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A VALUE ON $'AN$

JEAN-FRANÇOIS MERTENS[†] AND ABRAHAM NEYMAN[‡]

ABSTRACT. We prove here the existence of a value (of norm 1) on the spaces $'NA$ and even $'AN$, the closure in the variation distance of the linear space spanned by all games $f \circ \mu$, where μ is a non-atomic, non-negative finitely additive measure of mass 1 and f a real-valued function on $[0, 1]$ which satisfies a much weakened continuity at zero and one.

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1. Introduction

Aumann and Shapley (1974) proved the existence of a unique value on the space *bv′NA*, the closure in the variation norm of the linear space spanned by all games $f \circ \mu$, where μ is a non-atomic probability measure and f a real-valued function on $[0, 1]$ which is of bounded variation, continuous at $0 = f(0)$ and at 1. Neyman (1981) proved that this unique value is also an asymptotic value, and that the asymptotic approach fails when f is of unbounded variation: for some $\{0, 1\}$-valued function on $[0, 1]$, which is continuous at 0 and 1 and vanishes outside a countable set, $f \circ \mu$ does not have an asymptotic value. Tauman (1979) proved however that the axiomatic approach works also for games of unbounded variation: there exists a value of norm 1 on the space spanned by all games of the form $f \circ \mu$ where μ is a non-atomic probability measure and f is integrable and continuous at 0 and 1. The present paper removes the integrability assumption and weakens that of continuity: we prove the existence of a value of norm 1 on the spaces *′AN*, the closure in the variation distance of the linear space spanned by all games $f \circ \mu$, where μ is a non-atomic, non-negative finitely additive measure of mass 1 and f a real-valued function on $[0, 1]$ which satisfies a much weaker continuity at 0 and 1. Under this value, $f \circ \mu$ is mapped to $f(1)\mu$. Moreover, even when the player set is standard Borel, there are other values of norm 1 on *′AN*, that differ already on smooth functions of a finitely additive and non-atomic measure.

2. Preliminaries

Let $(I, \mathscr{C})$ be a measurable space. The members of the set I are called *players*, those of $\mathscr{C}$, *coalitions.* A game is a real-valued function v on $\mathscr{C}$ such that $v(\emptyset) = 0$. The linear space of all games is denoted G. A game $v \in G$ is *finitely additive* if $v(S \cup T) = v(S) + v(T)$ whenever S and T are two disjoint coalitions.

A game v is *monotone* if $v(S) \leq v(T)$ whenever $S \subset T$. The *variation* of a game $v \in G$, $\|v\|$, is the supremum of the variation of v over all increasing chains $S_1 \subset S_2 \subset \cdots \subset S_n$ in $\mathscr{C}$. A game $v \in G$ has bounded variation if $\|v\| < \infty$. The space of all games of bounded variation, *BV*, is a Banach space. The variation metric given by $d(v_1, v_2) = \mathsf{min}\{1, \|v_1 - v_2\|\}$ defines a distance (and hence a topology) on G.

FA (resp. M) is the set of additive (resp. countably additive) $v \in BV$. *AN* (resp. *NA*) is the set of non-atomic elements of *FA* (resp. M). Given a set of games Q, Q_+ denotes the monotone games in Q, and Q_1 all games v in Q_+ with $v(I) = 1$.

Denote by $\mathscr{G}$ the group of automorphisms (i.e., one-to-one measurable mappings θ from I onto I with θ^{-1} measurable) of the underlying space $(I, \mathscr{C})$. Each θ in $\mathscr{G}$ induces a linear mapping θ^* of G onto itself, defined by $(\theta^*(v))(S) = v(\theta^{-1}(S))$. A set of games Q is called *symmetric* if $\theta^*(Q) = Q$ for all θ in $\mathscr{G}$.

Definition 1. *Let Q be a symmetric linear subspace of G.*

A map $\varphi\colon Q \to G$ is called positive *if $\varphi(Q_+) \subseteq G_+$;* symmetric *if for every $\theta \in \mathscr{G}$ $\varphi \circ \theta^* = \theta^* \circ \varphi$; and* efficient *if for every v in Q, $(\varphi(v))(I) = v(I)$.*

A value *on Q is a symmetric, positive and efficient linear map from Q to FA.*

When $Q \subseteq BV$, the above definition of a value coincides with that in (Aumann and Shapley, 1974). It is a natural extension to include also spaces of games that are not necessarily subsets of BV.

The *upper (lower) average* of a function from an interval of $\mathbb{R}$ to $\mathbb{R}$ is its upper (lower) Denjoy-Perron (or other) integral divided by the length of that interval.

Let $'$ be the set of all functions $f\colon [0,1] \to \mathbb{R}$ with the following weakened continuity at 0 and 1: the upper and lower averages of f over the intervals $[0,\varepsilon]$ and $[1-\varepsilon,1]$ converge as $\varepsilon \to 0+$ to $f(0) = 0$ and $f(1)$ respectively. The subspace of all polynomials is denoted p. The subspace of all functions with bounded variation in $'$ is denoted bv'. The subspace of all integrable functions f that are continuous at 0 and 1 is denoted In'. Given subsets x of $'$ and Y of G, xY is the closed linear subspace of G spanned by the games $f\circ\gamma$ with $f \in x$ and $\gamma \in Y_1$.

Obviously $pNA \subset bv'NA \subset In'NA \subset {'AN}$. Aumann and Shapley (1974) and Tauman (1979) prove the existence of a value on $bv'NA$ and $In'NA$ respectively.

3. The Theorem

The objective of the present paper is:

Theorem 1. *There exists a value of norm* 1 *on $'AN$.*

We show first that whenever $\sum_{i=1}^n f_i \circ \mu_i$ is bounded, with $\mu_i \in AN_1$, $f_i \in {'}$, and $\mu_i \neq \mu_j$ for $j \neq i$, all the f_i's are bounded. Using an extension of Lebesgue measure to all sets we show next that $\|\sum_{i=1}^n f_i(1)\mu_i\| \leq \|\sum_{i=1}^n f_i \circ \mu_i\|$. Therefore the map $\sum_{i=1}^n f_i \circ \mu_i \mapsto \sum_{i=1}^n f_i(1)\mu_i$ defines a value of norm 1 on $'AN$.

