האוניברסיטה העברית בירושלים
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A FOURIER-THEORETIC PERSPECTIVE ON THE CONDORCET PARADOX AND ARROW'S THEOREM

by

GIL KALAI

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מרכז לחקר הרציונליות

CENTER FOR THE STUDY OF RATIONALITY

Feldman Building, Givat-Ram, 91904 Jerusalem, Israel
PHONE: [972]-2-6584135 FAX: [972]-2-6513681
E-MAIL: ratio@math.huji.ac.il
URL: http://www.ratio.huji.ac.il/

A Fourier-Theoretic Perspective on the Condorcet Paradox and Arrow's Theorem.

Gil Kalai
Institute of Mathematics, *
Hebrew University, Jerusalem, Israel

Abstract

We describe a Fourier-theoretic formula for the probability of rational outcomes for a social choice function on three alternatives. Several applications are given.

1 Introduction

The Condorcet "Paradox" demonstrates that the majority rule can lead to a situation in which the society prefers A on B , B on C and C on A. Arrow's Impossibility Theorem (1951) asserts that under certain conditions if there are at least three alternatives, then every non-dictatorial social choice gives rise to a non-rational choice function, namely there exists a profile such that the social choice is not rational.

A *profile* is a finite list of linear orders on a finite set of alternatives. (We will consider the case of three alternatives.) We consider social choice functions which, given a profile of n order relations R_i on a set X of m alternatives, yield an asymmetric relation R on the alternatives for the society. Thus, $R = F(R_1, R_2, \dots R_n)$ where F is the social choice function. If aR_ib we say that the i-th individual prefers alternative a over alternative b. If aRb we say that the society prefers alternative a over alternative b. The social choice is rational if R is an order relation on the alternatives.

A principal condition for social choice functions: "Independence of Irrelevant Alternatives" asserts that for every two alternatives a and b the society's choice between a and b depends only on the individual preferences

*The author is a member of the Center of Rationality and Interactive Decisions and of the Center of Theoretical Computer Science and Discrete Mathematics at HU

between these two alternatives. In other words, the set $\{i : aR_ib\}$ determines whether aRb.

There is an extensive literature on the probability of non-rational outcomes in voting schemes when individual preferences are uniform and independent (see Gehrlein (1997)). We will consider this probability for general social choice functions on three alternatives.

Our main result is a Fourier-theoretic formula for the probability of non-rational outcomes for an arbitrary social choice function on three alternatives. Several applications are given two of which we will mention here. A social choice function is *neutral* if it is invariant under permutations of the alternatives. We call a neutral social choice function *symmetric* if the choice is invariant under some transitive group of permutations on $\{1, 2, \ldots, n\}$. (Note: the social choice need not be invariant under all permutations of the individuals.) For example, an electoral voting system (such as that in the the United States of America) in which all states have the same number of voters and electors is symmetric.

Theorem 1.1. *The probability of a rational outcome for a symmetric social choice on three alternatives is less than 0.9192.*

The second application demonstrates that if the outcomes of a neutral social choice function for random profiles are rational with high probability then the social choice is approximately a dictatorship.

Theorem 1.2. *There exists an absolute constant K such that the following assertion holds: For every $\epsilon > 0$ and for every neutral social choice function if the probability that the social choice is non-rational is smaller than ϵ, then there exists a dictator such that for every pair of alternatives the probability that the social choice differs from the dictator's choice is smaller than $K \cdot \epsilon$.*

This theorem relies on a result which is proved in Friedgut, Kalai and Naor (2001) concerning Boolean functions whose spectrum is concentrated on the first two "levels".

For general references on social choice theory see Fishburn (1973), Sen (1986) and Peleg (1984).

2 Fourier expansion on the discrete cube

Consider the discrete cube $\Omega_n = \{0, 1\}^n$ endowed with the uniform probability measure $\mathbf{P}$. We will identify elements in Ω_n with subsets S of $[n] = \{1, 2, \ldots, n\}$.

