Locally finite knowledge structures

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Abstract: With respect to the $S5$ multi-agent epistemic logic, we define a cell to be a minimal subset of knowledge structures known in common semantically by all the agents. A cell has finite fanout if at every knowledge structure every agent considers only a finite number of other knowledge structures to be possible. A set of formulas in common knowledge is finitely generated if the common knowledge of some finite subset implies the common knowledge of the whole set. For every finitely generated set of formulas held in common knowledge at some knowledge structure either this set determines uniquely a finite cell or there are uncountable many cells of finite fanout (and also uncountably many cells of uncountable size) at which exactly this set of formulas is known in common. The situation is very different, however, for sets of formulas held in common knowledge that are not finitely generated – if there are uncountably many corresponding cells then either none of these cells or all of them could have finite fanout.

Key words: Cantor sets, Baire Category, Modal Logic, Common Knowledge

1 Introduction

In situations of incomplete information, what could be simpler than all agents in all situations knowing that there are only a finite number of other possibilities (and of course that the structure of possibilities is common knowledge)? This appears to be the condition closest to the common knowledge of a finite number of possibilities. Given that there are at least two agents, the answer is that it can be exceedingly complicated, far beyond that of the common knowledge of a finite number of possibilities.

The property we are interested in, that in every possible situation every agent considers only a finite number of other points to be possible, we call *finite fanout* (Fagin 1994). In Simon (1999) we showed that there exists a semantic model for the $S5$ multi-agent logic with finite fanout for which the tautologies are the only formulas known in common. Since there are also such semantic models without finite fanout, we showed that finite fanout, though a property of some form of common knowledge, is not carried by the formulas held in common knowledge.

Throughout this paper we will assume the $S5$ multi-agent logic, and a finite number of agents and primitive propositions.

Let Ω be the space of maximally consistent sets of formulas that are generated by the agents and primitive propositions; a point of Ω is also called a *knowledge structure*, (Fagin, Halpern, and Vardi 1991). For every agent j let $\mathcal{Q}^j$ be its knowledge partition of Ω, where the knowledge of an agent is defined to be its knowledge of formulas. Define a *possibility set* to be a member of $\mathcal{Q}^j$ for some agent j. Let $\mathcal{Q} := \wedge_j \mathcal{Q}^j$ be the meet partition of the $\mathcal{Q}^j$. A member of $\mathcal{Q}$ we call a *cell.* To every cell $C \in \mathcal{Q}$ corresponds a set of formulas $F(C)$, the set of formulas held in common knowledge at any point in C. A collection of formulas held in common knowledge is *finitely generated* if the common knowledge of a finite subset implies the common knowledge of the whole collection. Every semantic model with finite fanout for the same agents and primitive propositions is equivalent to a cell of Ω, once duplications of essentially equivalent points are removed; (see the next section).

In this paper we prove that if C is a cell and $F(C)$ is finitely generated then either C is finite and there is no other cell C' with $F(C) = F(C')$ **or** there is a continuum of distinct cells C' with continuum cardinality such that $F(C') = C$ and there is a continuum of distinct cells C' of finite fanout such that $F(C') = C$.

What happens if the set of formulas held in common knowledge is not finitely generated? Almost anything is possible. If there are uncountably many distinct cells with the same set of formulas in common knowledge, all of them or none of them may have finite fanout. Furthermore we investigate the difference between uncountably many cells with the same set of formulas in common knowledge and uncountably many distinct semantic models that live inside the same cell.

Behind our main results is a hierarchical construction of Ω. Every formula has a "depth", an inductively defined natural number representing the extent that the knowledge operators of the agents are used to define the formula. (Formulas of depth zero are those of the conventional propositional calculus, constructed without the knowledge operators of any agents.) For every natural number i there is a finite semantic model Ω_i that represents the knowledge of the agents up to the depth of i. Furthermore Ω is the inverse limit of the Ω_i, meaning that a point in Ω is defined by a sequence of extensions from Ω_i to Ω_{i+1} for all the i. If the set of formulas held in common knowledge is finitely generated, then these formulas have a maximal depth d. By exploiting the choices in how one could extend a point in Ω_i to Ω_{i+1} for some of the i that are greater than d, one can construct cells in the limit of the process that do or do not have finite fanout. If the corresponding set of formulas held in common knowledge is not finite generated, the lack of a maximal depth for a generating subset renders the hierarchical construction meaningless for our purposes.

Constructing uncountably many cells with finite fanout is more difficult than constructing cells of uncountable size, for two reasons. In order to construct a cell of finite fanout dense in all of Ω, the average size of the possibility sets radiating from any starting point must grow very fast. It is much easier to satisfy the density requirement if these possibility sets are allowed to be infinite in size – even easier if some of them are Cantor sets. Second, the formulas in common knowledge can be a hindrance. To construct a cell of finite fanout dense in Ω, we spliced together an infinite sequence of finite semantic models (Simon 1999). For each formula that could be true was associated one of these models and a point in the model where this formula was true. The added connections to the other semantic models were distant enough from this point that after the splicing each formula was still true at its corresponding point. To copy the same approach with a fixed formula in common knowledge one must keep this formula true at **all** points after the new connections are made, not only in some select places. We suspect that this can be done, but then one must show also that uncountably many distinct cells can be reached in this way.

What is important about our main results? Finite fanout implies that the semantic model is countable in size – because a countable union of countable sets is also countable. Because one can build or reconstruct a structure of finite fanout inductively, working from the reference of a given starting point, one could suspect that all structures of finite fanout could be defined by the formulas held in common knowledge. But our result shows that this is impossible – there are too many structures of finite fanout inhabiting essentially the same space. That such complexity is possible for cells with uncountable size is easier to believe.

Our results further an understanding of the radical difference between knowledge and common knowledge, essential to understanding the problems of games of incomplete information. The importance of this distinction has been demonstrated by the discovery of a game played on a Mertens-Zamir belief space (Mertens and Zamir, 1985) with a common prior such that there are equilibria but none that are measurable with respect to the completion of the common prior, (Simon, 2001b). When reduced to its combinatorial structure, this belief space corresponds to the union of uncountably many cells of finite fanout.

The rest of the paper is organized as follows. Section 2 provides background information. Sections 3 and 4 contain the proofs of the two main claims, Theorem 1 and Theorem 2. The last section discusses the lack of finite generation.

2 Background

2.1 Formulas

Construct the set $\mathcal{L}(X, J)$ of formulas using the finite sets X and J in the following way:
1) If $x \in X$ then $x \in \mathcal{L}(X, J)$,
2) If $g \in \mathcal{L}(X, J)$ then $(\neg g) \in \mathcal{L}(X, J)$,
3) If $g, h \in \mathcal{L}(X, J)$ then $(g \wedge h) \in \mathcal{L}(X, J)$,

4) If $g \in \mathcal{L}(X,J)$ then $k_j g \in \mathcal{L}(X,J)$ for every $j \in J$,
5) Only formulas constructed through application of the above four rules are members of $\mathcal{L}(X,J)$.
We write simply $\mathcal{L}$ if there is no ambiguity. We define $g \vee h$ to be $\neg(\neg g \wedge \neg h)$ and $g \rightarrow h$ to be $\neg g \vee h$. $E_L(f) = E_L^1(f)$ is defined to be $\wedge_{j \in J} k_j f$, $E^0(f) := f$, and for $i \geq 1$, $E^i(f) := E(E^{i-1}(f))$.

A formula $f \in \mathcal{L}(X,J)$ is common knowledge in a subset of formulas $A \subseteq \mathcal{L}(X,J)$ if $E^n f \in A$ for every $n < \infty$.

Throughout this paper, the multi-agent epistemic logic $S5$ will be assumed. For a discussion of the $S5$ logic, see Cresswell and Hughes (1968); and for the multi-agent variation, see Halpern and Moses (1992) and also Bacharach, et al, (1997).

A set of formulas $\mathcal{A} \subseteq \mathcal{L}(X,J)$ is called *complete* if for every formula $f \in \mathcal{L}(X,J)$ either $f \in \mathcal{A}$ or $\neg f \in \mathcal{A}$. A set of formulas is called *consistent* if no finite subset of this set leads to a logical contradiction, meaning a deduction of f and $\neg f$ for some formula f. We define

$$\Omega(X,J) := \{S \subseteq \mathcal{L}(X,J) \mid S \text{ is complete and consistent}\}.$$

For every agent $j \in J$ we define its *knowledge partition* $\mathcal{Q}^j(X,J)$ to be the partition of $\Omega(X,J)$ generated by the inverse images of the function $\beta^j : \Omega \rightarrow 2^{\mathcal{L}(X,J)}$, the set of subsets of $\mathcal{L}(X,J)$, defined by $\beta^j(z) := \{f \in \mathcal{L}(X,J) \mid k_j f \in z\}$. We will write $\mathcal{Q}^j$ if there is no ambiguity. A *possibility set* is defined to be a member of $\mathcal{Q}^j$ for some $j \in J$ and a cell is a member of the meet partition $\mathcal{Q} := \wedge_{j \in J} \mathcal{Q}^j$.

The following central lemma is in Simon (1999), but all the components of the proof can be found in other papers (Lemma 4.1 of Halpern and Moses 1992, Aumann 1999):

Lemma A: For any cell C of $\Omega(X,J)$ $\{f \in \mathcal{L}(X,J) \mid f$ is common knowledge in z for some $z \in C\} = \{f \in \mathcal{L}(X,J) \mid f$ is common knowledge in z for all $z \in C\} = \{f \in \mathcal{L}(X,J) \mid f \in z$ for all $z \in C\}$.