Lyapunov's (1940) classical convexity theorem asserts that the range of a vector $\vec{\mu} = (\mu_1,\ldots,\mu_n)$, $\{\vec{\mu}(S) \mid S \in \mathscr{C}\}$, of non-atomic probability measures, is convex (and compact); equivalently, for every ideal coalition χ (a measurable function $\chi : I \to [0,1]$) there is a coalition $T \in \mathscr{C}$ with $\vec{\mu}(T) = \vec{\mu}(\xi)$.

We make repeated use of the following generalizations and application of Lyapunov's theorem: given a vector of non-atomic finitely additive measures $\vec{\mu} = (\mu_1,\ldots,\mu_n)$, (1) for every ideal coalition ξ, there is a coalition T with $\vec{\mu}(T) = \vec{\mu}(\xi)$, and more generally, (2) for every increasing sequence of ideal coalitions $\chi_1 \leq \ldots \leq \chi_m$ there is an increasing sequence of coalitions $S_1 \subset \ldots \subset S_m$ such that $\vec{\mu}(S_j) = \vec{\mu}(\chi_j)$ (Mertens, 1990), and (3) there is a coalition S such that $\mu_i(S) \neq \mu_j(S)$ for all pairs i, j with $\mu_i \neq \mu_j$ (otherwise the range of $\vec{\mu}$ is contained in the union of the hyperplanes $x_i = x_j$ where i, j are the pairs such that $\mu_i \neq \mu_j$, which contradicts the convexity of the range of $\vec{\mu}$ unless all the measures μ_i are identical).

Lemma 1. *Assume that $\mu_1, \ldots, \mu_n \in AN_1$ are different, and that $f_1, \ldots, f_n \in {}'$. Then if $v = \sum_{i=1}^n f_i \circ \mu_i$ is bounded, so is each f_i.*

Proof. As each function f_i is in $'$, there is $0 < \delta < 1/3$ such that for every $0 < \varepsilon \leq \delta$ the upper and lower averages of each function f_i over the intervals $[0, 2\varepsilon]$ and $[1 - 2\varepsilon, 1]$ are within 1 of $f(0)$ and $f(1)$ respectively. Therefore, for every δ_j with $0 < |\delta_j| \leq 1$ and every $y \in (0, \delta] \cup [1 - \delta, 1)$ the upper and lower averages of $a \mapsto f_j(y + a\delta_j)$ over the interval $0 < a < \min\{y, 1 - y\}$ are bounded in absolute value by $3/\delta_j + |f_j(1)|$. There exists $S \in \mathscr{C}$ with $\mu_i(S) \neq \mu_j(S)$ whenever $i \neq j$. Fix $1 \leq i \leq n$ and a sequence $(x_k)_{k=1}^\infty$ in $(0, \delta] \cup [1 - \delta, 1)$. Set $\delta_j = \mu_j(S) - \mu_i(S)$. For every $a \leq \min\{x_k, 1 - x_k\}$, $aS + (x_k - a\mu_i(S))I$ is an ideal coalition. On the one hand,

$$\mu_i(aS + (x_k - a\mu_i(S))I) = x_k$$

and so $f_i(\mu_i(aS + (x_k - a\mu_i(S))I)) = f_i(x_k)$. On the other hand, for every $j \neq i$ $\mu_j(aS + (x_k - a\mu_i(S))I) = x_k + a\delta_j$, and thus the upper and lower averages of $a \mapsto f_j(\mu_j(aS + (x_k - a\mu_i(S))I))$ over $0 < a < \min\{x_k, 1 - x_k\}$ are bounded in absolute value by $3/\delta_j + |f_j(1)|$. Hence the upper and lower averages of the map $a \mapsto \sum_{j \neq i} f_j(\mu_j(aS + (x_k - a\mu_i(S))I))$ over $0 < a < \min\{x_k, 1 - x_k\}$ are bounded in absolute value by $\sum_{j \neq i} 3/\delta_j + |f_j(1)|$. As the game $\sum_{j=1}^n f_j \circ \mu_j$ is bounded, the upper and lower averages of the map $a \mapsto \sum_{j=1}^n f_j(\mu_j(aS + (x_k - a\mu_i(S))I)) = f_i(x_k) + \sum_{j \neq i} f_j(\mu_j(aS + (x_k - a\mu_i(S))I))$ over $0 < a < \min\{x_k, 1 - x_k\}$ are bounded, implying that the sequence $f_i(x_k)$ is bounded. So each f_i is bounded on $[0, \delta]$ and on $[1 - \delta, 1]$.

Define $\alpha_i = \inf\{\, x \in [0, 1] \mid f_i \text{ is bounded on } [x, 1] \,\}$. As f_i is bounded on $[1 - \delta, 1]$, $\alpha_i \leq 1 - \delta$. As f_i is bounded on $[0, \delta]$, either $\alpha_i = 0$ in which case f_i is bounded on $[0, 1]$, or $\alpha_i \geq \delta$. Assume $x = \max_{1 \leq i \leq n} \alpha_i \geq \delta$, and set $I = \{i \mid \alpha_i = x\}$. Let $i \in I$ with $\mu_i(S) \leq \mu_j(S)$ for every $j \in I$. There exists a sequence $(x_k)_{k=1}^\infty$ converging to x such that $|f_i(x_k)| \to \infty$ as $k \to \infty$. Fix $a > 0$ sufficiently small so that $2a < \min(x, 1 - x)$ and $\alpha_j < x - 2a$ whenever $j \notin I$ and $a < |\mu_k(S) - \mu_j(S)|$ whenever $k \neq j$. Then $aS + (x_k - a\mu_i(S))I$ is an ideal coalition whenever $(1 + x)/2 > x_k > x/2$. On the one hand,

$$\mu_i(aS + (x_k - a\mu_i(S))I) = x_k$$

and so $f_i(\mu_i(aS + (x_k - a\mu_i(S))I)) = f_i(x_k)$ is unbounded. On the other hand, for every $j \neq i$ $\lim_{k \to \infty} \mu_j(aS + (x_k - a\mu_i(S))I) =$