For two real functions f and g defined on Ω_n their inner product is

$$< f, g >= \sum 2^{-n} f(S) g(S).$$

The 2-norm of f is thus equal to: $\|f\|_2 = \sqrt{< f, f >} = (\sum 2^{-n} f^2(S))^{1/2}$. The Cauchy-Schwarz inequality asserts that

$$< f, g >\leq \|f\|_2 \|g\|_2.$$

For a real function defined on $\{0,1\}^n$ consider the Fourier-Walsh expansion of f,

$$f = \sum_{S \subset [n]} \widehat{f}(S) u_S,$$

where, $u_S(T) = (-1)^{|S \cap T|}$. Since the 2^n functions u_S form an orthogonal basis for the space of real functions on Ω_n, $\widehat{f}(S) =< f, u_S >$.

The Parseval formula asserts that $\|f\|_2^2 = \sum \widehat{f}^2(S)$, and more generally,

$$< f, g >= \sum \widehat{f}(S) \widehat{g}(S).$$

Let f be a Boolean function defined on Ω_n, namely $f : \Omega_n \to \{0,1\}$. In this case f is simply a characteristic function of a subset A of Ω_n. Denote by $\mathbf{P}(A) = |A|/2^n$ and note that in this case the Parseval formula asserts that:

$$\|f\|_2^2 = \mathbf{P}(A) = \sum \widehat{f}^2(S).$$

Note that $\widehat{f}(\emptyset) =< f, u_\emptyset >= \mathbf{P}(A)$ since $u_\emptyset(S) = 1$ for every S. We will write u_i for $u_{\{i\}}$.

3 The probability of irrational social choice for three alternatives

Consider a social choice function which, given a profile of n order relations R_i, $i = 1, 2, \ldots, n$ on three alternatives, yields an asymmetric relation R for the society. Thus $R = F(R_1, R_2, \ldots R_n)$ where F is the social choice function. aR_ib indicates that the i-th individual prefers alternative a over alternative b. aRb indicates that the society prefers alternative a over alternative b. The social preference relations are not assumed to be rational (=order relations).

Let a, b and c be the alternatives. The social preference between a and b depends only on the individual preferences between a and b and therefore can be described by a Boolean function f of n variables $x_1, x_2 \ldots x_n$ as follows: Set $x_i = 1$ if aR_ib and $x_i = 0$ otherwise. In addition, let aRb if and only if $f(x_1, \ldots x_n) = 1$. Similarly, let $g = g(y_1, y_2, \ldots, y_n)$ and $h = h(z_1, z_2, \ldots, z_n)$ be the Boolean functions which describe the society's preferences between b and c and between c and a, respectively.

Let $p_1 = \mathbf{P}\{x : f(x) = 1\}$ $(= \widehat{f}(\emptyset))$, $p_2 = \mathbf{P}\{x : g(x) = 1\}$ $(= \widehat{g}(\emptyset))$ and $p_3 = \mathbf{P}\{x : h(x) = 1\}$ $(= \widehat{h}(\emptyset))$.

We will call the social choice function *balanced* if $p_1 = p_2 = p_3 = 1/2$.

We will now consider a Boolean function on $3n$ variables $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $z_1, \ldots, z_n$. (Write $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$.)

Let Ψ $(=\Psi_3(n))$ be the subset of the $3n$-dimensional discrete cube corresponding to these variables which arise from a rational profile namely where each of the triples (x_i, y_i, z_i) is not equal to $(0, 0, 0)$ or $(1, 1, 1)$ for $i = 1, 2, \ldots n$. Note that $\mathbf{P}(\Psi) = (6/8)^n$.

