Utility Equivalence in Auctions *

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Abstract

Auctions are considered with a (non-symmetric) independent-private-value model of valuations. It shall be demonstrated that a utility equivalence principle holds for an agent if and only if such agent has a constant absolute risk-attitude.

1 Introduction

Most of the research in Auction Theory focuses on the seller's perspective. The most well-known examples are the Revenue Equivalence Theorem[1], which provides conditions under which a seller is indifferent between various auctions, and the Optimal Auction Theorem (Myerson(1981)), which characterizes auction mechanisms that maximize the seller's revenue. However, when following Myerson's proof for the Revenue Equivalence Theorem, it can be seen that the Revenue Equivalence Principle follows from a Utility Equivalence Principle for risk neutral agents, and that these two principles almost coincide. In other words, the seller is indifferent

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[1]Vickrey (1961), Ortega-Reichert(1968), Holt (1980), Harris and Raviv (1981), Myerson(1981), Riley and Samuelson (1981).

between two auction mechanisms if and only if every potential buyer is indifferent between them. Matthews (1987) was the first attempt to compare auction mechanisms from the buyers point of view, when the buyers were not risk neutral. Matthews compared first- and second-price auctions, and showed relationships between monotonicity properties of the Arrow-Pratt measure of risk aversion ($\frac{-u''(x)}{u'(x)}$) and the preferences of agents over these two auction mechanisms[2]. In particular, Mattehews showed that when an agent has constant absolute risk aversion, she is indifferent between first- and second- price auctions. This theorem was generalized by Monderer and Tennenholtz (2000a) to all k-price auctions, and to agents that may have constant absolute risk attitude (CART)[3]. In this discussion we prove a general utility equivalence principle, that holds for agents with constant absolute risk attitude. Furthermore, we show that this equivalence principle holds if and only if the agents have CART.

We shall consider a seller that wishes to sell a single[4] item by an auction mechanism to one of n potential buyers. We assume the (non symmetric) independent-private-value model of valuations[5]. Each potential buyer a is characterized by her utility for money function u^a, and by her valuation structure (distribution of types). The set of possible types of a is an interval $[\alpha^a, \beta^a]$. However, we assume the most general structure of distribution functions, and in particular our model treats atoms as well as atomless distributions. The auction mechanism is typically described by a set of messages, one set for each agent, and by functions (of vector of message)

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[2]Maskin and Rily(1984) discussed the revenue of the seller in first price auctions with risk averse agents.

[3]An agent has constant absolute risk attitude if her Arrow-Pratt measure is a constant function. It is well-known that such an agent has a utility function, which has the form

$$u(x) = c\frac{1 - e^{-\lambda x}}{\lambda}, \quad \text{for all } x \text{ in } R,$$

where $c > 0$, and $\lambda \in R$. By convention, $u(x) = cx$ when $\lambda = 0$. If $\lambda \geq 0$, the agent has constant absolute risk aversion (CARA), and if $\lambda \leq 0$, the agent has constant absolute risk seeking (CARS).

[4]This work does not deal with mechanisms for selling several items –combinatorial auctions mechanisms. Lately, Krishna and Perry (1998) generalized Myerson's utility equivalence to such auctions, keeping the assumption of risk neutrality.

[5]This assumption was made in all previous works that dealt with utility (or revenue) equivalence. However, there are several issues in auction theory that have been analyzed without the independence assumption. The first , and classical work to remove this assumption was Milgrom and Weber (1982).

that define the probability of winning the object by each agent, and the expected payment functions for each agent[6]. The auction mechanism together with the valuation structure define a Bayesian game – the auction game.
For a fixed equilibrium profile in this game, let $Q^a(t)$, $t \in [\alpha^a, \beta^a]$, be the probability that agent a wins the object in equilibrium given that her valuation is t, and let $U^a(t)$ be the expected utility of this agent in equilibrium. Myerson (1981) showed that,

$$U^a(t) = \int_{z=\alpha^a}^{t} u'(0)Q^a(z)dz + U^a(\alpha^a) \quad \text{for every } t \in [\alpha^a, \beta^a].$$

Therefore , for every auction games A and B and equilibrium profiles in this games, such that $Q^a_A(t) = Q^a_B(t)$ for every $t \in [\alpha^a, \beta^a]$,

$$U^a_A(t) - U^a_B(t) = U^a_A(s) - U^a_B(s) \quad \text{for every } t, s \in [\alpha^a, \beta^a].$$

We refer to this result as Myerson's utility equivalence theorem[7]. We show that:

- Myerson's utility equivalence theorem holds only for risk neutral buyers.