$x+a(\mu_j(S)-\mu_i(S))$. Note that $x+a(\mu_j(S)-\mu_i(S)) > \alpha_j+a^2$ whenever $j \neq i$ and hence for k sufficiently large

$$\mu_j(aS + (x_k - a\mu_i(S))I) > \alpha_j + a^2$$

and therefore for every $j \neq i$ the sequence $f_j(\mu_j(aS+(x_k-a\mu_i(S))I)) = f_j(x_k+a(\mu_j(S)-\mu_i(S)))$ is bounded. Thus $\sum_{k=1}^n f_k\circ\mu_k$ is an unbounded game. ∎

The next is a "classic" corollary of the Markov-Kakutani theorem:

Lemma 2. *Let E be the space of all real-valued functions on $\mathbb{R}$ that are majorized in absolute value by some Lebesgue-integrable function. There exists a translation invariant positive linear functional on E extending the Lebesgue integral.*

Proof. Define $p(f)$ for $f \in E$ as the upper-integral: $\inf\{\int g dx \mid g \in \mathscr{L}_1,\ g \geq f\}$. Notice that $p(f+g) \leq p(f)+p(g)$ and $p(\alpha f) = \alpha p(f)$ whenever $f, g \in E$ and $\alpha \geq 0$, and thus the Hahn-Banach theorem yields the existence of a linear functional φ on E with $\varphi \leq p$: φ is a positive linear functional and extends the Lebesgue integral.

For $f \in E$ let $\|f\| = p(|f|)$: this turns E into a semi-normed space. The set of all positive linear functionals that extend the Lebesgue integral is a weak*-compact convex subset C of the unit ball of the dual E', and $C \neq \emptyset$ as just argued.

Let, for $t \in \mathbb{R}$ and $f \in E$, $T_t(f)\colon x \mapsto f(x+t)$: this is an abelian group of isometries of E; the transposes T_t^* are continuous linear maps from C to itself; hence by the Markov-Kakutani theorem (Dunford and Schwartz, 1958, p. 456) there exists a common fixed point in C of all T_t^*: this is a translation invariant extension. ∎

In what follows we fix such a translation invariant extension, $\mathscr{L}$, and for a bounded function g on $\mathbb{R}$, and $a \leq b$ in $\mathbb{R}$, let $\int_a^b g(x)L(dx) = \mathscr{L}(g\mathbb{1}_{[a,b)})$, where $\mathbb{1}_A(x) = 1$ if $x \in A$ and 0 otherwise. The crucial step is the following:

Proposition 1. *For every $n \in \mathbb{N}$, $f_1, \ldots, f_n$ in $'$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ in $(AN_1)^n$,*

$$\left\|\sum\nolimits_{i=1}^n f_i(1)\mu_i\right\| \leq \left\|\sum\nolimits_{i=1}^n f_i\circ\mu_i\right\|$$

Proof. Set $v = \sum_{i=1}^n f_i\circ\mu_i$ and $\varphi v = \sum_{i=1}^n f_i(1)\mu_i$. We must prove $\|\varphi v\| \leq \|v\|$.

We can assume that the right hand member ($\|v\|$) is finite; hence that $\sum_{i=1}^n f_i\circ\mu_i$ is bounded. Since w.l.o.g. $\mu_i \neq \mu_j$ for $i \neq j$, lemma 1 shows then that f_i is bounded.

Obviously, $\varphi v \in AN \subset FA$. For each $u \in FA$, $\|u\| = \sup_{S\in\mathscr{C}}|u(S)| + |u(S^c)|$. It suffices thus to prove that for every coalition $S \in \mathscr{C}$, $|\varphi v(S)| + |\varphi v(S^c)| \leq \|v\|$.

For each positive integer m let $S_0 \subset S_1 \subset \cdots \subset S_m$ and $S_0^c \subset S_1^c \subset \cdots \subset S_m^c$ be measurable subsets of S and $S^c = I \setminus S$ respectively with $\vec{\mu}(S_j) = \frac{j}{m+1}\vec{\mu}(S)$ and $\vec{\mu}(S_j^c) = \frac{j}{m+1}\vec{\mu}(S^c)$. For every $0 \le t \le \frac{1}{m+1}$ let I_t be a measurable subset of $I \setminus (S_m \cup S_m^c)$ with $\vec{\mu}(I_t) = t\vec{\mu}(I)$. Define the increasing sequence of coalitions $T_0 \subset T_1 \subset \dots T_{2m}$ by $T_0 = I_t$, $T_{2j-1} = I_t \cup S_j \cup S_{j-1}^c$ and $T_{2j} = I_t \cup S_j \cup S_j^c$, $j = 1, \dots, m$. Obviously, $\|v\| \ge \sum_{j=1}^{2m} |v(T_j) - v(T_{j-1})| \ge |\sum_{j=0}^{m-1} v(T_{2j+1}) - v(T_{2j})| + |\sum_{j=1}^{m} v(T_{2j}) - v(T_{2j-1})|$. Set $\varepsilon = \frac{1}{m+1}$. Note that $\frac{1}{\varepsilon}\int_0^\varepsilon \sum_{j=0}^{m-1}[\sum_{i=1}^n f_i(t+\varepsilon j+\varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t+j\varepsilon)]L(dt) = \frac{1}{\varepsilon}\int_0^{1-\varepsilon}[\sum_{i=1}^n f_i(t+\varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t)]L(dt) \underset{m\to\infty}{\longrightarrow} \varphi v(S)$, and similarly $\frac{1}{\varepsilon}\int_0^\varepsilon \sum_{j=1}^{m}[\sum_{i=1}^n f_i(t+\varepsilon j) - \sum_{i=1}^n f_i(t+j\varepsilon-\varepsilon\mu_i(S^c))]L(dt) = \frac{1}{\varepsilon}\int_\varepsilon^1[\sum_{i=1}^n f_i(t) - \sum_{i=1}^n f_i(t-\varepsilon\mu_i(S^c))]L(dt) \underset{m\to\infty}{\longrightarrow} \varphi v(S^c)$. As $v(T_{2j+1}) - v(T_{2j}) = \sum_{i=1}^n f_i(t+\varepsilon j+\varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t+j\varepsilon)$, and $v(T_{2j}) - v(T_{2j-1}) = \sum_{i=1}^n f_i(t+2j\varepsilon) - \sum_{i=1}^n f_i(t+2j\varepsilon-\varepsilon\mu_i(S^c))$, we deduce that for each fixed $0 \le t \le \varepsilon$, $|\sum_{j=0}^{m-1}[\sum_{i=1}^n f_i(t+\varepsilon j+\varepsilon\mu_i(S)) - \sum_{i=1}^n f_i(t+j\varepsilon)]| + |\sum_{j=1}^m[\sum_{i=1}^n f_i(t+\varepsilon j) - \sum_{i=1}^n f_i(t+j\varepsilon-\varepsilon\mu_i(S^c))]| \le \|v\|$ and therefore $|\varphi v(S)| + |\varphi v(S^c)| \le \|v\|$. ■