We will introduce the following notation: For real functions f, g on Ω_n let

$$<< f, g >>= \sum_{\emptyset \neq S \subset [n]} \widehat{f}(S)\widehat{g}(S)(-1/3)^{|S|-1}. \tag{3.1}$$

Let $W = W(f, g, h)$ be the probability of obtaining a non-rational outcome from profiles of the n individuals when the individual preferences are uniform on the six orderings of the alternatives and independent. First, note that

$$W(f, g, h) = \tag{3.2}$$

$$= \mathbf{P}\{\Psi\}^{-1} \cdot \sum_{(x,y,z) \in \Psi} 2^{-3n}(f(x)g(y)h(z) + (1 - f(x))(1 - g(y))(1 - h(z))).$$

Indeed, the outcome of the social choice function which is described by $f(x)$, $g(y)$ and $h(z)$ is rational if and only if the vector $(f(x), g(y), h(z))$ is not equal to $(1, 1, 1)$ and not to $(0, 0, 0)$. Therefore, if the social choice function yields an order relation for the society, then $f(x)g(y)h(z) + (1 - f(x))(1 - g(y))(1 - h(z) = 0$ and if it yields a non-rational outcome, then $f(x)g(y)h(z) + (1 - f(x))(1 - g(y))(1 - h(z) = 1$.

Theorem 3.1.

$$W(f, g, h) = ((1 - p_1)(1 - p_2)(1 - p_3) + p_1p_2p_3- \tag{3.3}$$

$$-(<< f, g >> + << g, h >> + << h, f >>)/3.$$

Proof of the theorem: Let $A = A_n = \chi_\Psi$ and $B = f(x)g(y)h(z)$. We need to compute the inner product of A and B and we will carry out this computation using the Fourier transform.

$$\sum_{(x,y,z)\in\Psi} 2^{-3n} f(x)g(y)h(z) =< A, B >= \sum \widehat{A}(u)\widehat{B}(u). \qquad (3.4)$$

It is left to determine the Fourier coefficients of A and B. In our case the Fourier coefficients are indexed by 0-1 vectors of length $3n$ or equivalently by subsets of the variables. In our case, we have $3n$ variables: $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$. We represent subsets of these $3n$ variables by a triple (S_1, S_2, S_3) of subsets of $[n]$. S_1 will correspond to a subset of the n variables x_i, $i = 1, \ldots, n$ and similarly S_2 and S_3 will correspond to the y_i's and z_i's, respectively.

We start with the Fourier coefficients of B. For three subsets S_1, S_2 and S_3 of $[n]$ let $\widehat{B}(S_1, S_2, S_3)$ be the Fourier coefficients that correspond to S_1, S_2, S_3. The multiplicative form of B, $B(x, y, z) = f(x)g(y)h(z)$, implies a similar multiplicative form for the Fourier coefficients:

$$\widehat{B}(S_1, S_2, S_3) = \widehat{f}(S_1)\widehat{g}(S_2)\widehat{h}(S_3).$$

Note that A also has a multiplicative structure as the product of n expressions. $\Psi_3(n)$ is simply the Cartesian product of n copies of $\Psi_3(1)$. Direct computation for $n = 1$ shows that $\widehat{A}_1(\emptyset) = 3/4$, $\widehat{A}_1(U) = -1/4$ when $|U| = 2$ and $\widehat{A}_1(U) = 0$ otherwise. It follows that $\widehat{A}(S_1, S_2, S_3)$ is a product of n expressions one for each i, $i = 1, 2, \ldots, n$. The contribution of i is given by the Fourier coefficients of A_1, namely it is $3/4$ if i does not belong to S_1, S_2 and S_3, it is $-1/4$ if i belongs to two out of the three S_j's and it is 0 otherwise. In summary, $\widehat{A}(S_1, S_2, S_3) = 0$ unless the triple of sets (S_1, S_2, S_3) is special, i.e. every index i belongs to zero or two sets S_i and for special triples:

$$\widehat{A}(S_1, S_2, S_3) = (-1/4)^{|S_1 \cup S_2 \cup S_3|}(3/4)^{n-|S_1 \cup S_2 \cup S_3|}.$$

It follows that

$$\sum 2_\Psi^{-3n} f(x)g(y)h(z) =$$
$$= \sum \widehat{f}(S_1)\widehat{g}(S_2)\widehat{h}(S_3)(-1/4)^{|S_1 \cup S_2 \cup S_3|}(3/4)^{n-|S_1 \cup S_2 \cup S_3|}$$

where the the right hand side is summed over all special triples (S_1, S_2, S_3).