We state a utility equivalence theorem, which holds for an agent if and only if she has constant absolute risk-attitude. Since the buyers are not necessarily risk neutral, we split the expected utility function U^a, $U^a(t) = U^a_g(t) + U^a_l(t)$, where $U^a_g(t), U^a_l(t)$ are the expected gain and loss utility functions.

We prove that for a CART agent, a, there exists two functions: $g(Q^a, U^a_l, t)$ and $f(t)$ such that $f(t) > 0$ for every type t, and

$$U^a(t) = \int_{z=\alpha^a}^{t} g(Q^a, U^a_l, z)dz + f(t)U^a(\alpha^a) \quad \text{for every } t \in [\alpha^a, \beta^a].$$

From this result we deduce our utility equivalence principle, which generalizes Myerson's utility equivalence principle:

[6]There are two such payment functions for each agent. One function describes the payment paid by her, when she wins the item, and the other one gives her payment when she does not win. This splitting of payments is necessary when dealing with agents that are not risk neutral.

[7]Note that Myerson's theorem implies that for every two auction games A and B such that $Q^a_A(t) = Q^a_B(t)$ for every $t \in [\alpha^a, \beta^a]$ and $U^a_A(\alpha^a) = U^a_B(\alpha^a)$,

$$U^a_A(t) = U^a_B(t) \quad \text{for every } t \in [\alpha^a, \beta^a].$$

- Let a be a $CART$ agent. There exists a positive function h such that:
  For every auction games A and B and equilibrium profiles in these games, for which $Q^a_A = Q^a_B$ and $U^a_{l,A} = U^a_{l,B}$,
  $$h(t)(U^a_A(t) - U^a_B(t)) = h(s)(U^a_A(s) - U^a_B(s)), \quad \text{for every } t, s \in [\alpha^a, \beta^a].$$

We further prove that :

- Our utility equivalence principle holds for CART agents only.

An important consequence of the utility equivalence principle is:

- Let a be a $CART$ agent. For every two auction games and associated equilibrium profiles, if $Q^a_A = Q^a_B$, $U^a_{l,A} = U^a_{l,B}$,and $U^a_A(\alpha_a) = U^a_B(\alpha_a)$, then
  $$U_A(t) = U_B(t), \quad \text{for every } t \in [\alpha^a, \beta^a].$$

In order to prove the utility equivalence theorem we analyze the equilibrium structure in auction games. The properties which are obtained have their own significance and are proved for the *most general utility functions*.
We show that for every equilibrium in an auction game, U^a_A is a Liptchitz non-decreasing function. In addition we state an equation that should be satisfied in equilibrium (the equilibrium equation). All explicit formulas for the bidding functions in equilibrium may be derived from this equation (see e.g, Maskin and Riley(1984) and Monderer and Tennenholtz (2000a)).

2 Preliminaries

This section presents the basic notations and assumptions that will be used to describe the auction environment throughout this discussion. This environment includes a single owner (a seller), who wishes to sell an item to one of $n \geq 1$ agents (potential buyers) through an action mechanism.

2.1 The agents description

The set of agents is denoted by N.
We assume $N = \{1, 2, ..., n\}, n \geq 1$. Every agent a has a Von-Neumann Morgenstern utility function for money, $u^a(x), -\infty < x < \infty$, such that

- $u^a(0) = 0$.
- u^a is twice differentiable.
- $(u^a)'(x) > 0$ for every $x \in R$,

where R denotes the set of the real numbers. Throughout the paper, whenever possible, we will omit the agent superscript.