Proof of the Theorem. Consider the linear space Q generated by all games of the form $f \circ \mu$ where $f \in {}'$ and $\mu \in AN_1$. Any $v \in Q$ is of the form $\sum_{i=1}^n f_i \circ \mu_i$ where $f_i \in {}'$ and $\mu_i \in AN_1$. Define $\varphi\colon Q \to AN$ by $\varphi(\sum_{i=1}^n f_i \circ \mu_i) = \sum_{i=1}^n f_i(1)\mu_i$. The proposition implies that φ is well defined, i.e., independent of the representation. Indeed, if $v = \sum_{i=1}^n f_i \circ \mu_i = \sum_{k=1}^m g_k \circ \nu_k$, $0 = \sum_{i=1}^n f_i \circ \mu_i - \sum_{k=1}^m g_k \circ \nu_k \in BV$, and thus by the proposition $\sum_{i=1}^n f_i(1)\mu_i = \sum_{k=1}^m g_k(1)\nu_k$. Efficiency, linearity and symmetry follow now from the definition of φ. Finally, the proposition implies that $\|\varphi v\| \le \|v\|$, so φ can be extended to a linear, efficient and symmetric map $\varphi\colon {}'AN \to AN$ $(\subseteq FA)$ such that $\|\varphi v\| \le \|v\|$. This last property and efficiency imply that φ is positive. ■

4. Comments

4.1. **Continuity at** 0 **and** 1**.** Previous papers on scalar-measure games $f \circ \mu$ assumed continuity of f at 0 and 1 — and this was understood as the definition of $'$. This concept is used however only in the definitions of In' and bv' (cf. above); the former is subsumed by the present paper, and the definition of the latter is not changed here, since functions in bv anyway have limits at 0 and 1.

We could have used any other concept of integral to define the space $'$ — in fact, the only properties we use are linearity, monotonicity, and translation and scale covariance. But the Denjoy integral is applicable to a wider class of functions than any other classical integration theory (Riemann, Lebesgue, …); hence it implies a bigger space $'$. For example, for $\alpha < \beta^+$, $x^{-\alpha}\cos(x^{-\beta}) \in {}'$, while using Lebesgue instead of Denjoy-Perron (or at least Newton) in the definition would further require $\alpha < 1$: the additional absolute summability requirement is clearly irrelevant (and would amount to again sneaking some bv requirement into the definition, this time on the primitive).

A further extension: apply our result to the symmetrized game ($v \mapsto \hat{v}$ where $\hat{v}(S) = \frac{1}{2}(v(S) + v(I) - v(\complement S))$, obtaining thus a value on the sum of the present space and that of all anti-symmetric games ($v(S) = v(\complement S)$). Then, for $f \circ \mu$ to belong to this space, it would suffice that $\lim_{\delta \to 0+} \frac{1}{\delta} \int_0^\delta [f(1-y) - f(y)]dy = f(1)$ in the sense of upper- and lower- Denjoy-integrals (and $f(0) = 0$) — thus defining a larger $'$.

4.2. ***'AN* $\cap$ *BV* = *bv'AN*?** We suspect that maybe $'AN \cap BV = bv'AN$ (equivalently: $'NA \cap BV = bv'NA$), and conceivably even the stronger result: $\sum_i f_i \circ \mu_i \in BV$, where $f_i \in {}'$ and μ_i are distinct elements of AN_1, implies $f_i \in bv'$ $\forall i$. Here we reduce these problems to the case where the f_i's are continuous, and are smooth in the interior of $[0,1]$.

Lemma 3. *If* $v = \sum_{i=1}^n f_i \circ \mu_i \in BV$, *with* $f_i \in {}'$, $\mu_i \in AN_1$, $\mu_i \neq \mu_j$ *for* $i \neq j$, *then:*

(1) $\exists h_i$ *which are continuous on* $[0,1]$ *and* C^∞ *on* $]0,1[$ *such that* $f_i - h_i \in bv'$.

(2) v *is "continuous in variation" at* 0 *(and similarly at* 1*), i.e.* $\forall \varepsilon > 0\ \exists \delta > 0$*: for any ideal coalition* χ *with* $\mu_i(\chi) \leq \delta\, \forall i$ *the variation of* v *on* $[0, \chi]$ *is* $\leq \varepsilon$.

Proof. By lemma 1, all f_i are bounded.

Step 1: $\forall \varepsilon > 0$ f_i has bounded variation on $[\varepsilon, 1-\varepsilon]$:

Fix $S \in \mathscr{C}$ such that $\mu_i(S) \neq \mu_1(S)$ for $i \neq 1$, and set $\rho = \min_{i \neq 1} |\mu_i(S) - \mu_1(S)|$.

The function f_i^ε, defined on $[\varepsilon, 1-\varepsilon]$ by (with $L(d\theta)$ as above)

$$f_i^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^\varepsilon f_i(x + \theta(\mu_i(S) - \mu_1(S)))L(d\theta)$$

is Lipschitz of constant $\frac{2K}{\varepsilon\rho}$ where $K \geq \sup_{0 \leq x \leq 1} |f_i(x)|$ if $i \neq 1$, and $f_1^\varepsilon = f_1$.