The theorem follows when we evaluate $\sum_\Psi (f(x)g(y)h(z) + (1 - f(a))(1 - g(y))(1 - h(z)))$ and take into account that $\widehat{(1-f)}(S) = -\widehat{f}(S)$ if $S \neq \emptyset$ and $\widehat{(1-f)}(\emptyset) = 1 - \widehat{f}(\emptyset)$.

This completes the proof. $\square$

Note that if $f = g = h$ then Relation (3.3) reduces to

$$W = (p^3 + (1-p)^3 - \sum_{S \neq \emptyset} \widehat{f}^2(S)(-1/3)^{|S|-1}). \tag{3.5}$$

In this case, $p = p_1 = p_2 = p_3$.

4 The probability of the Condorcet Paradox

4.1 A rough computation

Consider the case in which $n = 2m+1$ is odd and f, g and h are the majority function. Let $G(n,3)$ be the probability of a rational outcome in this case and let $G(3) = \lim_{n \to \infty} G(n,3)$. It is known that

$$G(3) = 3/4 + (3/(2 \cdot \pi)) \cdot ArcSin(1/3) \approx .91226.$$

Gulibaud stated this formula without a proof in a footnote of a paper in the 1960's. Many people have since reproduced the result (see Gehrlein (1997)).

Define $d_m = \sum_{k=1}^{2m+1} \widehat{f}^2(\{k\})$. The value of $\widehat{f}(\{k\})$ is $\binom{2m}{m} 2^{-2m-1}$ and therefore

$$d_m = (\binom{2m}{m} 2^{-2m-1})^2 \cdot (2m+1).$$

Note that d_m is a decreasing sequence which tends to $1/2\pi$. Since

$$f(1-x_1, \dots, 1-x_n) = 1 - f(x_1, \dots, x_n), \tag{4.1}$$

it follows that $\widehat{f}(S) = 0$ whenever $|S| > 0$ is even.

Proposition 4.1.

$$0.9092 \leq G(3) \leq 0.9192$$

Proof: Relation 3.3 asserts that in our case the probability of non-rational outcomes is given by

$$1/4 - \sum_{S \neq \emptyset} \widehat{f}^2(S)(-1/3)^{|S|-1}.$$

It follows directly that

$$1 - G(3,n) \leq 1/4 - d_m. \tag{4.2}$$

and therefore, $1 - G(3) \leq 1/4 - 1/2\pi$.

On the other hand,

$$\sum\{\widehat{f}^2(S) : |S| \geq 3\} = 1/2 - \widehat{f}^2(\emptyset) - \sum_{k=1}^{n} \widehat{f}^2(\{k\}) = 1/4 - d_m$$

and therefore,

$$1 - G(3, n) \leq 1/4 - d_m - (1/4 - d_m)/9 = \tag{4.3}$$

$$= 2/9 - 8/9 d_m \leq 2/9 - 8/9 \cdot 1/2\pi.$$

4.2 Gulibaud's formula

The Fourier coefficients of the majority function are known and we will briefly present them here. Let $n = 2m + 1$. By Levenshtein's (1995) formula 46, we have

$$2^n \widehat{f}(k) = \sum_{s=0}^{m} K_{n,s}(k) = K_{2m,m}(k-1).$$

Here $K_{n,s}$ is the s'th Krawchouk polynomial on n points. Now,

$$K_{2m,m}(x) = (-1)^m 2^m / m!(x-1)(x-3)...(x-2m+1)$$

and substituting we obtain,

$$\widehat{f}^2(k) = (1/2^{4m+2})(2^{2m})/(m!)^2 (k-2)^2 (k-4)^2 ... (k-2m)^2.$$

This leads finally to

$$\sum\{\widehat{f}^2(S) : |S| = k\} = 1/2^{4m+2}(2m+1)/k \binom{2m}{m} \binom{k-1}{(k-1)/2} \binom{2m-k+1}{(2m-k+1)/2}.$$

For k fixed and n tending to infinity this expression tends to $\binom{k-1}{(k-1)/2} 2^{-k-1} k^{-1} \pi^{-1}$.