We define a topology for Ω, the same as in Samet (1990). For every $f \in \mathcal{L}$ define $\alpha(f) := \{z \in \Omega \mid f \in z\}$. Let $\{\alpha(f) \mid f \in \mathcal{L}\}$ be the base of open sets of Ω. (The set of open sets is defined to be the arbitrary unions of base elements. A topology is defined by the fact that $\alpha(f) \cap \alpha(g) = \alpha(f \wedge g)$). The topology of a subset A of Ω will be the relative topology for which the open sets of A are $\{A \cap O \mid O$ is an open set of $\Omega\}$. A dense subset of D is a subset F such that every open set that intersects D also intersects F. An isolated point of a set D is a point $x \in D$ such that there exists a open set O with $\{x\} = O \cap D$.

Due to Lemma A, we have a map F from the meet partition $\mathcal{Q}$ to subsets of formulas defined by $F(C) := \{f \mid f$ is common knowledge in any (equivalently all) members of $C\}$.

For any subset of formulas $T \subseteq \mathcal{L}$ define $\underline{Ck}(T) := \{f \in \mathcal{L} \mid$ there exists an $i < \infty$ and a finite set $T' \subseteq T$ with $(\wedge_{t \in T'} E^i(t)) \rightarrow f$ a tautology $\}$. We define $\mathcal{T}(X, J) = \{\underline{Ck}(T) \mid T \subseteq \mathcal{L}(X, J)\} \setminus \{\mathcal{L}(X, J)\}$, and we say that T generates $\underline{Ck}(T)$. If there is no ambiguity, we can write simply $\mathcal{T}$. $\underline{Ck}(T)$ is the set of formulas whose common knowledge is implied by the common knowledge of the formulas in T.

An $S \in \mathcal{T}$ is finitely generated if there exists a finite subset $T \subseteq S$ such that $\underline{Ck}(T) = S$. For every set of formulas $T \subseteq \mathcal{L}$ define the set

$$\mathbf{Ck}(T) := \{z \in \Omega \mid \text{ every member of } T \text{ is common knowledge in } z\}.$$

For any $T \subseteq \mathcal{L}$, $\mathbf{Ck}(T)$ is a closed set, since the $\mathbf{Ck}(T)$ is the intersection of the sets $\alpha(E^l f)$ for all $l < \infty$ and all formulas f in T.

A cell C is defined to be *centered* if and only if there is no other cell C' with $F(C') = F(C)$.

In Simon (1999) we proved that if a cell C is not centered then there are uncountably many other cells C' such that $F(C') = F(C)$. Since $\underline{Ck}(T) = F(C)$ means that the cell C is dense in $\mathbf{Ck}(T)$; the cell C being not centered is equivalent to the existence of uncountably many other cells C' that are also dense in $\mathbf{Ck}(F(C)) = \mathbf{Ck}(T)$.

2.2 Semantic Models

For this paper a semantic model is a quintuple $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ where J is a set of agents, for each $j \in J$ $\mathcal{P}^j$ is a partition of the set S, X is a set of primitive propositions, and $\psi : X \rightarrow 2^S$ is a map from X to the subsets of S, such that for every $x \in X$ the set $\psi(x)$ is interpreted to be the subset of S where x is true. We define a map $\alpha^{\mathcal{K}} : \mathcal{L}(X, J) \rightarrow 2^S$ inductively on the structure of the formulas in the following way:
Case 1 $f = x \in X$**:** $\alpha^{\mathcal{K}}(x) := \psi(x)$.
Case 2 $f = \neg g$**:** $\alpha^{\mathcal{K}}(f) := S \setminus \alpha^{\mathcal{K}}(g)$,

Case 3 $f = g \wedge h$**:** $\alpha^{\mathcal{K}}(f) := \alpha^{\mathcal{K}}(g) \cap \alpha^{\mathcal{K}}(h)$,
Case 4 $f = k_j(g)$**:** $\alpha^{\mathcal{K}}(f) := \{s \mid s \in P \in \mathcal{P}^j \Rightarrow P \subseteq \alpha^{\mathcal{K}}(g)\}$.
We define a map $\phi^{\mathcal{K}} : S \to \Omega(X, J)$ (see Fagin, Halpern, and Vardi 1991) by

$$\phi^{\mathcal{K}}(s) := \{f \in \mathcal{L}(X, J)) \mid \ s \in \alpha^{\mathcal{K}}(f)\}.$$

We are justified in using again the notation α for the following reason. Consider the map $\overline{\psi} : X \to 2^{\Omega}$ defined by $\overline{\psi}(x) := \{z \in \Omega \mid x \in z\}$. We have a semantic model $\Omega = (\Omega; J; \mathcal{Q}^1, \ldots, \mathcal{Q}^n; X; \overline{\psi})$. (Due to its canonical nature, we index this semantic model with Ω.)

Theorem: For every $f \in \mathcal{L}(X, J)$, f is a theorem of the multi-agent S5 logic if and only if f is a tautology. Furthermore, $\phi^{\Omega}(z) = z$ for every $z \in \Omega$.

For a proof of the first part of this theorem, see Halpern and Moses (1992) and Cresswell and Hughes (1968), and for how the second part follows from the first part see Aumann (1999). We will call this result the "Completeness Theorem."

For a semantic model $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$, if $s \in \alpha^{\mathcal{K}}(f)$, or equivalently $f \in \phi^{\mathcal{K}}(s)$, we say that f is true at s with respect to $\mathcal{K}$. We say that f is valid with respect to the semantic model $\mathcal{K}$ if f is true at s with respect to $\mathcal{K}$ for every $s \in S$. The semantic model is *connected* if the meet partition $\wedge_{j \in J} \mathcal{P}^j$ is a singleton (equal to $\{S\}$). We define a *connected component* of a semantic model to be a member of this meet partition. Two points $s, s' \in S$ are *adjacent* if they share some member of $\mathcal{P}^j$ for some $j \in J$. We define the *adjacency distance* between any two points s and s' in S as min $\{d \mid$ there is a sequence $s = s_0, \ldots, s_d = s'$, a function $a : \{1, \ldots, d\} \to J$ and sequence of sets $D_1 \in \mathcal{P}^{a(1)}, \ldots, D_d \in \mathcal{P}^{a(d)}$ such that for all $1 \leq i \leq d \quad s_i$ and s_{i-1} both belong to $D_i\}$, with zero distance between any point and itself and infinite distance if there is no such sequence from s to s'. Such a sequence we call an *adjacency path.*

Given a semantic model $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ and a subset $A \subseteq S$, we define another semantic model $\mathcal{V}^{\mathcal{K}}(A) := (A; J; (\mathcal{P}^j|_A \mid j \in J)); X; \psi|_A)$ where for all $j \in J$ $\mathcal{P}^j|_A := \{F \cap A \mid F \cap A \neq \emptyset$ and $F \in \mathcal{P}^j\}$ and for all $x \in X$ $\psi|_A(x) = \psi(x) \cap A$. If there is no ambiguity concerning the initial model $\mathcal{K}$, we can replace $\mathcal{V}^{\mathcal{K}}(A)$ by $\mathcal{V}(A)$.

Now we can see why a semantic model with the finite fanout property is essentially a cell with finite fanout. It is easy to prove that for every possibility set P of an agent $j \in J$ in a semantic model $\mathcal{K}$ that $\phi^{\mathcal{K}}(P)$ is a dense subset of some member of $\mathcal{Q}^j$ (Lemma 5, Simon 1999). Fagin (1994) proved that a cell has a unique extension to all canonical semantic models corresponding to the transfinite ordinal numbers beyond the first infinite ordinal if and only if it is of finite fanout, and that representation in all these canonical semantic models characterizes the interactive knowledge of the agents. Combining these results, restricting ourselves to Ω is sufficient for understanding all semantic models with finite fanout.

2.3 Canonical Finite Models

We define the *depth* of a formula inductively on the structure of the formulas. If $x \in X$, then depth $(x) := 0$. If $f = \neg g$ then depth $(f) :=$ depth (g); if $f = g \wedge h$ then depth (f):= max (depth (g) , depth (h)); and if $f = k_j(g)$ then depth (f):= depth (g) +1.

For every $0 \leq i < \infty$ we define $\mathcal{L}_i := \{f \in \mathcal{L} \mid \text{depth } (f) \leq i\}$ and define Ω_i to be the set of maximally consistent subsets of $\mathcal{L}_i$. If there may be ambiguity, we will write $\Omega_i(X, J)$. We must perceive an Ω_i in two ways, as a semantic model in its own right and as a canonical projective image of Ω inducing a partition of Ω through inverse images. We define $\pi_i : \Omega \to \Omega_i$ to be the canonical projection $\pi_i(z) := z \cap \mathcal{L}_i$. Due to an application of Lindenbaum's Lemma, the maps π_i are surjective. For any semantic model $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ and $i \geq 0$ we define $\phi_i^{\mathcal{K}} : S \to \Omega_i(X, J)$ by $\phi_i^{\mathcal{K}}(s) := \phi^{\mathcal{K}}(s) \cap \mathcal{L}_i(X, J) = \pi_i(\phi^{\mathcal{K}}(s))$.