For $\varepsilon \leq x_0 < \cdots < x_k \leq 1 - \varepsilon$, and $\alpha_i = \mu_i(S) - \mu_1(S)$,

$$\begin{aligned}
\sum_{j=1}^k \left| f_1(x_j) - f_1(x_{j-1}) \right| &\leq \sum_{j=1}^k \left| \sum_{i=1}^n f_i^\varepsilon(x_j) - \sum_{i=1}^n f_i^\varepsilon(x_{j-1}) \right| \\
&\quad + \sum_{j=1}^k \left| \sum_{i=2}^n f_i^\varepsilon(x_j) - \sum_{i=2}^n f_i^\varepsilon(x_{j-1}) \right| \\
&\leq \sum_{j=1}^k \left| \sum_{i=1}^n \frac{1}{\varepsilon} \int_0^\varepsilon \left[f_i\big(x_j + \theta\alpha_i\big) - f_i\big(x_{j-1} + \theta\alpha_i\big) \right] L(d\theta) \right| + n\frac{2K}{\varepsilon\rho} \\
&\leq \frac{1}{\varepsilon} \int_0^\varepsilon \sum_{j=1}^k \left| \sum_{i=1}^n \left[f_i\big(x_j + \theta\alpha_i\big) - f_i\big(x_{j-1} + \theta\alpha_i\big) \right] \right| L(d\theta) + n\frac{2K}{\varepsilon\rho}
\end{aligned}$$

As for every $0 < \theta < \varepsilon$, the sequence $x_j + \theta S - \theta\mu_1(S)$ is a chain of ideal coalitions, the right-hand side is bounded by $\left\| \sum f_i \circ \mu_i \right\| + 2nK/\varepsilon\rho$. Therefore f_1 has bounded variation on $[\varepsilon, 1-\varepsilon]$.

Step 2: $\exists g_i \in bv'$ such that $h_i = f_i - g_i$ is locally absolutely continuous on $]0,1[$.

Let $h_i^0(x) = \int_{\frac{1}{2}}^x f_i'(y)dy$ (the absolutely continuous part of f), $g_i = f_i - h_i^0$, $f_i^\varepsilon(x) = f_i(\varepsilon + (1-2\varepsilon)x) - f_i(\varepsilon)$, and similarly g_i^ε; by step 1, $f_i^\varepsilon \in bv$. If f_i is continuous at ε and $1-\varepsilon$, $f_i^\varepsilon \in bv'$. Given a chain $S_1 \subset \cdots S_k$, $\chi_i = \varepsilon + (1-2\varepsilon)S_i$ is a chain of ideal coalitions and $f_i^\varepsilon(\mu_i(S_j)) = f_i(\mu_i(\chi_j))$, so:

$$\left\|\sum f_i^\varepsilon \circ \mu_i\right\| \leq \left\|\sum f_i \circ \mu_i\right\|$$

By Aumann and Shapley (1974, 8.17, p. 65), $\|\sum f_i \circ \mu_i\| \geq \|\sum f_i^\varepsilon \circ \mu_i\| \geq \sum \|g_i^\varepsilon\|$ and thus $\|g_i^\varepsilon\|$ is bounded in ε and $g_i \in bv$. Therefore, g_i has limits at 0 and 1. Hence defining $\bar{g}_i(x) = g_i(x) - \lim_{y\to 0+} g_i(y)$, $\bar{g}_i(0) = 0$, and $\bar{g}_i(1) = \lim_{x \to 1-} \bar{g}_i(x)$, $\bar{g}_i \in bv'$. Setting $h_i = f_i - \bar{g}_i$ we conclude that h_i is absolutely continuous on $]0,1[$.

Step 3: Smoothing h_i.

For $n = 1, \cdots$, let h_i^n be a smooth function on an open neighborhood of $[2^{-n}, 1-2^{-n}]$ that coincides on $[2^{-(n-1)}, 1-2^{-(n-1)}]$ with h_i^{n-1} and at 2^{-n} and $1-2^{-n}$ with h_i, and whose variation distance to h_i on this open neighborhood is $\leq 1-2^{-n}$. Then h_i^∞ is C^∞ on $]0,1[$, and with $g_i = h_i - h_i^\infty$, $\|g_i\| \leq 1$, so g_i has limits at 0 and 1: extend g_i to $[0,1]$ by those limits, then subtract $g_i(0)$ from it: we have a function $g_i \in bv'$ such that $h_i - g_i$ is C^∞ on $]0,1[$.

Step 4: Continuity of h_i.

For (1), it remains to prove continuity at 0 and 1, say of h_1 at 0. Otherwise, e.g., $\limsup_{x\to 0+} h_1(x) > 0$ (or change the sign of the game). Then choose $0 < \beta < \limsup_{x \to 0+} h_1(x)$, and a sequence x_i decreasing to 0 such that $h_1(x_i) > \beta$. Let $y_i = \min\{\, x \mid h_1(y) \geq \beta/2 \text{ for } x \leq y \leq x_i \,\}$. By continuity, the min is achieved and $y_i \leq x_i$, and $h_1 \in ', h_1(0) = 0$ imply $y_i > 0$. So, for a subsequence, $x_{i+1} < y_i$.

Let $\chi(z) = z\big((1-\theta\mu_1(S))I + \theta S\big)$, and $H_i(z) = \int_0^1 (h_i \circ \mu_i)(\chi(z))d\theta$: $H_1 = h_1$ and for $i \neq 1$ $H_i(z) = \frac{1}{z(\mu_i(S) - \mu_1(S))} \int_z^{z(1+\mu_i(S)-\mu_1(S))} h_i(x)dx$ converges with z to 0 since $h_i \in '$. Assume w.l.o.g. that $\forall x \leq x_1, |H_i(x)| < \beta/(8n)$.

Now let $\chi_{2k-1} = \chi(x_k)$, $\chi_{2k} = \chi(y_k)$: $\forall \theta \in [0,1]$, the χ_k form a decreasing chain of ideal coalitions (assuming $x_1 \leq 1/2$). Hence $\forall \theta, \sum_k |v(\chi_{2k-1}) - v(\chi(x_k))| \leq \|v\|$, so, by Jensen, taking expectations inside w.r.t. θ, $\sum_k |\sum_i (H_i(x_k) - H_i(y_k))| \leq \|v\|$. But $H_1(x_k) - H_1(y_k) = h_1(x_k) - h_1(y_k) \geq \beta/2$ by construction, and for $i \neq 1$ $H_i(x_k) - H_i(y_k) > -\beta/(4n)$, so that $\sum_i (H_i(x_k) - H_i(y_k)) > \beta/4$, a contradiction.