It follows that

$$1 - G(3) = 1/4 - \sum_{i=0}^{\infty} \binom{2i}{i} \cdot 2^{-2i-1} \cdot 1/(2i+1) \cdot 1/\pi \cdot (1/3)^{2i} =$$

$$= 1/4 - 1/2\pi - 1/108\pi - 1/2160\pi \ldots.$$

5 Symmetric social choice on three alternatives

5.1 Symmetric forms of social choice

Recall that a neutral social choice function is symmetric if the choice is invariant under some transitive group of permutations on $\{1, 2, \ldots, n\}$.

Theorem 5.1. *For a symmetric social choice on three alternatives the probability of a rational outcome is less than 0.9192*

Proof: Neutrality implies that $f = g = h$ and that f satisfies $f(1 - x_1, 1 - x_2, \ldots, 1 - x_n) = 1 - f(x_1, x_2, \ldots, x_n)$. It follows that $\widehat{f}(S) = 0$ if S is even and non-empty.

Symmetry among the voters implies that the Fourier coefficients $\widehat{f}(\{k\})$ are equal for $k = 1, 2, \ldots, n$.

At this point all that is required is the fact that if a Boolean function is symmetric then $\sum_{k=1}^{n} |\widehat{f}(\{k\})|$ is maximized by its value on the majority function. This follows at once from the fact that $\sum_{k=1}^{n} \widehat{f}(\{k\}) = \sum_{S \subset [n]} 2^{-n} f(S)(2|S| - n)$. Therefore the sum of the Fourier coefficients, $\sum \widehat{f}(\{k\})$, is maximized for the majority function and so is its absolute value.

Under the assumption that all the Fourier coefficients $\widehat{f}(\{k\})$ are equal the absolute value of each is maximized for the majority function.

It follows that the probability of a non-rational outcome is at least

$$1/4 - d_m - (1/4 - d_m)/9 = 2/9 - 8/9 d_m \geq 2/9 - 8/9 \cdot 1/2\pi.$$

□

This argument applies even if f, g and h are distinct as long as they satisfy relation 4.1.

It appears to be true that under the conditions of Theorem 5.1 the probability for a rational outcome is always at most $G(3)$. For this we need:

Conjecture 5.1. $\sum_{|S| \leq k} \widehat{f}^2(S)$ is maximized for every fixed k, when n tends to infinity, for the majority function.

See Bourgain (2001) for a related result.

It follows from Theorem 5.1 that the probability that a symmetric social choice on $3m$ alternatives leads to a rational outcome is at most 0.91^m and therefore it rapidly approaches zero. (**Problem:** How rapidly?)

Another immediate consequence of relation (3.3) is the following proposition (which also deserves a direct combinatorial proof):

Proposition 5.2. *1. If the social choice is neutral (invariant under permutations of the alternatives) then the probability of a rational outcome is at least 3/4.*

2. If the social choice is balanced (namely, $p_1 = p_2 = p_3 = 1/2$), then the probability of a rational outcome is at least 2/3.

Proof: 1. This follows from the fact that since f, g and h satisfy relation (4.1), all their Fourier coefficients vanish for even sized sets. An example of equality is the following. The ordering between a and b is determined according to one individual, the ordering between b and c by a second individual and the ordering between c and a by a third individual.

2. If $p_1 = p_2 = p_3 = 1/2$, the smallest possible value of $<< f, g >>$ is obtained if all the contributions are coming from a single set S containing two elements. An example of equality is obtained when aRb holds precisely when this is the preference of one out of the two first voters and the same holds for bRc and cRa. (This example is not entirely kosher since it violates the condition that if all members of the society prefer a on b, then so does the society. However, if we add this condition and let n grow we obtain a sequence of kosher examples for which the probability of rational choice approaches 2/3.)□

It may also be true that if f, g and h are monotone and $p_1 = p_2 = p_3 = 1/2$ then the probability of a rational outcome is at least 3/4. To show this we need to prove that for monotone Boolean functions f and g, $<< f, g >>$ is non-negative. This seems to be related to the FKG inequality (which asserts that when f and g are monotone, $\sum_{\emptyset \neq S \subset [n]} \widehat{f}(S)\widehat{g}(S) \geq 0$.) Talagrand (1996) may be relevant in this context.