For every $0 \leq i < \infty$ we consider the semantic model $\Omega_i = (\Omega_i; J; (\overline{\mathcal{F}}_i^j \mid j \in J); X; \overline{\psi}_i)$, where $\overline{\psi}_i = \pi_i \circ \overline{\psi}$ and for $i > 0$ the partition $\overline{\mathcal{F}}_i^j$ of Ω_i is induced by the inverse images of the function $\beta_i^j : \Omega_i \to 2^{\mathcal{L}_{i-1}(X,J)}$ defined by $\beta_i^j(w) := \{f \in \mathcal{L}_{i-1}(X, J) \mid k_j(f) \in w\}$. We define $\overline{\mathcal{F}}_0^j = \{\Omega_0\}$ for every $j \in J$.

Now we consider Ω_i again as a canonical projective image. $\mathcal{G}_i$ is defined to be the partition of Ω induced by the inverse images of π_i, $\mathcal{G}_i := \{\pi_i^{-1}(w) \mid w \in \Omega_i\}$. By the definition of Ω, the join partition $\vee_{i=1}^{\infty}\mathcal{G}_i$ is the discrete partition of Ω, meaning that it consists of singletons. Let $\mathcal{F}_i^j$ be the partition on Ω, coarser than $\mathcal{G}_i$, defined by $\mathcal{F}_i^j := \{\pi_i^{-1}(B) \mid B \in \overline{\mathcal{F}}_i^j\}$. From the definitions of the Ω_i and the $\mathcal{F}_i^j$ it follows that $\vee_{i=0}^{\infty}\mathcal{F}_i^j = \mathcal{Q}^j$.

Since X and J are finite, there are several important properties of the semantic models Ω_i, all of which are used in this paper.

(i) Ω_i is finite for every $0 \leq i < \infty$. (For a more general statement, see Lismont and Mongin 1995.)

(ii) For every $w \in \Omega_i$ we can define a formula $\mathbf{f}(w)$ of depth i or less such that $\alpha^{\Omega_i}(\mathbf{f}(w)) = \{w\}$, meaning that the formula $\mathbf{f}(w)$ is true with respect to Ω_i only at $w \in \Omega_i$. This follows from the finiteness of Ω_i. For any subset $A \subseteq \Omega_i$ define $\mathbf{f}(A) := \vee_{w \in A} \mathbf{f}(w)$, a formula that is true with respect to Ω_i only in the subset A.

(iii) It is easy to extend a member of Ω_i to a member of Ω_{i+1}. Fix $0 \leq i < \infty$ and $w \in \Omega_i$. For every $j \in J$ define $\overline{F}_i^j$ by $w \in \overline{F}_i^j \in \overline{\mathcal{F}}_i^j$. If $(M_i^j \mid j \in J)$ are subsets of $(\overline{F}_i^j \mid j \in J)$, respectively, such that
1) $w \in M_i^j$ for every $j \in J$, and
2) for every $B \in \mathcal{G}_{i-1}$ $\quad \overline{F}_i^j \cap \pi_i(B) \neq \emptyset$ implies that $M_i^j \cap \pi_i(B) \neq \emptyset$,
then there is a unique $v \in \Omega_{i+1}$ such that $\pi_i \circ \pi_{i+1}^{-1}(v) = w$ and for every $u \in \Omega_i \quad \neg k_j \neg \mathbf{f}(u) \in v$ if and only if $u \in M_i^j$. Furthermore, this is the only way to extend a member of Ω_i to a member of Ω_{i+1}; this is Lemma 4.2 of Fagin, Halpern, and Vardi (1991). For any $i \geq 0$ and $v \in \Omega_k$ with $k > i$ we define $M_i^j(v) := \{u \in \Omega_i \mid \neg k_j \neg \mathbf{f}(u) \in v\}$. Notice that if $w \in F \in \overline{\mathcal{F}}_i^j$ then $M_{i-1}^j(w)$ is equal to $\pi_{i-1} \circ \pi_i^{-1}(F)$, which could be a proper subset of the member of $\overline{\mathcal{F}}_{i-1}^j$ that contains $\pi_{i-1} \circ \pi_i^{-1}(w)$.

(iv) For every formula $f \in \mathcal{L}_i$ and $l \geq i \quad \pi_l^{-1}(\alpha^{\Omega_l}(f)) = \alpha^{\Omega}(f)$. This follows from (iii) and the Completeness Theorem. (See also Lemma 2.5 in Fagin, Halpern, and Vardi 1991.)

(v) As a semantic model, every Ω_i is connected. This was proven first by Fagin, Halpern, and Vardi (1991) and it can be proven in several ways (for example from Proposition 1 of Simon, 1999).

2.4 The Common Knowledge of a Formula

Fagin, Halpern and Vardi (1991) investigated what happens when the agents have common knowledge of a finite set of formulas, equivalently the common knowledge of a single formula. Following their definition for "closed," for $i > 0$ we define a non-empty subset $A \subseteq \Omega_i$ to be *semantically closed* (Simon 2001a) if for every $j \in J$, every $B \in \mathcal{G}_{i-1}$ and every $w \in A$ if $\pi_i^{-1}(w) \subseteq F \in \mathcal{F}_i^j$ and $F \cap B \neq \emptyset$ then $F \cap B \cap \pi_i^{-1}(A) \neq \emptyset$. Any non-empty subset of Ω_0

is allowed to be semantically closed. Let $f \in \mathcal{L}$ be a formula with $d =$ depth (f). Fagin, Halpern, and Vardi (1991) proved that $\mathbf{Ck}(\{f\})$ is not empty if and only if the subset $\alpha^{\Omega_d}(f)$ is a semantically closed subset of Ω_d and that there exists a cell dense in $\mathbf{Ck}(\{f\})$ (equivalently $\underline{Ck}(\{f\}) = F(C)$ for some cell C) if and only if the semantic model $\mathcal{V}(\alpha^{\Omega_d}(f))$ is connected. For all $i \geq d =$ depth (f) we define $\Omega_i^f := \pi_i(\mathbf{Ck}(\{f\}))$. It follows from property (iv) that $\Omega_i^f \subseteq \alpha^{\Omega_i}(E_{i-d}(f))$. Define $\overline{\mathcal{F}}_i^j(f)$ by $\overline{\mathcal{F}}_i^j(f) := \{F \cap \Omega_i^f \mid F \in \overline{\mathcal{F}}_i^j\}$. Define the semantic model $\Omega_i^f = (\Omega_i^f; J; (\overline{\mathcal{F}}_i^j(f) \mid j \in J); X; \overline{\psi}|_{\Omega_i^f})$ where $\overline{\psi}|_{\Omega_i^f}(x) = \overline{\psi}(x) \cap \Omega_i^f$. Likewise define $\mathcal{F}_i^j(f)$ by $\mathcal{F}_i^j(f) := \{\pi_i^{-1}(F) \mid F \in \overline{\mathcal{F}}_i^j(f)\}$ and define $\mathcal{G}_i(f)$ by $\mathcal{G}_i(f) := \{G \in \mathcal{G}_i \mid G \subseteq \alpha^{\Omega}(E_{i-d}(f))$.

Most importantly, Fagin, Halpern, and Vardi (1991) showed how to create extensions of Ω_i^f to Ω_{i+1}^f for all $i \geq d =$ depth (f), with only an additional requirement to the rules of (iii) governing extensions from Ω_i to Ω_{i+1}: We must require that the $\overline{F}_i^j$ are in $\overline{\mathcal{F}}_i^j(f)$, which means that the $(M_i^j \mid j \in J)$ are also subsets of Ω_i^f. For the existence of such subsets is needed the semantically closed property. The ability to extend establishes also the opposite containment and conclusion $\Omega_i^f = \alpha^{\Omega_i}(E_{i-d}(f))$ for all $i \geq d =$ depth (f).

Fix $w \in \Omega_i^f$ with $i \geq d =$ depth (f) and with $\alpha^{\Omega_d}(f)$ semantically closed, and let $\overline{F}_i^j$ be the member of $\overline{\mathcal{F}}_i^j(f)$ containing w. The choice of $M_i^j(p_{i+1}(w)) = \overline{F}_i^j$ for agent j was called the "least-information" extension in Fagin, Halpern, and Vardi (1991). Define $p_{i+1}^f(w)$ to be that unique member of Ω_{i+1}^f such that $\pi_i(p_{i+1}^f(w)) = w$ and $M_i^j(p_{i+1}(w)) = \overline{F}_i^j$ for every $j \in J$. If f is a tautology, then it was called the "no-information" extension, and in this case we write p_{i+1} instead of p_{i+1}^f.

We define a formula $f \in \mathcal{L}(X, J)$ with $d =$ depth (f) and $|J| \geq 2$ to be *generative* if and only if $\alpha^{\Omega_d}(f)$ is semantically closed, $\mathcal{V}(\alpha^{\Omega_d}(f))$ is connected, and there exists more than one cell dense in $\mathbf{Ck}(\{f\})$, (meaning that these cells are uncentered). In Simon (2001a), Theorem 1 states that the following are equivalent:

a) the formula f is generative,

b)there is an uncentered cell C such that $F(C) = \underline{Ck}(\{f\})$, meaning that there are uncountably many such cells, (equivalently uncountably many uncentered cells dense in $\mathbf{Ck}(\{f\})$),

c) $\underline{Ck}(f) = F(C)$ for some cell, but $\underline{Ck}(f)$ is not a maximal member of $\mathcal{T}$,

d) there is a cell dense in $\mathbf{Ck}(\{f\})$ that is not finite.