Step 5: Continuity in variation.

By (1), it suffices to prove this when $f_i = h_i$, since a game $g \circ \mu$ with $g \in bv'$ and $\mu \in AN_1$ is clearly continuous in variation at 0 and 1, and this continuity is preserved when summing games. If the result were not true, there would be a sequence χ_k and $\varepsilon > 0$ such that

$\mu(\chi_k) \to 0$ (with $\mu = \sum_i \mu_i$) and $\forall k$ $\mathrm{Var}(v)[0,\chi_k] > \varepsilon$. Now fix a chain χ'_j with variation $> \|v\| - \varepsilon$. Let $\chi^k_j = \mathsf{max}(\chi'_j, \chi_k)$. Observe that $0 \le \mu_i(\chi^k_j) - \mu_i(\chi'_j) \le \mu_i(\chi^k) \to 0$; hence, by continuity of h_i (step 4), for k sufficiently large the variation of v on the chain χ^k_j is $> \|v\| - \varepsilon$. Take a chain $\chi''_l \le \chi_k$ with variation $> \varepsilon$. Then the variation of v on the chain consisting of the χ''_l followed by the χ^k_j is $> \|v\|$: a contradiction. ∎

Corollary 1. *$'AN \cap BV \subseteq bv'AND$ where $bv'AND$ is the closed space spanned by $bv'AN$ and all games of bounded variation that vanish on an AN-diagonal neighborhood.*

Proof. By lemma 3 any game in $'NA \cap BV$ is approximated by the sum of a game in $bv'AN$ and a game $v = \sum_{i=1}^n f_i \circ \mu_i$ where the f_i are continuous on $[0,1]$ and smooth on its interior. It suffices to prove that $v \in bv'AND$. Fix $\varepsilon > 0$, and by the previous lemma take $\delta > 0$ such that $\mathrm{Var}(v)[0,\chi] < \varepsilon$ whenever $\mu(\chi) < \delta$. Let $g\colon \mathbb{R} \to [0,1]$ be a smooth monotone function with $g(x) = 0$ for $x \le \delta/2$ and $g(x) = 1$ for $x \ge \delta$. Define $w = (g \circ \mu) \times v$. It follows that for χ with $\mu(\chi) \ge \delta$, $w(\chi) = v(\chi)$, and for χ with $\mu(\chi) = \delta$, $\mathrm{Var}(w)[0,\chi] \le 2\mathrm{Var}(v)[0,\chi]$. Therefore the variation of $v - w$ is bounded by 3ε. Also, w is smooth on a neighborhood of $[0,1/2]$. Handling the neighborhood of 1 similarly, we can approximate the game v by a game w which is smooth on a neighborhood of the diagonal, hence in $bv'AND$. ∎

The equalities $'NA \cap BV = bv'NA$ and $'AN \cap BV = bv'AN$ may depend on the space of players $(I, \mathscr{C})$. To state $'NA \cap BV = bv'NA \Leftrightarrow {'AN} \cap BV = bv'AN$, given that spaces depend on $(I, \mathscr{C})$:

Proposition 2. *If $'NA \cap BV = bv'NA$ ($\neq \{0\}$) or $'AN \cap BV = bv'AN$ ($\neq \{0\}$) for some $(I, \mathscr{C})$ where NA or AN respectively are $\neq \{0\}$, then both equalities hold for all $(I, \mathscr{C})$.*

Proof. Step 1: Fix a player set $(I, \mathscr{C})$ and finitely many elements $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_k$ in AN_1. We prove first that if f_i, $1 \le i \le n$, are continuous on $[0,1]$ and smooth on its interior, and $g_j = g^s_j + g^{ac}_j \in bv'$, $1 \le j \le k$, with g^{ac}_j absolutely continuous and g^s_j singular, then

$$\Big\|\sum\nolimits_{i=1}^n f_i \circ \mu_i - \sum\nolimits_{j=1}^k g^{ac}_j \circ \nu_j\Big\| \le \Big\|\sum\nolimits_{i=1}^n f_i \circ \mu_i - \sum\nolimits_{j=1}^k g_j \circ \nu_j\Big\|$$

Indeed, let $\Omega\colon \chi_0 \le \ldots \le \chi_m$ be an increasing chain of ideal coalitions so that $\|\sum_{i=1}^n f_i \circ \mu_i - \sum_{j=1}^k g^{ac}_j \circ \nu_j\|_\Omega > \|\sum_{i=1}^n f_i \circ \mu_i - \sum_{j=1}^k g^{ac}_j \circ \nu_j\| - \varepsilon$. Using the continuity of the functions f_i and g^{ac}_j we may assume w.l.o.g. that there is $\delta > 0$ such that $\delta < \chi_0 \le \chi_m < 1 - \delta$. Consider the increasing path of ideal coalitions $\chi(t) = \chi_j + (mt - j)(\chi_{j+1} - \chi_j)$ for $j/m \le t \le (j+1)/m$, $0 \le t \le 1$. The function $t \mapsto \sum_{i=1}^n f_i(\mu_i(\chi(t))) - \sum_{j=1}^k g_j(\nu_j(\chi(t)))$ is a sum of an absolutely continuous function $t \mapsto \sum_{i=1}^n f_i(\mu_i(\chi(t))) - \sum_{j=1}^k g^{ac}_j(\nu_j(\chi(t)))$ and a singular function $t \mapsto -\sum_{j=1}^k g^s_j(\nu_j(\chi(t)))$ and therefore its variation over $[0,1]$ is $\ge \|\sum_{i=1}^n f_i \circ \mu_i - \sum_{j=1}^k g^{ac}_j \circ \nu_j\|_\Omega \ge \|\sum_{i=1}^n f_i \circ \mu_i - \sum_{j=1}^k g^{ac}_j \circ \nu_j\| - \varepsilon$.