6 A Fourier-theoretic proof of Arrow's theorem (under neutrality)

There several proofs for Arrow's theorem in the literature and a large number of extensions and variations. Many of the proofs are similar to Arrow's original one and use direct and simple combinatorial arguments (see, for example, Geanakoplos (1997)). Baryshnikov (1997) found a topological proof for Arrow's theorem in a context which unified combinatorial impossibility theorems with topological social choice theory, an area initiated by Chichilinsky (1972). Saari (1997) presented a geometric proof.

There is no loss of generality to assume for Arrow's theorem that the number of alternatives is three. The theorem makes the following assumption stated in our language on the social choice: $f(x, x, \dots, x) = x$ for $x = 0, 1$ and this condition holds also for g and h.

We will make the further assumption that f, g and h are balanced, namely that $p = p_1 = p_2 = p_3 = 1/2$. This is the case if the social choice is neutral, i.e. invariant under permutations of the alternatives.

We require the following lemma:

Lemma 6.1. *If f is a Boolean function and if $\hat{f}(S) = 0$ when $|S| > 1$ then $f = 0$ or $f = 1$ or $f(x_1, x_2, \dots, x_n) = x_i$ for some i or $f(x_1, x_2, \dots, x_n) = 1 - x_i$ for some i.*

Proof 1: Let $p = \|f\|_2^2 = \mathbf{P}\{x : f(x) = 1\}$. $\hat{f}(\emptyset) = p$ and therefore $\sum \hat{f}^2(\{i\}) = p - p^2$. Assume that $p \geq 1/2$, otherwise replace f with $1 - f$. ¿From the convexity of the function $\phi(t) = t^2$ it follows that $\sum |\hat{f}(\{i\})| \geq \sqrt{p - p^2}$ with equality only if there is one non-zero $\hat{f}(i)$.

Suppose that $p \neq 0, 1$. Let x be a vector which is 1 for every i with $\hat{f}(\{i\}) \leq 0$ and 0 otherwise. Then $f(x) = p + \sum |\hat{f}(\{i\})| \geq p + \sqrt{p - p^2}$ and the only way that this equals 1 is if $p = 1/2$ for one value of i, $\hat{f}(\{i\}) = 1/2$ and all other $\hat{f}(\{j\})$'s are zero.

Proof 2 (by Ehud Friedgut): f is of the form

$$f = c + \sum_i \hat{f}(i) u_i.$$

Since f is Boolean,

$$c = |f|_1 = |f|_2^2 = c^2 + \sum \hat{f}(i)^2.$$

Using this and the fact that for $i \neq j$ $u_i u_j = u_{\{i,j\}}$, we have that

$$0 = f^2 - f = \sum_i (2c - 1)\hat{f}(i) u_i + \sum_{i \neq j} \hat{f}(i)\hat{f}(j) u_{\{i,j\}}.$$

From the uniqueness of the Fourier expansion we conclude that $\hat{f}(i)\hat{f}(j) = 0$ for $i \neq j$, hence for at most one value of i $\hat{f}(i) \neq 0$, and if there exists such an i then

$$f = \pm(\frac{1}{2} - \frac{1}{2} u_i).$$

□

Proof of Arrow's theorem under neutrality: We use the notation of Section 3. Let

$$\bar{f} = \sum_{\emptyset \neq S} \hat{f}(S) u_S \quad \text{and} \quad f' = \sum_{\emptyset \neq S} \hat{f}(S)(-1/3)^{|S|-1} u(S).$$

Similarly define $\bar{g}, g'$, $\bar{h}, h'$. Note that $\|\bar{f}\|_2^2 = 1/4$ and $\|f'\|_2^2 \leq 1/4$ with equality only if all non-zero Fourier coefficients $\widehat{f}(S)$ are for $|S| \leq 1$. (For the general case $\|\bar{f}\|_2^2 = p_1 - p_1^2$.)