We will call any member of Ω_i an *i-atom*, or an *atom* of Ω_i^f if it also belongs to Ω_i^f.

3 Uncountably many cells with uncountable cardinality

Our goal is to prove **Theorem 1:** If the formula f is generative then there is a continuum of distinct cells dense in $\mathbf{Ck}(\{f\})$ such that each has continuum cardinality.

If f is generative and $i \geq$ depth (f) define an $F \in \overline{\mathcal{F}}_i^j(f)$ to be *proto-generative* (for f) if there exists at least one $v \in \Omega_{i-1}^f$ such that the number of members in $F \cap \pi_i \circ \pi_{i-1}^{-1}(v)$ is at least 2; and define such an $F \in \overline{\mathcal{F}}_i^j(f)$ to be *generative* (for f) if for every $v \in \Omega_{i-1}^f$ such that $F \cap \pi_i \circ \pi_{i-1}^{-1}(v) \neq \emptyset$ then the cardinality of this intersection is at least 2. Define an atom $w \in \Omega_i^f$ to be proto-generative (respectively generative) for an agent j if it is contained in a member of $\overline{\mathcal{F}}_i^j(f)$ that is proto-generative (respectively generative).

If f is generative with depth d then there must be a proto-generative member of $\overline{\mathcal{F}}_d^j(f)$ for some $j \in J$, since otherwise all extensions from Ω_d^f to Ω_{d+1}^f would be determined uniquely, and the same would be true for all the following Ω_i^f for all $i > d$, and we would have a contradiction to Theorem 1 of Simon (2001a).

Lemma 1: Let f be generative and let $i \geq d =$ depth (f).

a) If $F \in \overline{\mathcal{F}}_i^j(f)$ is proto-generative then every $G \in \overline{\mathcal{F}}_{i+1}^k(f)$ with $k \neq j$ and $\pi_{i+1}^{-1}(G) \cap \pi_i^{-1}(F) \neq \emptyset$ is also proto-generative.

b) Let $F \in \overline{\mathcal{F}}_i^j(f)$ be given. If every $G \in \overline{\mathcal{F}}_i^k(f)$ such that $k \neq j$ and $G \cap F \neq \emptyset$ is proto-generative, then every $F' \in \overline{\mathcal{F}}_{i+1}^j(f)$ extending F is generative.

c) If there are at least three agents then there is a level $\hat{i} \geq d$ such that for all $k \geq \hat{i}$ all k-atoms of Ω_k^f are generative for all agents. If there are exactly two agents, then there is a level $\hat{i} \geq d$ such that for all $k \geq \hat{i}$ all k-atoms of Ω_k^f are generative for either one or the other agent.

d) For every level $i \geq d$ and every $N > 0$ there is a level $k > i$ such that for all $k' \geq k$ if $F \in \overline{\mathcal{F}}^j_{k'}(f)$ was created from the repeated use of the least information extensions from the level i to the level k', then the cardinality of F is at least N.

Proof:

a) Let $F' \in \overline{\mathcal{F}}^j_{i+1}(f)$ be any extension of F intersecting $G \in \overline{\mathcal{F}}^k_{i+1}(f)$, and let $B \in \mathcal{G}_i(f)$ be any member such that $\pi^{-1}_{i+1}(F') \cap \pi^{-1}_{i+1}(G) \cap B \neq \emptyset$. Since there was at least two ways for Agent j to extend F that included the possibility of $\pi_i(B) \in \Omega^f_i$ (and because in extending $\pi_i(B)$ the agents choose their sets M^j_i independently) we conclude that $|G \cap \pi_{i+1}(B)| \geq 2$.

b) Let B be any member of $\mathcal{G}_i(f)$ such that $\pi^{-1}_{i+1}(F') \cap B \neq \emptyset$, and let $G \in \overline{\mathcal{F}}^k_i(f)$ be such that $F \cap G$ contains $\pi_i(B)$. Because G is proto-generative there must be at least two elements of Ω^f_{i+1} in $F' \cap \pi_{i+1}(B)$.

c) By Parts a) and b) for all $i \geq$ depth (f) the set of points $O_i := \{z \in \mathbf{Ck}(\{f\}) \mid$ for all $k \geq i$ $\pi_k(z)$ is proto-generative for some agent$\}$ is equal to the clopen set $\{z \in \mathbf{Ck}(\{f\}) \mid \pi_k(z)$ is proto-generative for some agent and some k with depth $(f) \leq k \leq i\}$. For the sake of contradiction, let us suppose that there is a $z \in \mathbf{Ck}(\{f\})$ not in any O_i. We conclude through induction that z is the only extension of $\pi_d(z)$ in $\mathbf{Ck}(\{f\})$, and therefore z is an isolated point of $\mathbf{Ck}(\{f\})$. By Theorem 1 of Simon (2001a) there can be no isolated point of $\mathbf{Ck}(\{f\})$, so that the union of all the O_i must cover $\mathbf{Ck}(\{f\})$. The compactness of $\mathbf{Ck}(\{f\})$ implies that there must be an i such that O_i covers all of $\mathbf{Ck}(\{f\})$. Also from Parts a and b if O_i covers all of $\mathbf{Ck}(\{f\})$ then for all $k \geq i+2$ all k atoms of Ω^f_k are generative for some agent. The claim concerning more than two agents is now transparent.

d) From Part c there is a level i such that for every agent j the cardinality of the possibility sets for Agent j resulting from least information extensions are at least doubling, either on every stage or on every other stage. □

For any generative formula $f \in \mathcal{L}$ define gen (f) to be the first level $i \geq$ depth (f) such that if there are two agents then all members of Ω^f_i are generative for at least one agent, and if there are at least three agents then all members of Ω^f_i are generative for all agents.

For the rest of this section let a generative $f \in \mathcal{L}$ be fixed. Let $2^{\mathbf{N_0}}_\infty$ be the set of subsets of the whole numbers $\mathbf{N_0} = \{0, 1, 2, \cdots\}$ with infinite cardinality ($S \in 2^{\mathbf{N_0}}_\infty$ implies $S \subseteq \mathbf{N_0}$ and $|S| = \infty$). For every pair $i, k \in S$

with $k \geq i \geq$ depth (f) we will define a map $p_k^{S,f} : \Omega_i^f \to \Omega_k^f$. If $i \in S \in 2_\infty^{\mathbf{N_0}}$ define $n_S(i) := \inf \{k \in S \mid k > i\}$, the first member of S strictly larger than i. If $i \in S$ and $w \in \Omega_i^f$ then define $p_{n_S(i)}^{S,f}(w) := \phi_{n_S(i)}^{\Omega_i^f}(w)$ and define $p_i^{S,f}(w) := w$. $p_{n_S(i)}^{S,f}(w)$ is an extension of w, meaning that $\pi_i \circ \pi_{n_S(i)}^{-1}(p_{n_S(i)}^{S,f}(w)) = w$ (Lemma 1 of Simon 2001a). For every $k \in S$ and $w \in \Omega_i^f$ with $k \geq i \in S$ and $p_k^{S,f}(w) \in \Omega_k$ already defined, define $p_{n_S(k)}^{S,f}(w)$ to be $p_{n_S(k)}^{S,f}(p_k^{S,f}(w))$. Lastly, for all $i \in S \in 2_\infty^{\mathbf{N_0}}$ and $w \in \Omega_i$ define $p^{S,f} : \Omega_i \to \Omega$ by

$$p^{S,f}(w) := \bigcap_{l \in S,\ l > i} \pi_l^{-1} \circ p_l^{S,f}(w).$$

For any $i \in S \in 2_\infty^{\mathbf{N_0}}$ and $w \in \Omega_i^f$ we call $p^{S,f}(w)$ the alienated extension of w with respect to S and f. Define p^S to be $p^{S,f}$ for any tautology $f \in \mathcal{L}$.

An alienated extension involves an infinite number of least-information extensions. For all $0 \leq i < \infty$ and $w \in \Omega_i^f$ it is easy to confirm that $\phi_{i+1}^{\Omega_i^f}(w) = p_{i+1}^f(w)$, meaning also that $p^{\mathbf{N_0},f}$ is the infinite repetition of the least-information extension. Define the map p^f to be $p^{\mathbf{N_0},f}$ and p to be $p^{\mathbf{N_0}}$.

For any $S \in 2_\infty^{\mathbf{N_0}}$ and positive k define $n_S^k(i)$ by $n_S^1(i) = n_S(i)$ and $n_S^k(i) = n_S \circ n_S^{k-1}(i)$.

Lemma 2: If $S \in 2_\infty^{\mathbf{N_0}}$ and f is generative, then all alienated extensions with respect to S and f share the same dense cell of $\mathbf{Ck}(\{f\})$.

Proof: If $i \geq$ depth (f) and both w and w' are members of Ω_i^f such that both are contained in the same member of $\overline{\mathcal{F}}_i^j(f)$, then from induction and the definition of $\phi^{\Omega_i^f}$ $p^{S,f}(w)$ and $p^{S,f}(w')$ are both contained in the same member of $\mathcal{Q}^j$.