Step 2: Every game $v = f \circ \vec{\mu} \in bv'NA$, where $\vec{\mu} = (\mu_1, \dots, \mu_n)$ is a vector of NA_+ elements, can be approximated by a game $w = \sum g_i \circ \nu_i$ such that all $\nu_i \in NA_+$ are dominated by $\mu = \sum \mu_j$: indeed, if $w = \sum g_i \circ \nu_i$ with $\nu_i \in NA_+$, let $\nu = \sum \nu_j$ and $B = \{\, x \mid d\mu/d(\mu+\nu)(x) > 0 \,\}$ and set $\tilde{\nu}_i(C) = \nu(C \cap B)$. Let $\tilde{w} = \sum g_i \circ \tilde{\nu}_i$. Then the variation of $v - \tilde{w}$ over a chain $C_1 \subseteq \cdots \subseteq C_n$ equals that of $v - w$ over $C_1 \cap B \subseteq \cdots \subseteq C_n \cap B$; hence $\|v - \tilde{w}\| \le \|v - w\|$. Modify g_i to be left-continuous at $\tilde{\nu}_i(I)$ if needed.[1]

Step 3: We now prove that if $'NA \cap BV = bv'NA \neq \{0\}$ for $(I', \mathscr{C}')$, then $'NA \cap BV = bv'NA$ for $(I, \mathscr{C})$. Fix $\nu \in NA_1$ on $(I', \mathscr{C}')$. It suffices, by lemma 3, to prove that $v = \sum_{i=1}^n f_i \circ \mu_i \in bv'NA$ when $v \in BV$, $\mu_i \in NA_1(I, \mathscr{C})$, and the f_i are continuous on $[0,1]$ and smooth on its interior. Let μ be the average of the μ_i. Let f be a Radon-Nikodym derivative of the vector μ_i w.r.t. μ. Further, for each atom x_k of the distribution of f under μ, let $I_k = f^{-1}(x_k)$, and construct, for each rational $r \in [0,1]$, a measurable subset I_k^r of I_k with $I_k^r \subseteq I_k^s$ for $r < s$ and $\mu(I_k^r) = r\mu(I_k)$. Let $\mathscr{C}_0$ be the separable sub-σ-field of $\mathscr{C}$ spanned by f and the I_k^r. Similarly, let $\mathscr{C}_0'$ be a separable sub-σ-field of $\mathscr{C}'$ on which ν is still non-atomic. The separable measure algebras $\langle \mathscr{C}_0, \mu \rangle$ and $\langle \mathscr{C}_0', \nu \rangle$ are isomorphic. This isomorphism induces an isometry h from $L_1(\mathscr{C}_0, \mu)$ to $L_1(\mathscr{C}_0', \nu)$. The isomorphism h induces maps H and H' acting on all measures absolutely continuous with respect to μ respectively ν to those absolutely continuous with respect to ν and μ respectively such that $H(\xi) = h(\frac{d\xi}{d\mu}|_{\mathscr{C}_0}) d\nu$ and $H'(\eta) = h^{-1}(\frac{d\eta}{d\nu}|_{\mathscr{C}_0'}) d\mu$. The isometry of the L_1 spaces induces one of their duals L_∞ and therefore preserves all (relevant, i.e., $\mathscr{C}_0$-measurable) chains. It maps the game v to a game $h(v) = \sum_{i=1}^n f_i \circ H(\mu_i) \in BV$ on $(I', \mathscr{C}')$. Therefore by our assumption $h(v) \in bv'NA$ and thus, by step 2, it can be approximated by a game $w = \sum_{j=1}^k g_j \circ \nu_j$ with ν_j dominated by ν. Therefore, the game $h^{-1}(w) = \sum_{j=1}^k g_j \circ H'(\nu_j)$ approximates the game v.

Step 4: It remains to show that $\forall (I, \mathscr{C}) \exists (I', \mathscr{C}')$ such that $'AN \cap BV = bv'AN \neq \{0\}$ on $(I, \mathscr{C})$ iff $'NA \cap BV = bv'NA \neq \{0\}$ on $(I', \mathscr{C}')$. For this, let I' be the Stone space S of $B(I, \mathscr{C})$, i.e., a compact space whose algebra of continuous functions, $C(S)$, is isomorphic to $B(I, \mathscr{C})$. Endow S with the (Baire) σ-field $\mathscr{C}'$ spanned by the continuous functions. Every measure $\lambda \in AN(I, \mathscr{C})$ thus becomes a continuous linear functional $h(\lambda)$ on $C(S)$, i.e., by Riesz's theorem h induces an isometry between $AN(I, \mathscr{C})$ and $NA(I', \mathscr{C}')$. It maps a game $v = \sum_{i=1}^n f_i \circ \mu_i$ on $(I, \mathscr{C})$ to a game $h(v) = \sum_{i=1}^n f_i \circ h(\mu_i)$ on $(I', \mathscr{C}')$ with $\|h(v)\| = \|v\|$ when the functions f_i are continuous, and thus $h(v) \in BV$. If $\sum_{j=1}^k g_j \circ \nu_j$ approximates $h(v)$ then $\sum_j g_j \circ h^{-1}(\nu_j)$ approximates v which proves the "if" part, (without any need for step 2). For the converse, h^{-1} maps the game $v = \sum_{i=1}^n f_i \circ \mu_i$ on $(I', \mathscr{C}')$ to a game $h^{-1}(v) = \sum_{i=1}^n f_i \circ h^{-1}(\mu_i)$

[1]Tauman (1982) has a much sharper result for the case $v \in pNA$.

on $(I, \mathscr{C})$ with $\|h^{-1}(v)\| \leq \|v\|$, and thus $h^{-1}(v) \in BV$. Assuming $'AN \cap BV = bv'AN$, there is a finite sum $\sum_{j=1}^{k} g_j \circ \nu_j$ with continuous functions g_j that approximate $h^{-1}(v)$. As the functions f_i are also continuous, $\sum_{j=1}^{k} g_j \circ h(\nu_j)$ approximate v: the variation of the game $v - \sum_{j=1}^{k} g_j \circ h(\nu_j)$ over a chain of ideal coalitions in $(I', \mathscr{C}')$ is approximated by its variation over a chain of continuous ideal coalitions and thus $\|v - \sum_j g_j \circ h(\nu_j)\| = \|h^{-1}(v) - \sum_j g_j \circ \nu_j\|$. ■