Now, $<< f, g >>$ is by the Cauchy-Schwarz inequality at most $\sqrt{(\|f'\|_2^2)\|\bar{g}\|_2^2}$. This quantity is at most 1/4 with equality only if f' is proportional to g and hence $f' = f = g$. This happens only if all the Fourier coefficients of f are on the 0 and 1 levels and since f is not constant and $f(0,0,\ldots,0) = 0$ by lemma 6.1, $f = g = x_i$ for some i.

Therefore, for $W = 0$ we require that $<< f, g >> + << f, h >> + << g, h >>= 3/4$ and it follows that $f = x_i$, $g = x_j$ and $h = x_k$ and that $i = j = k$. □

Remark: I do not have a proof along these lines for the general case of Arrow's theorem or for a stability result in the non-balanced case. We need to show that $<< f, g >> + << g, h >> + << f, h >>= 3p_1p_2p_3 + 3(1 - p_1)(1 - p_2)(1 - p_3)$ can occur only if $p_1 = p_2 = p_3 = 1/2$. Our argument shows that this is the case when

$$\sqrt{p_1(1-p_1)p_2(1-p_2)} + \sqrt{p_2(1-p_2)p_3(1-p_3)} + \sqrt{p_1(1-p_1)p_3(1-p_3)} \leq$$
$$\leq 3p_1p_2p_3 + 3(1-p_1)(1-p_2)(1-p_3).$$

(This inequality fails, for example, for $p_1 = p_2 = 1/5$ and $p_3 = 1$.)

For the general case improved upper bounds on the strange "inner product" $<< f, g >>$ (in terms of p_1 and p_2) are needed.

7 Stability of Arrow's theorem

Lemma 6.1 can be extended to the following description of Boolean functions whose spectrum is concentrated on the first two levels:

Theorem 7.1. *If f is a Boolean function, $\|f\|_2^2 = p$ and if $\sum_{|S|>1} \widehat{f}^2(S) \leq \delta$ then either $p < K'\delta$ or $p > 1 - K'\delta$ or $\|f(x_1, x_2, \ldots, x_n) - x_i\|_2^2 \leq K\delta$ for some i or $\|f(x_1, x_2, \ldots, x_n) - (1 - x_i)\|_2^2 \leq K\delta$ for some i.*

Here, K' and K are absolute constants. Two proofs will be given in Friedgut, Kalai and Naor (2001).

Equipped with Theorem 7.1, our proof for Arrow's theorem yields, with minor changes, the following result:

Theorem 7.2. *For every $\epsilon > 0$ and for every balanced social choice function on three alternatives, if the probability that the social choice is non-rational is smaller than ϵ than there is a dictator such that the probability that the social choice differs from the dictator's choice is smaller than $K \cdot \epsilon$.*

It follows that for balanced social choice functions the following assertion holds: For every $\epsilon > 0$, as the number of alternatives tends to infinity, if

- For every pair of alternatives there is no dictator such that the probability that the social choice differs from the dictator's choice is smaller than ϵ,

then

- The probability for a rational outcome tends to zero.

(This is due to the fact that for every triple of the alternatives, the probability of a rational outcome is bounded away from 1 and disjoint triples are independent.)

8 Concluding remarks

8.1 More alternatives and an interesting graph invariant

Extending our formula from three to more alternatives leads naturally to the problem of identifying the Fourier coefficients for the analog of Ψ_3 for more than three alternatives.

Given a directed graph D on the vertex set $1, 2, \ldots, n$, for every permutation π on $[n]$ consider the number of *inversions* of π among the edges of D, namely the number of directed edges $i \to j$ such that $\pi(i) > \pi(j)$. Denote this number by $i_D(\pi)$. Call a permutation even if $i_D(\pi)$ is even and odd otherwise. Let $sg(D)$ be the difference between the number of odd permutations on the graph minus the number of even permutations. Note that up to a sign this number depends only on the underlying undirected graph G.