Now, given any $i, k \in S$ and $b \in \Omega_i^f$ and $d \in \Omega_k^f$, the adjacency distance between $p^{S,f}(b)$ and $p^{S,f}(d)$ in Ω is no more than the adjacency distance between $p_{\max(i,k)}^{S,f}(b)$ and $p_{\max(i,k)}^{S,f}(d)$ in $\Omega_{\max(i,k)}^f$. The rest follows by the connectedness of Ω_i^f for every $i \geq$ depth (f). □

Define the formula $g_i^f := \mathbf{f}(\phi_{i+1}^{\Omega_i^f}(\Omega_i^f))$ of depth $i+1$, the formula true with respect to Ω_{i+1}^f only in the image $\phi_{i+1}^{\Omega_i^f}(\Omega_i^f)$. As we will see, the common knowledge of g_i^f is closely linked to the semantic model Ω_i^f (see also Theorem 4.23 of Fagin, Halpern, and Vardi 1991).

Lemma 3: g_i^f is common knowledge in the semantic model Ω_i^f. If $i \in S \in 2^{\mathbf{N_0}}_\infty$, $i \geq$ depth (f), and $i+1, i+2, \ldots, i+l \notin S$ for some $l \geq 1$, then $p^{S,f}(\Omega_i^f) \subseteq \alpha^\Omega(E^l(g_i^f))$.

Proof: Because Ω_i^f is finite and connected, $\phi^{\Omega_i^f}(\Omega_i^f)$ is a cell. Because $\phi^{\Omega_i^f}(\Omega_i^f) \subseteq \alpha^\Omega(g_i^f)$, Property (iv) and Lemma A imply that g_i^f is common knowledge in the cell $\phi^{\Omega_i^f}(\Omega_i^f)$. If $E^l(g_i^f)$, a formula of depth $i+l+1$, were not true at some point of $\phi_{i+l+1}^{\Omega_i^f}(\Omega_i^f)$ then also by Property (iv) we must have that g_i^f is not common knowledge at some point of $\phi^{\Omega_i^f}(\Omega_i^f)$, a contradiction.

By Simon (2001a) Ω_i^f and $\phi^{\Omega_i^f}(\Omega_i^f)$ are equivalent as semantic models, so that g_i^f is also common knowledge in the semantic model Ω_i^f. □

Lemma 4: If f is generative, $i \geq$ gen (f), and $i+1$ is in $S \in 2^{\mathbf{N_0}}_\infty$, then Eg_i^f is not true at any point of $p^{S,f}(\Omega_{i+1}^f)$.

Proof: Because $\phi_{i+2}^{\Omega_{i+1}^f}(w) = p_{i+2}^f(w)$ for any $w \in \Omega_{i+1}^f$, given $w \in \Omega_{i+1}^f$ it suffices to find some $j \in J$ such that $k_j(g_i^f)$ is not true at $p_{i+2}^f(w)$. Let $j \in J$ be such that $w \in \Omega_{i+1}^f$ is generative for Agent j and let $w \in \pi_{i+1}(F) \in \overline{\mathcal{F}}_{i+1}^j(f)$ with $F \in \mathcal{F}_{i+1}^j(f)$. For every $v \in \Omega_i^f$ there is only one member of Ω_{i+1}^f in the subset $\pi_{i+1}(\pi_i^{-1}(v))$ where g_i^f is true. But for $v := \pi_i \circ \pi_{i+1}^{-1}(w) \in \Omega_i^f$ there are at least two $u \in \Omega_{i+1}^f$ with $u \in \pi_{i+1}(F) \cap \pi_{i+1}(\pi_i^{-1}(v))$ (including at least the possibility of $u = w$). The $F' \in \mathcal{F}_{i+2}^j(f)$ containing $\pi_{i+2}^{-1}(p_{i+2}^f(w))$ must have a non-empty intersection with $\pi_{i+1}^{-1}(u)$ for all $u \in \Omega_{i+1}^f$ with $u \in \pi_{i+1}(F)$, and therefore F' is not contained in $\alpha^\Omega(g_i^f)$. □

Proof of Theorem 1: Define a map $\beta : 2^{\mathbf{N_0}} \to 2^{\mathbf{N_0}}_\infty$ by $\beta(S) := \{0, 1, 2, 4, 8, \cdots\} \cup \{2^i + 1, \cdots, 2^{i+1} - 1 \mid i \in S\}$. Define an equivalence relation on $2^{\mathbf{N_0}}$ by $S \sim T$ if and only if there exists an $m \in \mathbf{N_0}$ such that $S \backslash \{0, 1, 2, \cdots m\} = T \backslash \{0, 1, 2, \cdots, m\}$. The co-sets of this equivalence relation have the cardinality of the continuum.

Let $d = 2^k \geq$ gen (f). Due to Lemma 2, it suffices to show for some $w \in \Omega_d^f$ that if S and T are both subsets of $\mathbf{N_0}$ with $S \not\sim T$ then $p^{\beta(S),f}(w)$ does not share the same cell as $p^{\beta(T),f}(w)$. For the sake of contradiction, let us suppose that the adjacency distance in $\mathbf{Ck}(\{f\})$ between $p^{\beta(S),f}(w)$ and $p^{\beta(T),f}(w)$ equals a finite number $l < \infty$. Because $S \not\sim T$ there exists an $i > \max(log_2((l+2)), k)$ such that $i \in S$ and $i \notin T$, or vice versa. By symmetry, let us assume that $i \in S$ and $i \notin T$. Lemma 3 applied to

$p^{\beta(T),f}(w)$ implies that $p^{\beta(T),f}(w) \in \alpha^{\Omega}(E^{l+1}g^f_{2^i})$. But because the adjacency-distance between $p^{\beta(S),f}(w)$ and $p^{\beta(T),f}(w)$ is l we have that $p^{\beta(S),f}(w) \in \alpha^{\Omega}(E(g^f_{2^i}))$, a contradiction to Lemma 4. That some possibility sets of every such constructed cell are homeomorphic to Cantor sets follows from Lemma 1 and the fact that for every S there is some $j \in J$ and an infinite subset D of $\{1, 2, 4, 8, \ldots\}$ such that for every $i \in D$ $p^{\beta(S),f}_i(w)$ is generative for j. $\square$

4 Finite Fanout

Now we construct uncountably many cells of finite fanout dense in $\mathbf{Ck}(\{f\})$ for any generative $f \in \mathcal{L}$. Let such an f be fixed for the rest of this section.

For any $w \in \Omega^f_i$ define $F^j_i(w)$ to be that member of $\mathcal{F}^j_i(f)$ containing w.

From Proposition 2 of Simon (1999) with finitely many agents a cell is compact if and only if it has finite diameter. Therefore by Theorem 1 of Simon (2001a) all cells dense in $\mathbf{Ck}(\{f\})$ do not have finite diameter, and this implies also that there is no bound on the diameters of the Ω^f_i.

Let $S \in 2^{\mathbf{N_0}}_{\infty}$ satisfy

1) $\inf S > \text{gen } (f) + 2$

2) for every $i \in S$ $n_S(i)$ (the next member of S after i) is large enough so that any $F \in \overline{\mathcal{F}}^j_{n_S(i)}(f)$ created by the least information extension from i to $n_s(i)$ has cardinality strictly greater than Ω^f_i (possible by Lemma 1),

3) for every $i \in S$ the adjacency diameter of $\Omega^f_{n_S(i)}$ is strictly greater than twice the size of the set $\{k \in S \mid k \leq i\}$ plus 3, and

4) $n_S(i) > i + 5$ for all $i \in S$.

Let T be any infinite subset of S with $\inf T = \inf S$. For every $i \geq \inf T$ we define inductively two subsets A_i and B_i of Ω^f_i. We start with any element w_0 of $\Omega^f_{\inf T}$ and let $B_{\inf T} = \{w_0\}$ and $A_{\inf T} = \emptyset$. We assume that A_k and B_k have been defined for all $\inf T \leq k < i$, and show how to define A_i and B_i. First, we define a extension function $\gamma_i : A_{i-1} \cup B_{i-1} \to A_i \cup B_i$ for all $i > \inf T$; it suffices to determine the sets $M^j_{i-1}(\gamma_i(w))$. If $w \in A_{i-1}$ then $M^j_{i-1}(\gamma_i(w)) := (A_{i-1} \cup B_{i-1}) \cap F^j_{i-1}(w)$. If $w \in B_{i-1}$ and $F^j_{i-1}(w)$ contains some member of A_{i-1}, then $M^j_{i-1}(\gamma_i(w)) := (A_{i-1} \cup B_{i-1}) \cap F^j_{i-1}(w)$; otherwise if $A_{i-1} \cap F^j_{i-1}(w) = \emptyset$, then $M^j_{i-1}(\gamma_i(w)) := F^j_{i-1}(w)$. If $i \in T$ we define B_i to be the set $\{p^f_i(w) \mid w \in \Omega^f_{i-1} \backslash (A_{i-1} \cup B_{i-1}),\ w \in F^j_{i-1}(b)$ for some $b \in B_{i-1}$ and $j \in J$ with $F^j_{i-1}(b) \cap A_{i-1} = \emptyset\}$ and we define A_i to be the set

$\gamma_i(A_{i-1} \cup B_{i-1})$. If $i \notin T$ we define B_i to be the set $\gamma_i(B_{i-1})$. and we define A_i to be the set $\gamma_i(A_{i-1})$. For any $i > \inf T$, $l \geq 0$, and $w \in A_{i-1} \cup B_{i-1}$ we define $\gamma_{i+l}(w) = \gamma_{i+l} \circ \ldots \circ \gamma_i(w)$ and we define $\gamma(w) := \cap_{k=i}^{\infty} \pi_k^{-1} \gamma_k(w)$. We define C to be $\{\gamma(w) \mid i \geq \inf T, w \in A_i\}$.