Remarks: 1) To have $AN_1 \neq \emptyset$, it is necessary and sufficient that $\#\mathscr{C} = \infty$. The necessity is clear; for the sufficiency, $\#\mathscr{C} = \infty$ implies the existence of a sequence x_n in I such that $n \neq m \Rightarrow \exists C \in \mathscr{C} : x_n \in C, x_m \notin C$, and hence of a countable measurable partition C_n such that $\forall n, x_n \in C_n$. For $f \in B(I, \mathscr{C})$ let $p(f) = \limsup_{n\to\infty} \sum_{i=1}^{n} f(x_i)/n$, and choose by Hahn-Banach a linear functional μ on $B(I, \mathscr{C})$ with $\mu \leq p$. The linear functional μ is a finitely additive measure on $(I, \mathscr{C})$. For every n $\mu(\left(\bigcup_{k=0}^{\infty} C_{kn+i}\right) \leq 1/n$ and $\left(\bigcup_{k=0}^{\infty} C_{kn+i}\right)_{i=1}^{n}$ is a finite measurable partition of I. Therefore, μ is non-atomic.

2) We have no such characterization for NA for general $(I, \mathscr{C})$, but: For the existence of a non-atomic, non-null regular borel measure on a Hausdorff space X, it is necessary and sufficient that X has a compact perfect subset. For the necessity, by regularity we have then such a measure with compact support, and by non-atomicity this support must be perfect. For the sufficiency we can assume X compact and perfect, observe then that the set of probability measures having at least one atom of mass $\leq 1/n$ is closed in the weak* topology, and with empty interior by perfectness. and use the Baire category theorem.

Proposition 3. *For every player set with $\#\mathscr{C} = \infty$ the following are equivalent:*
1) If $\sum_{i=1}^{n} f_i \circ \mu_i \in BV$ with $(f_i, \mu_i) \in ' \times AN_1$ and $\mu_i \neq \mu_j$ for $i \neq j$, then $\forall i, f_i \in bv'$.
2) If $F\colon [0,1]^2 \to \mathbb{R}\colon (x, y) \mapsto \sum_{i=1}^{n} f_i(a_i x + (1 - a_i)y)$ has bounded variation, where $f_i \in C([0,1])$ are smooth on $]0,1[$ and $0 < a_1 < \cdots < a_n < 1$, then $\forall i, f_i \in bv'$.

Proof. $\#\mathscr{C} = \infty$ implies that there are two mutually singular measures in AN_1. So $1 \Rightarrow 2$ is obvious. For $2 \Rightarrow 1$, assume $v = \sum_{i=1}^{n} f_i \circ \mu_i \in BV$ with $(f_i, \mu_i) \in ' \times AN_1$ and $\mu_i \neq \mu_j$ for $i \neq j$. We have to prove that each f_i has bounded variation. By lemma 3 we can assume w.l.o.g. that the f_i are continuous on $[0,1]$ and smooth on its interior. There is a coalition $S \in \mathscr{C}$ such that $0 \neq \mu_i(S) \neq \mu_j(S) \neq 1$ if $i \neq j$. Set $a_i = \mu_i(S)$. As v has bounded variation over ideal coalitions and $v(xS + y\complement S) = F(x, y) = \sum_{i=1}^{n} f_i(a_i x + (1 - a_i)y)$, the function F has bounded variation over the square $[0,1]^2$ and thus by (2), each f_i has bounded variation. ■

Proposition 4. *If the player set is standard Borel, the equality* $'NA \cap BV = bv'NA$ *implies that there is a unique value on* $'NA$.

Proof. Let φ be a value on $'NA$. By assumption any game $v \in {}'NA$ is a sum of a game $u \in bv'NA$ and a finite sum $\sum_{i=1}^{n} f_i \circ \mu_i$ with $f_i \in {}'$ and $\mu_i \in NA_1$. Thus, if ψ is the unique value on $bv'NA$, $\varphi v = \psi u + \sum_{i=1}^{n} f_i(1)\mu_i$, i.e. φv is uniquely defined. ■

Even when $(I, \mathscr{C})$ is standard Borel, there are many values of norm 1 on $'AN$, that differ already on pAN. E.g., decompose every μ in AN_+ as $\mu^1 + \mu^2$, with μ^1 carried by a countable set and μ^2 vanishing on each such set. Fix a smooth increasing path (cf. Hart, 1973; Haimenko, 2000) $\gamma\colon [0,1] \to [0,1]^2$ with $\gamma(0) = 0$, $\gamma(1) = 1$. For simplicity, assume γ is affine in a neighborhood of 0 and of 1. For f continuous on $[0,1]$ and C^1 on its interior, let $[\varphi_\gamma(f \circ \mu)](S) = \int_0^1 f'(\gamma_1(t)\mu^1(1) + \gamma_2(t)\mu^2(1))[\gamma_1'(t)\mu^1(S) + \gamma_2'(t)\mu^2(S)]dt$, as a Denjoy integral. For $f \in bv'$, let $[\varphi_\gamma(f \circ \mu)](S) = \mathsf{lim}_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 [f(\mathsf{min}(1, \gamma_1(t)\mu^1(1) + \gamma_2(t)\mu^2(1) + \epsilon)) - f(\gamma_1(t)\mu^1(1) + \gamma_2(t)\mu^2(1))][\gamma_1'(t)\mu^1(S) + \gamma_2'(t)\mu^2(S)]dt$. Observe that both formulas coincide on the intersection of those 2 spaces, hence define by linearity $\varphi_\gamma(f \circ \mu)$ uniquely for all f in their sum X. Fix a Hamel basis for X, and complete it to a Hamel basis of $'$: the additional basis vectors span a space Y, such that every $f \in {}'$ has a unique decomposition $f = f^1 + f^2$ with $f^1 \in X$ and $f^2 \in Y$. Define then $\varphi_\gamma(f \circ \mu) = \varphi_\gamma(f^1 \circ \mu) + f^2(1)\mu$. Now $\| \sum_i \varphi_\gamma(f_i \circ \mu_i)\| \le \| \sum_i f_i \circ \mu_i \|$: indeed, if the right hand member is finite, lemma 3.1 implies that $f_i \in X \, \forall i$; the verification is then straightforward. This defines therefore φ_γ as a value of norm 1 on a dense subspace of $'AN$, hence on $'AN$.

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