When there are m alternatives, the Fourier coefficients of the characteristic function of Ψ_m are indexed by graphs G on n vertices and are given up to a constant factor by $sg(G)$.

Brendan McKay suggested recording the inversion number statistics as follows: For a directed graph D with n vertices define $g(D, x) = 1/n! \cdot \sum_k a_k x^k$ where a_k is the number of permutations π, with $i_\pi(D) = k$. $|sg(G)|$ is equal to $n! \cdot |g(D, -1)|$ which only depends on the underlying undirected graph. The parameters $sg(G)$ and $g(D, x)$ appear to be of independent interest. Compare also Foata and Zeilberger (1996) and White and Williamson (2001).

8.2 The superiority of majority

It will be interesting to use the Fourier-theoretic approach to study the old question regarding the probability that a given alternative is preferred over any other by more than half the voters. When the number of alternatives is larger than three, the Fourier-theoretic formulas are quite messy. It is known that for a fixed number of voters, if the number of alternatives tends to infinity, the probability of such a preferred alternative existing tends to zero. Bell (1981) proved that the tournament representing the social preferences has a Hamiltonian cycle with probability that tends to one as the number of alternatives tends to infinity. Precise probabilistic computations seem challenging.

For arbitrary symmetric social choice functions, we conjecture that the probability that there is an alternative which is preferred on all others tends to zero as the number of alternatives tends to infinity. It may be true that the probability of a Hamiltonian cycle for the social preferences tends to one when the number of alternatives tends to infinity. For these conjectures to hold perhaps all that is needed is that the social choice does not coincide with a dictatorship with probability of at least $1 - \epsilon$. For the symmetric case, it may even be true that these probabilities are maximized for the case of majority.

I do not have a counterexample to the following bold conjecture:

Conjecture 8.1. Let X be a set of m alternatives and let the number of individuals n be an odd integer. Let η be an arbitrary probability distribution on the set of orderings on X. Consider random profiles where the order relations for the individuals are drawn independently according to η. For a symmetric social choice function F, let $p(F)$ be the probability that F leads to a rational social choice. Then $p(F)$ is maximized for the majority function.

Finally, consider neutral social choice functions where the *influence* of each individual is prescribed (see, Kahn, Kalai and Linial (1988)). It may be true that in this case the "most rational" social choice functions (in terms of the probability for a rational outcome) are those based on weighted majority functions.

8.3 Relation to PCP

The following remark is based on collaboration with Ehud Friedgut, Shmuel Safra and Uri Zwick. There are interesting connections between Theorems 3.1,7.1 and the theory of probabilistically checkable proofs and especially a certain test developed by Hastad (1999). Compare also with Parnas, Ron and Samorodnitsky (2000).

Hastad described a probabilistic test, based on sampling three values of a function f which allows to distinguish between a Boolean function which is a dictatorship and a Boolean function whose value is not determined with high probability by a bounded number of variables.

Our main result can be seen as a probabilistic test for checking, based on sampling one instance for each of three balanced Boolean functions f, g and h, whether all these functions are determined (at least approximately) by a single variable x_j.

Consider a social choice function as above and ask the following question: What is the probability for the uniform distribution over rational preferences of the individuals that aRb and bRc or bRa and cRb? For a dictatorship the answer is $1/3$. Using a similar arguments to those used here it follows that if the social choice function is not close to being a dictatorship then the answer is larger than and bounded away from $1/3$. If $f = g$ this yields a probabilistic test based on sampling two values $f(x)$ and $f(y)$ for dictatorship. (Here, the distribution of x and y is given by the following: the probabilities for (x_i, y_i) to be $(0,0), (0,1), (1,0), (1,1)$ are (1/6, 1/3, 1/3, 1/6), respectively.)

We intend to study further possible applications for hardness of approximations.

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