Lemma 5 : γ_i is well defined for every $i \geq \inf T$ and if $b \in B_i$ is adjacent in Ω_i^f to $a \in A_i$, sharing the same member of $\overline{\mathcal{F}}_i^j(f)$, and k is the largest member of T less than or equal to i, then $a = \gamma_i(b')$ for some $b' \in B_{k-1}$ with $F_{k-1}^j(b') \cap A_{k-1} = \emptyset$.

Proof: We prove both claims together by induction on i. $\gamma_{\inf T+1}(w_0) = p_{\inf T+1}^f(w_0)$ is well defined. We assume that γ_k is well defined for all $\inf T + 1 \leq k < i$. Let $w \in A_{i-1} \cup B_{i-1}$, and for any given $j \in J$ let us assume that $v \in \Omega_{i-2}^f$ satisfies $\pi_{i-2}^{-1}(v) \cap \pi_{i-1}^{-1}(F_{i-1}^j(w)) \neq \emptyset$. We need to show that $\pi_{i-1}(\pi_{i-2}^{-1}(v)) \cap M_{i-1}^j(\gamma_i(w)) \neq \emptyset$. If $i-1 \notin T$ and $i > \inf T$ then the well definition of γ_{i-1} shows the same for γ_i, so for the following cases, we assume that $i - 1 \in T$.

Case 1; $w \in A_{i-1}$ **and** $v \in A_{i-2} \cup B_{i-2}$**:** $\gamma_{i-1}(v)$ is in $F_{i-1}^j(w)$ because v and $\pi_{i-2} \circ \pi_{i-1}^{-1}(w)$ share the same member of $\overline{\mathcal{F}}_{i-2}^j(f)$.

Case 2; $w \in A_{i-1}$ **and** $v \notin A_{i-2} \cup B_{i-2}$**:** This is possible only if $\pi_{i-2} \circ \pi_{i-1}^{-1}(w) \in B_{i-2}$ and $F_{i-2}^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(w)) \cap A_{i-2} = \emptyset$. Since $v \in F_{i-2}^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(w))$ we have $p_{i-1}^f(v) \in B_{i-1} \cap F_{i-1}^j(w)$.

Case 3; $w \in B_{i-1}$ **and** $F_{i-1}^j(w) \cap A_{i-1} \neq \emptyset$**:** Let $a \in F_{i-1}^j(w) \cap A_{i-1}$. By the second part of the induction hypothesis $\pi_{i-2} \circ \pi_{i-1}^{-1}(a) \in B_{i-2}$ with $F_{i-2}^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(a)) \cap A_{i-2} = \emptyset$. It follows that v is in $F_{i-2}^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(a))$ and whether or not v was in B_i that there is an extension of v in $A_{i-1} \cup B_{i-1}$.

Case 4; $w \in B_{i-1}$ **and** $F_{i-1}^j(w) \cap A_{i-1} = \emptyset$**:** Since $w \in p_{i-1}^f(\Omega_{i-2})$ we have that $p_{i-1}^f(v) \in F_{i-1}^j(w)$.

For the second part of the claim, suppose for the sake of contradiction that $b' := \pi_{k-1} \circ \pi_i^{-1}(a) \in A_{k-1}$. b' shares the same member of $\overline{\mathcal{F}}_{k-1}^j(f)$ with $c := \pi_{k-1} \circ \pi_i^{-1}(b) \in \Omega_{k-1} \backslash (A_{k-1} \cup B_{k-1})$. For every $j \in J$ and $D \in \mathcal{G}_{k-2}(f)$ if $\pi_k^{-1}(F_k^j(\gamma_k(b')))$ intersects D then it intersects D in only one member of $\mathcal{G}_{k-1}(f)$. If b' is generative for j then by Lemma 1 it is different from $\pi_k^{-1}(F_k^j(p_k^f(c))) = \pi_k^{-1}(F_k^j(\pi_k \circ \pi_i^{-1}(b)))$, a contradiction. If there are only two agents and b' is not generative for j then by the well definition of γ_{k-1} c must have been in $A_{k-1} \cup B_{k-1}$, also a contradiction. So we conclude that b' was in B_{k-1}. Furthermore, if $F_{k-1}^j(b') \cap A_{k-1} \neq \emptyset$ then either $\pi_k \circ \pi_i^{-1}(a)$

and $\pi_k \circ \pi_i^{-1}(b)$ would not share the same member of $\overline{\mathcal{F}}_k^j(f)$ (the case of $F_{k-1}^j(b')$ generative) or from the well definition of γ_k $\pi_{k-1} \circ \pi_i^{-1}(b)$ would be in $A_{k-1} \cup B_{k-1}$ (the case of $F_{k-1}^j(b')$ not generative), both contradictions. □

The second part of Lemma 5 shows that if $i \in T$, $b \in B_{i-1}$ and $\gamma_i(b) = a \in A_i$ then for every $k \geq 1$ the only members of $A_{n_T^k(i)} \cup B_{n_T^k(i)}$ adjacent to $\gamma_{n_T^k(i)}(a)$ are already in the set $\gamma_{n_T^k(i)}(A_i \cup B_i) \subseteq A_{n_T^k(i)}$. Therefore $C = \{\gamma(w) \mid i \geq \inf T, w \in A_i\}$ is a cell with finite fanout.

Lemma 6: If w and w' in Ω_i^f for $i \geq \inf T$ are not in B_i then there is an adjacency path $w = w_0, w_1, \ldots, w_l = w'$ between w and w' such that $w_m \notin B_i$ for all $1 \leq m \leq l-1$.

Proof: We proceed by induction on i. Consider an adjacency path $v_1, v_2, \ldots, v_l$ in Ω_{i-1}^f with $v_1 = \pi_{i-1} \circ \pi_i^{-1}(w)$ and $v_l = \pi_{i-1} \circ \pi_i^{-1}(w')$. We assume that v_k and v_{k+1} share the same member $F_{i-1}^{j_k}$ of $\overline{\mathcal{F}}_{i-1}^{j_k}(f)$ for all $1 \leq k \leq l-1$, and that $j_k \neq j_{k+1}$ for every consecutive pair $k, k+1$. We will define an extension $\beta(v_k) \in \Omega_i^f$ for every such k. Let $M_{i-1}^{j_k}(\beta(v_k)) = M_{i-1}^{j_k}(\beta(v_{k+1}))$ be an appropriate subset of $F_k^{j_k} \in \overline{\mathcal{F}}_{i-1}^{j_k}(f)$ containing v_k and v_{k+1} (with a non-empty intersection with $\pi_{i-1}(D)$ for every $D \in \mathcal{G}_{i-2}(f)$ with $\pi_{i-1}^{-1}(F_k^{j_k}) \cap D \neq \emptyset$). Let $M_{i-1}^j(\beta(v_1)) = M_{i-1}^j(w)$ for at least one $j \neq j_1$ and $M_{i-1}^j(\beta(v_l)) = M_{i-1}^j(w')$ for at least one $j \neq j_{l-1}$. If $|J| \geq 3$ then for any $M_{i-1}^j(v_k)$ not yet defined let $M_{i-1}^j(v_k)$ be an appropriate subset of $F_{i-1}^j(v_k)$. If $l = 1$, then we have our choice of which agent to use to connect w_1 to w and which to connect w_1 to w'. $w, \beta(v_1), \ldots, \beta(v_l), w'$ is an adjacency path connecting w and w', allowing possibly for the identity of w and $\beta(v_1)$ or of w' and $\beta(v_l)$. We must show that these extensions can be done so that for all $1 < k < l$ no extension of v_k is in B_i.

Case 1; there are at least three agents: Looking at any v_k, let $j \in J$ be any agent other than j_k or j_{k-1}. Since all levels involved are generative for all agents, the selection of $M_{i-1}^j(\beta(v_k))$ can be made so that $\beta(v_k)$ is not in B_i.

Case 2; there are only two agents ($J = \{1, 2\}$) and $i \notin T$: If v_1 and v_l are not in B_{i-1} then we can apply induction and assume that the path between v_1 and v_l is in $\Omega_{i-1}^f \backslash B_{i-1}$. Since no extensions of these points in Ω_i^f could be in B_i, we would be done. So let us assume that $v_1 = \pi_{i-1} \circ \pi_i^{-1}(w)$ is in B_{i-1}. Let k be the largest member of T that is less than i. We know from Lemma 5 that there is a $\hat{j} \in \{1,2\}$ such that v_1 is a member of B_{i-1} because $\pi_{k-1} \circ \pi_{i-1}^{-1}(v_1)$ shared the same member of $\overline{\mathcal{F}}_{k-1}^{\hat{j}}(f)$ with a $b \in B_{k-1}$ (but no member of A_{k-1}). Define $b' = \gamma_{i-2}(b)$ (with $b' = b$ if $k = i-1$). Because $\pi_{i-1}^{-1}(F_{i-1}^{\hat{j}}(v_1)) \cap \pi_{i-2}^{-1}(b') \neq \emptyset$ it follows that $F_i^{\hat{j}}(w)$ contains some extension $u \in \Omega_i^f$ of this b', and furthermore no member of either B_i or B_{i-1} is an extension of this b'. Therefore we can replace w by $u \notin B_i$, since w and u are adjacent. If necessary we replace w' with an appropriate u', and exploit the induction assumption on Ω_{i-1}^f as described above.

