ($\pi_0$ is well defined, because $n\geq 3$).

(6.6) In all other cases let $\pi_0(\eta^1,\ldots,\eta^n) = \pi(\sigma^1,\ldots,\sigma^n)$, where π has already been defined in Section 5

We claim that Γ_0 is an acceptable representation of E. The proof consists of several steps.

Step 1. Γ_0 is Nash consistent.

Let $R^N\in V^N$. Choose $a\in H(R^N)$ and define $\eta^i = (R^N,a,0,0,t_0^i,B_0^i,\varphi_0^i,\sigma^i), i\in N$. By (6.1) – (6.3) $\eta = (\eta^1,\ldots,\eta^n)$ is an NE of (Γ_0,R^N).

Step 2. Every NE of Γ_0 is Pareto optimal.

Let $R^N\in V^N$ and let $\eta = (\eta^1,\ldots,\eta^n)$ be an NE of (Γ_0,R^N). We distinguish the following possibilities.

(6.1*) $\eta^i = (\hat{R}^N,a,0,0,t_0^i,B_0^i,\varphi_0^i,\sigma^i)$ for all $i\in N$.

Then $\pi_0(\eta) = a$ and $a\in H(\hat{R}^N)$. By (6.2), $L(a,\hat{R}^i)\subseteq L(a,R^i)$ for all $i\in N$. Thus $a\in H(R^N)$, because H is Maskin monotonic. Hence a is Pareto optimal.

(6.2*) There exists $j\in N$ such that $\eta^i = (\hat{R}^N,a,0,0,t_0^i,B_0^i,\varphi_0^i,\sigma^i)$ for all $i\neq j$, $(R_j^N,a^j,q_1^j,q_2^j)\neq(\hat{R}^N,a,0,0)$, and $q_1^j = 0$.

Let $b = \pi_0(\eta)$. By (6.5) $A \subseteq L(b, R^i)$ for all $i \neq j$. Thus, b is Pareto optimal.

$$\text{(6.3*)} \qquad \begin{array}{l} \text{There exists } \ j \in N \ \text{ such that } \eta^i = (\hat{R}^N, a, \ 0, 0, t_0^i, B_0^i, \varphi_0^i, \sigma^i) \\ \text{for all } i \neq j, \ \ \textit{and} \ \ q_1^j = 1. \end{array}$$

Let $b = \pi_0(\eta)$. By (6.2) $L(a, \hat{R}^j) \subseteq L(b, R^j)$. Further, by (6.4), $B_0^j \subseteq L(b, R^i)$ for all $i \neq j$. Also, $L(a, \hat{R}^j) \in E(N \setminus \{j\})$ and $B_0^j \in E(j)$. Hence, by (4.2), $B_0^j \cup L(a, \hat{R}^j) = A$. Therefore, b is Pareto optimal.

$$\text{(6.4*)} \qquad \begin{array}{l} \text{There exist } \ j, h \in N, \ j \neq h, \text{ such that} \\ \eta^i = (\hat{R}^N, a, 0, 0, t_0^i, B_0^i, \varphi_0^i, \sigma^i) \text{for } i \in N \setminus \{j, h\}, \\ \eta^h = (\hat{R}^N, a, 0, 1, t_0^h, B_0^h, \varphi_0^h, \sigma^h), \ \text{ and } \ q_1^j = 1. \end{array}$$

Let $b = \pi_0(\eta)$. By (6.5) $A \subseteq L(b, R^j)$. Hence b is Pareto optimal.

$$\text{(6.5*)} \qquad \begin{array}{l} \text{There exist } \ h, j \in N, \ h \neq j, \ \text{ such that} \\ \eta^i = (\hat{R}^N, a, 0, 0, t_0^i, B_0^i, \varphi_0^i, \sigma^i) \\ \text{for all } i \in N \setminus \{h, j\}, \eta^h = (\hat{R}^N, a, 0, 1, t_0^h, B_0^h, \varphi_0^h, \sigma^h), \\ q_1^j = 0, \ \text{ and } (R_j^N, a^j, q_1^j, q_2^j) \neq (\hat{R}^N, a, 0, 0). \end{array}$$

Let $b = \pi_0(\eta)$. By (6.5) $A \subseteq L(b, R^i)$ for all $i \in N$. Hence b is Pareto optimal.

$$\text{(6.6*)} \qquad \begin{array}{l} \text{In all other cases } \ \pi_0(\eta^1, \ldots, \eta^n) = \pi(\sigma^1, \ldots, \sigma^n), \\ \text{where } \ \sigma = (\sigma^1, \ldots, \sigma^n) \ \text{ is an NE of } (\Gamma, R^N). \end{array}$$

By Section 5, $\pi(\sigma)$ is Pareto optimal.

Step 3. Γ_0 is a representation of E.

First we observe that if $S \subseteq N, |S| \geq 2$, then $E^{\Gamma_0}(S) = E^{\Gamma}(S) = E(S)$. Indeed, if $|S| \geq 2$, then $E^{\Gamma_0}(S) \supseteq E^{\Gamma}(S)$, because S can enforce the play of strategies of Γ. On the other hand $E^{\Gamma_0}(S) \subseteq E^{\Gamma}(S)$ as Γ_0 gives no extra power to S (over its power in Γ).