Case 3; there are only two agents ($J = \{1,2\}$) and $i \in T$: We assume that the path is of minimal length. Because all members of B_i are created as extensions of points not in $B_{i-1} \cup A_{i-1}$, they are all in $p_i^f(\Omega_{i-1}^f)$. If possible, for every k let $M_{i-1}^{j_k}(\beta(v_k)) = M_{i-1}^{j_k}(\beta(v_{k+1}))$ be any proper subset of $F_k^{j_k}$ containing v_k and v_{k+1}. If this is not possible, then let $M_{i-1}^{j_k}(\beta(v_k)) = M_{i-1}^{j_k}(\beta(v_{k+1})) = F_k^{j_k} \cap \Omega_{i-1}^f$. We must show that w_k is not in B_i for all k.

Case 3a; $l > 1$ and $1 < k < l$: By Lemma 1, either v_k is generative for Agent 1 or Agent 2. Without loss of generality assume that v_k is generative for Agent 1. If an extension of v_k is in B_i, it means that for some $j \in \{1,2\}$ the set $F_{i-1}^j(v_k) \in \overline{\mathcal{F}}_{i-1}^j(f)$ defines the least information extension for at least five stages, up to and including the stage $i-1$. Because the size of these sets is far more than 3, if $j = 1$ then there was sufficient freedom in defining $\beta(v_k)$ to prevent it from being in $p_i^f(\Omega_{i-1}^f)$. On the other hand, if $j = 2$ and v_k is not generative for Agent 2 then there could be a failure of w_k to avoid the set B_i only if $F_{i-1}^1(v_k) = \{v_k, v^*\}$ with both v_k and v^* extensions of some $u \in \Omega_{i-2}^f$ (with $v^* = v_{k+1}$ if $j_k = 1$ and $v^* = v_{k-1}$ if $j_k = 2$) If $F_{i-2}^2(u)$ was generative, then there would have been many more than two members of Ω_{i-1}^f in $F_{i-1}^1(v_k)$. If $F_{i-2}^2(u)$ was not generative, then $F_{i-1}^2(v_k)$ also contains v^* and the step between v_k and v^* of the adjacency path was superfluous.

Case 3b; $l > 1$ and $k = 1$ (equivalently $k = l$): We must assume that v_1 is generative for Agent j_1, since otherwise w_1 would equal w (which is not in B_i).

As with Case 3a, for w_1 to be in B_i it is necessary that the least information extension had defined the possibility set of one of the agents for at least five consecutive levels. If this agent was j_1, then certainly B_i could have been avoided. So in what follows we can assume that this agent was not Agent j_1.

Without loss of generality assume that $j_1 = 1 \in J$.

The only possibility that w_1 is in B_i is if $F^1(v_1) = F^1(v_2) = \{v_1, v_2\}$ where both v_1 and v_2 are extensions of some $u \in \Omega^f_{i-2}$. As with Case 3a, if u was not generative for Agent 2 then v_1 and v_2 would belong to the same member of $\mathcal{F}^2_{i-1}(f)$, and then the path between v_1 and v_2 would have been superfluous. Similarly to Case 3a, due to the size of $F^2_{i-2}(u)$ (defined by at least four consecutive least information extensions) we must assume that there were many more than two extensions of u contained in $F^1_{i-1}(v_1)$.

Case 3c; $l = 1$: $w \notin B_i$ implies that for one of the $j = 1$ or $j = 2$ the choice of a member of $\overline{\mathcal{F}}^j_i(f)$ was not appropriate for membership in B_i. If this is true for $j = 1$ for one of and $j = 2$ for the other of the pair w, w', then it is easy from the above to construct a member of $\Omega^f_i \backslash B_i$ that extends $\pi_{i-1} \circ \pi_i^{-1}(w) = \pi_{i-1} \circ \pi_i^{-1}(w')$ and is adjacent to both w and w'. Otherwise w and w' were already adjacent. □

Lemma 6 implies that the removal of B_i does not disconnect Ω^f_i. As we will see, it does not matter that perhaps $\Omega^f_i \backslash B_i$ may be connected through A_i, as we will be applying several consecutive least information extensions.

Lemma 7: If $i \in T$ and the shortest adjacency paths within $\Omega^f_i \backslash B_i$ between $w \in \Omega^f_i \backslash \pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ and $\pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)}) \subseteq \Omega^f_i$ are of length $k \geq 1$, then there is an $1 \leq l \leq k$ with $p_{n_T^{l+1}(i)}(w) \in B_{n_T^{l+1}(i)}$.

Proof: We proceed by induction on k. If $k = 1$, let $c \in \pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ be adjacent to $w \notin \pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ and let j be the member of J such that w and c share the same member of $\overline{\mathcal{F}}^j_i(f)$. $p^f_{n_T(i)}(c) \in B_{n_T(i)}$ can not share the same member of $\overline{\mathcal{F}}^j_{n_T(i)}(f)$ with a member of $A_{n_T(i)}$, since by Lemma 5 c would have shared the same member of $\overline{\mathcal{F}}^j_i(f)$ with a member of B_i and therefore $p^f_{n_T(i)}(w)$ would also be a member of $B_{n_T(i)}$, a contradiction to $k = 1$.

Assume the claim is true for $k - 1 \geq 1$. Let $v \in \Omega^f_i$ be the next element after c in one of the **shortest** adjacency paths within $\Omega^f_i \backslash B_i$ from some $c \in \pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ to w. Let v share with c a member of $\overline{\mathcal{F}}^j_i(f)$. By the same argument as above we have that $p^f_{n^2_T(i)}(v) \in B_{n^2_T(i)}$ and also $p^f_{n_T(i)}(v) \notin B_{n_T(i)}$ and $p^f_{n_T(i)}(u) \notin B_{n_T(i)}$ for all $u \in \Omega_i$ in this adjacency path from v to w, including $u = w$ (due to the minimality of the path). Therefore we have an adjacency path of length $k-1$ within $\Omega^f_{n_T(i)} \backslash B_{n_T(i)}$ between $p^f_{n_T(i)}(v) \in \pi_{n_T(i)} \circ \pi^{-1}_{n^2_T(i)}(B_{n^2_T(i)})$ and $p^f_{n_T(i)}(w)$. Whether or not it is one of the shortest adjacency paths of this kind we have our conclusion by the induction hypothesis. □

Lemma 8: The cell $C = \{\gamma(a) \mid i \in T, a \in A_i\}$ is dense in $\mathbf{Ck}(\{f\})$.

Proof: By Lemma 7 and the fact that B_i does not disconnect $\Omega^f_i \backslash B_i$ (Lemma 6), we need to show that $\pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ is not empty for every $i \in T$. (The density property follows because by choosing any level $i \in T$ and any $w \in \Omega^f_i$ we are showing that there is some level $\hat{i} \geq i$ with a member of $A_{\hat{i}}$ extending w.) We establish this by induction on $i \in T$. If $i = \inf T$ then $B_i = \{w_0\}$ and the claim follows because i is greater than gen (f). Assume the claim is true for any $i \in T$. By Lemma 7 all elements of $B_i \cup (\pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)}))$ are within an adjacency distance of $m := |T \cap \{1, 2, \ldots, i\}|$ from $\gamma_i(w_0)$, yet the diameter of Ω^f_i is at least $2m+1$ by the definition of S; this means that $\Omega^f_i \backslash \pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ is not empty. Also by Lemma 7, the non-emptiness of $\pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ and the existence of some $w \in \Omega^f_i \backslash \pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ that is of positive but finite adjacency distance from $\pi_i \circ \pi^{-1}_{n_T(i)}(B_{n_T(i)})$ implies the non-emptiness of $\pi_{n_T(i)} \circ \pi^{-1}_{n^2_T(i)}(B_{n^2_T(i)}) \subseteq \Omega^f_{n_T(i)}$. □

Theorem 2: If the formula f is generative then there is a continuum of cells with finite fanout that are dense in $\mathbf{Ck}(\{f\})$.

Proof: Notice that every possibility set of a constructed cell is created at some level $i \in T$ and was formed by the consecutive repetition of the least information extension since the last level in T. One can conclude from the definition of S and the sizes of the possibility sets in the constructed cell which subset T of S was used to construct the cell. q.e.d.

5 Infinite generation

Let us review the possibilities for cells of finite fanout. First, all finite cells are defined by the common knowledge of a single formula (Fagin, Halpern, and Vardi 1991). Combined with results from Simon (1999, 2001a) if a cell C has finite fanout it can come in only one of four forms:
1) C is finite, $F(C)$ is finitely generated and maximal in $\mathcal{T}$,

2) C is infinite, $F(C)$ is finitely generated and not maximal in $\mathcal{T}$, and C is uncentered,
3) C is infinite, $F(C)$ is infinitely generated and not maximal in $\mathcal{T}$, and C is centered.
4) C is infinite, $F(C)$ is infinitely generated, and C is uncentered.
How can one distinguish Case 3 from the others? A countable cell is centered if and only if it contains at least one isolated point, (Simon 1999), a straightforward application of Baire Category. Difficult to distinguish is Case 2 from Case 4, the distinction being that of finite vs infinite generation.