We shall now prove that $E^{\Gamma_0}(i) = E(i)$ for all $i \in N$. Let $i \in N$ and let $B \in E(i)$. Consider the following strategy of $i : \eta^i = (R_i^N, a^i, 1, q_2^i, t_0^i, B, \varphi_0^i, \sigma^i)$ where $\sigma^i = (m^i, \varphi^i, t^i, 0)$ and $m^i(S) = \{\{i\}, B\}$ for all $S \in N^i$. As the reader may easily verify, $\pi_0(\eta^i, \eta^{N\setminus\{i\}}) \in B$ for all $\eta^{N\setminus\{i\}} \in \Sigma_0^{N\setminus\{i\}}$. Thus, $B \in E^{\Gamma_0}(i)$ and $E^{\Gamma_0}(i) \supseteq E(i)$. As $|N\setminus\{i\}| \geq 2$, $N\setminus\{i\}$ can enforce the play of Γ. Hence i's power in Γ_0 does not exceed its power Γ, that is, $E^{\Gamma_0}(i) \subseteq E^{\Gamma}(i) = E(i)$. □

7 Acceptable GF's and simple games

Let $N = \{1, \ldots, n\}, n \geq 3$, be a society, and let (A, τ) be a compact Hausdorff space (of social states). As a direct corollary of Theorem 4.2 we obtain the following result.

Theorem 7.1. *Let G be a proper simple game with at most one vetoer. Then the EF $E(G)$ has an acceptable representation on V^N.*

Proof. As G is proper and monotonic, $E(G) = E$ satisfies (4.1) and (4.2). Also, $E^*(i) = \{A\}$ for every $i \in N$ who is not a vetoer. Therefore, (4.4) is satisfied. Finally, $\{A\}$ and $\kappa(A)$ are closed. □

For the sake of completeness we remark that the family of EF's that have acceptable representation is larger than the family described in Theorem 7.1.

We also remark that Theorem 7.1 provides stable voting procedures to committees without vetoers in a topological framework. Hence it complements the results of Keiding and Peleg (2001a) on existence of (strongly) stable representations of weak committees in economic environments.

8 Minimal liberalism

In this section we prove that every EF which satisfies minimal liberalism has no acceptable representation. Then we discuss the implications of this result. Let $N = \{1, \ldots, n\}$, $n \geq 3$ be a society and let (A, τ) be a compact Hausdorff topological space (of social states). An EF $E : P(N) \to P(\kappa(A))$ satisfies *minimal liberalism* (ML) if there exist $i, j \in N$, $i \neq j$, $B^i \in E(i)$, and $B^j \in E(j)$ such that $B^i \neq A \neq B^j$. The reader is referred to Peleg (1998) for a discussion of ML. The following impossibility theorem is true.

Theorem 8.1. *Let $E : P(N) \to P(\kappa(A))$ be an EF. If E satisfies ML, then E has no acceptable representation.*

Proof. Assume, on the contrary, that E has an acceptable representation. Then, by Theorem 4.2, E satisfies (4.2) and (4.4). Let $i, j \in N$, $i \neq j, B^i \in E(i)$ and $B^j \in E(j)$ such that $B^i \neq A \neq B^j$. By (4.2) $B^i \cup B^j = A$. Thus we may choose $x \in B^j \setminus B^i$ and $y \in B^i \setminus B^j$. Again by (4.2), $x \in B$ for all $B \in E(N \setminus \{i\})$, and $y \in B$ for all $B \in E(N \setminus \{j\})$. Hence $\{x\} \in E^*(i)$ and $\{y\} \in E^*(j)$. As $\{x\} \cap \{y\} = \emptyset$, we have obtained the desired contradiction (to (4.4)). $\square$

Let $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be a GF. Γ is *weakly* acceptable if for every $R^N \in V^N$ the game (Γ, R^N) has a Pareto optimal NE. Let $E : P(N) \to P(\kappa(A))$ be an $EF.E$ has a weakly acceptable representation if it satisfies (4.1), (4.4) and (4.5) (see Peleg et al. (2001)). Thus, in that model (i.e., the model of Peleg et al. (2001)) there is no conflict between existence of individual rights, Nash stability, and Pareto optimality (see, e.g., Example 3.10 of Peleg et al. (2001)). The role of Theorem 8.1 is to point out that strengthening of the requirement for Pareto optimality from weak acceptability to acceptability leads to GF's which do not allow individual rights (a GF Γ satisfies ML if E^Γ satisfies ML).

9 Concluding remarks

We have obtained a complete characterization of the effectivity functions which belong to some acceptable game form. Our result can be described as follows. A game form is weakly acceptable if for every profile of (continuous) preferences it has a Pareto optimal Nash equilibrium. An effectivity function can be represented by an acceptable game form if, and only if, it has a weakly acceptable representation and it satisfies, in addition, (4.2). (Notice that, by Peleg et al. (2001) an effectivity function has a weakly acceptable representation iff it satisfies (4.1) and (4.3).) Thus, the set of effectivity functions with an acceptable representation is a "small" subset of the set of effectivity functions with a weakly acceptable representation because (4.2) is a strong condition. Also, using Peleg et al. (2001), we have given sufficient direct conditions (i.e., conditions referring only to effectivity functions), for the existence of acceptable representations for effectivity functions (see our Theorem 4.2).

There are two byproducts of our characterization. First, in Section 8, we prove that every acceptable game form violates minimal liberalism (as formulated for game forms (see Deb et al. (1997)). Thus, we cannot represent constitutions that allow for individual rights, by acceptable game forms (see Peleg (1998), Keiding and Peleg (2001b), and Peleg et al. (2001) for representations of constitutions).

Second, by Theorem 7.1, if G is a proper simple game without vetoers, then its effectivity function has an acceptable representation. This result provides stable voting procedures to committees without vetoers in a topological framework. It complements the work of Keiding and Peleg (2001a) on existence of (strongly) stable voting procedures for weak committees in economic situations.

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