A few words must be said about the maximality of a member of $\mathcal{T}$. Maximality implies that it is not possible to hold more formulas in common knowledge. If $S \in \mathcal{T}$ is maximal then there must be a cell C with $F(C) = S$ and furthermore all such cells must be dense in $\mathbf{Ck}(S)$. As stated earlier, all maximal members of $\mathcal{T}$ that are finitely generated correspond to a single finite cell. The difference between finite generation and the lack thereof is most dramatic in the context of the maximality property.

We mention a few examples of Case 4, pathological from the perspective of the theory of finite generation. Simon (2000) contains an example of an infinite generating set of formulas $T \subseteq \mathcal{L}(X, \{1, 2, 3, 4\})$ such that $\underline{Ck}(T) \in \mathcal{T}$ is maximal with uncountably many cells contained (and dense) in $\mathbf{Ck}(T)$ and the cardinality of every possibility set of every agent is no more than 2 (a property stronger than finite fanout). Example 1 of Simon (2001a) is that of an an infinite generating set of formulas $U \in \mathcal{L}(X, \{1, 2, 3\})$ such that $\underline{Ck}(U) \in \mathcal{T}$ is maximal and with uncountably many cells contained (and dense) in $\mathbf{Ck}(U)$ and for the third agent all possibility sets of this agent are homeomorphic to Cantor sets.

Until now, this paper has been concerned with the existence and properties of uncountably many cells that share the same set of formulas in common knowledge. This phenomenum is qualitatively different from that of uncountably many distinct semantic models that map injectively to mutually distinct subsets of a cell of Ω. The main reason why these issues are different is that every possibility set of a cell is a compact set, a property not assumed of the image of a semantic model that maps injectively to Ω. This distinction comes into sharp contrast when considering the finite fanout property. A cell of finite fanout has the *surjective* property, meaning that all semantic models that map to it must map to it surjectively. A non-surjective cell may offer many possibilities for disconnected semantic models to map to this cell. But neither this cell nor these semantic models can have finite fanout. This is because the image of a possibility set of a semantic model in Ω must be a dense subset of a possibility set of Ω (Lemma 5, Simon 1999).

We present a cell that is centered and compact, meaning also by Proposition 2 of Simon (1999) that it has finite adjacency diameter, and yet there is a semantic model with uncountably many connected components that maps injectively to this cell. Furthermore, the corresponding set of formulas cannot be finitely generated, since the compactness of the cell C implies the maximality of these formulas in $\mathcal{T}$ and by Theorem 1 of Simon (2001a) this would imply that this cell must be finite.

To explain our claim, we must first describe Example 3 presented in Simon (2001a). This was an example of a compact cell homeomorphic to a Cantor set with an adjacency radius of 2. To construct this example we let Ω be $\Omega(X, \{1,2\})$ and defined a sequence of partitions in the following way:
for every $0 < i < \infty$ define $A_i = \{p_i(w) \mid w \in \Omega_{i-1}\}$. Define $\mathcal{P}_0 = \{\Omega\}$ and $\mathcal{P}_i = \mathcal{P}_{i-1} \vee \{\pi_i^{-1}(A_i)\ ,\ \Omega \backslash \pi_i^{-1}(A_i)\}$. We labeled the partitions by $\mathcal{B} = (\mathcal{P}_i \mid 0 \leq i < \infty)$ and we defined a semantic model

$$\mathcal{K}(\mathcal{B}) = (\Omega; (\mathcal{Q}^j \mid j \in \{1,2\}), \mathcal{P}_\infty; X; \overline{\psi})$$

where the partition $\mathcal{P}_\infty$ for the third agent is the limit of the partitions $\mathcal{P}_i$, (meaning that z and z' share the same member of $\mathcal{P}_\infty$ if and only if they share the same member of $\mathcal{P}_i$ for every $i < \infty$), with $\overline{\psi}$ and the $\mathcal{Q}^j$ for $j = 1,2$ the same used to define the semantic model Ω. The third agent can distinguish two points if and only if the no-information extension was applied on different stages. We showed that the set $\phi^{\mathcal{K}(\mathcal{B})}(\Omega)$ is a cell of $\Omega(\{1,2,3\})$ equivalent as a semantic model to $\mathcal{K}(\mathcal{B})$, and furthermore that the map $\phi^{\mathcal{K}(\mathcal{B})}$ is a homeomorphism between Ω and the cell that is its image.

Define $A := \{p^S(w) \mid S \in 2^{\mathbf{N_0}}_\infty, i \in S, w \in \Omega_i\} \subseteq \Omega$, the set of all alienated extensions with respect to the tautologies. Define $B := \phi^{\mathcal{K}(\mathcal{B})}(A)$, the image of the set A in $\Omega(X, \{1,2,3\})$. We will show that B defines a semantic model with uncountably many connected components.

To show this, we need some additional results from Simon (1999).

We define a subset $A \subseteq \Omega$ to be *good* if for every $j \in J$ and every $F \in \mathcal{Q}^j$ satisfying $F \cap A \neq \emptyset$ it follows that $F \cap A$ is dense in F. By Lemma 5 and Lemma 6 of Simon (1999) A is good if and only if for every $z \in A$ $\phi^{\mathcal{V}(A)}(z) = z$.

First we show that B is a good subset. Let $z = p^S(w) \in \Omega$ for some $i \in S \in 2^{\mathbf{N_0}}_\infty$ and $w \in \Omega_i$. Let $j \in \{1,2\}$, $z \in F \in \mathcal{Q}^j = \mathcal{Q}^j(X, \{1,2\})$, and $F \cap \pi_k^{-1}(v) \neq \emptyset$ for some $v \in \Omega_k$ with $k \in S$ and $k \geq i$. Since v shares the same member of $\overline{\mathcal{F}}^j_k$ with $\pi_k(z)$ we have that $p^S(v) \in F$. Otherwise let $z \in P \in \mathcal{P}_\infty$ and let $P \cap \pi_k^{-1}(v) \neq \emptyset$ for some $v \in \Omega_k$ with $k \in S$ and $k \geq i$. Likewise $p^S(v)$ is in P, since $\pi_k^{-1}(v)$ shares the same member of $\mathcal{P}_k$ with $z = p^S(\pi_k(z))$. By the above mentioned homeomorphism B is a good subset.

Fix $w_0 \in \Omega_0$. Next we assume that the adjacency distance between $p^S(w_0)$ and $p^T(w_0)$ within the semantic model $\mathcal{V}^{\mathcal{K}(\mathcal{B})}(A)$ is $l < \infty$ for some pair $S, T \in 2^{\mathbf{N_0}}_\infty$ with S and T both containing $\{0\}$. Let $p^S(w_0) = z_0, z_1, \ldots, z_l = p^T(w_0)$ be a path of members of A such that for every $0 \leq k \leq l-1$ z_k and z_{k+1} share the same member of $\mathcal{Q}^1$, $\mathcal{Q}^2$, or $\mathcal{P}_\infty$, and for every $0 \leq k \leq l$ $z_k = p^{S_k}(v_k)$ for $S_k \in 2^{\mathbf{N_0}}_\infty$ for all k, $v_k \in \Omega_{n_k}$ and $n_k \in S_k$ (with $S_0 = S$, $S_l = T$, $n_0 = n_l = 0$, and $v_0 = v_l = w_0$.) Let $N = \max_{0 \leq k \leq l}(n_k)$. If z_k and z_{k+1} share the same member of $\mathcal{P}_\infty$ then by the definition of $\mathcal{P}_i$ we have that $S_k \backslash \{0, 1, \ldots, N-1\} = S_{k+1} \backslash \{0, 1, \ldots, N-1\}$. Now assume that z_k and z_{k+1} share the same member of $\mathcal{Q}^1$, (respectively $\mathcal{Q}^2$.) If $i \geq \max(n_k, n_{k+1})$ it is not possible for i to be in $\mathbf{N_0} \backslash S_k$ without i being in $\mathbf{N_0} \backslash S_{k+1}$ (and vice versa). If such an $i \in \mathbf{N_0} \backslash S_k$ were in S_{k+1} then $\pi_{i+1} \circ p^{S_{k+1}}(v_{k+1})$ would be a no-information extension and therefore $\pi_{i+1} \circ p^{S_k}(v_k)$ could not share the same member of $\overline{\mathcal{F}}^1_{i+1}$ with it, given that $i \notin S_k$. (We use that all members of Ω_i for all $i \geq 0$ are generative for both agents.) That suffices for $S_k \backslash \{0, 1, \ldots, N-1\} = S_{k+1} \backslash \{0, 1, \ldots, N-1\}$ and furthermore that $S \backslash \{0, 1, \ldots, N-1\} = T \backslash \{0, 1, \ldots, N-1\}$. With $\sim$ defined on $2^{\mathbf{N_0}}_\infty$ as before, we see that $S \not\sim T$ implies that $p^S(w)$ and $p^T(w)$ cannot have a finite adjacency distance in the semantic model $\mathcal{V}^{\mathcal{K}(\mathcal{B})}(A)$.

The above argument that $S \backslash \{0, 1, \ldots, N-1\} = T \backslash \{0, 1, \ldots, N-1\}$ works only because all the points concerned are alienated extensions. With respect to the whole space Ω the semantic model $\mathcal{K}(\mathcal{B})$ is connected and has an adjacency radius of 2 (see Simon 2001a), meaning that there is a point such that all other points can be reached from this point by adjacency paths of length 2 or less!

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