In this paper we will mainly deal with agents who have *constant absolute risk attitude* ($CART$). We refer to such an agent as a $CART$ agent. The set of all utility functions of the $CART$ agents is denoted by $CART$. Recall that an agent has a $CART$ if and only if there exists a constant λ such that $\frac{u''(x)}{u'(x)} = \lambda$ for all x. Note that, for such agent, the Arrow-Pratt measure of risk attitude is constant and it is $-\lambda$. If $\lambda = 0$ the agent is *risk neutral* and the utility function has the form $u(x) = cx$ for some $c > 0$. If $\lambda < 0$, the agent has *constant absolute risk aversion* and $u(x) = c(1 - e^{\lambda x})$, $c > 0$. If $\lambda > 0$, the agent has *constant absolute risk seeking* and $u(x) = c(e^{\lambda x} - 1)$, $c > 0$.
The following is a useful characterization of $CART$.

Lemma 1 *$u \in CART$ if and only if there exists $\Gamma \in R$ such that*

$$u(a + b) = u(a) + u(b) + \Gamma u(a)u(b) \quad \forall a, b. \tag{2.1}$$

Proof: If $u \in CART$, then (2.1) is satisfied by $\Gamma = \lambda/u'(0)$.

Suppose there exists Γ such that (2.1) is satisfied. Differentiating both sides of (2.1) with respect to a yields:

$$u'(a + b) = u'(a)[1 + \Gamma u(b)] \quad \forall b.$$

Differentiating both sides again according to a yields:

$$u''(a + b) = u''(a)[1 + \Gamma u(b)] \quad \forall b.$$

By dividing the two equations (note that $(u)'(x) > 0$ for every $x \in R$) we get that $u''(x)/u'(x)$ is constant. Hence, $u \in CART$ ∎
We will use the following equality derived from the proof of Lemma 1:

$$u'(a) = u'(0)[1 + \Gamma u(a)] \quad \forall a. \tag{2.2}$$

We proceed to discuss the agents valuations. We use the (non-symmetric) independent-private-value model. In this model, every agent $a \in N$ knows her own valuation (type, willingness to pay), $t^a \in T^a$, where $T^a = [\alpha^a, \beta^a]$, $0 \le \alpha^a \le \beta^a$. This valuation is a realization of a random variable $\tilde{Z}^a$ which takes values in T^a and has a distribution function F^a [8]. Let $F(t) = \prod_{a=1}^{n} F^a(t^a)$ be the common distribution function on $T = \times_{a=1}^{n} T^a$. The triplet (N, T, F) is called a *valuation structure*.

2.2 The auction mechanism

The auction mechanism comprises of sets of messages, one for each agent, as well as rules that determine the winner and the payments.
An agent $a \in N$ has a message set M^a that contains a message e^a that is called a *null message*. Such a message is never actually sent, but if a does not send any actual message, the seller relates to it as if a sent e^a.
Let $M = \times_{a \in N} M^a$ be the set of vector of messages, and let $e \in M$ be the vector of null messages. We assume that

- $M^a \setminus \{e^a\}$ is a subset of some Euclidean space for every $a \in N$.[9]

Note that M^a is a metric space with the natural metric of Euclidean spaces, and with agreeing that the distance between e^a and a real message m is 1. Hence, M^a and M have a natural Borel structure. A subset B^a of M^a is *bounded* if $B^a \setminus \{e^a\}$ is bounded.
The rest of the auction mechanism is defined by three functions

$$\tau : M \to [0,1]^n, \quad x : M \to \Re^n, \quad y : M \to \Re^n.$$

If the agents send the vector of messages $m \in M$, the seller conducts a lottery to determine the winner. The probability that a is the winner is $\tau^a(m)$. The seller may keep the item to himself. Hence,

$$\sum_{a \in N} \tau^a(m) \le 1 \quad and \quad \tau^a(m) \ge 0, \quad \forall a \in N, \quad \forall m \in M.$$

[8]That is, $F^a(t^a) = Prob(\tilde{z}^a \le t^a)$. Note that our model covers both a continuous and a discrete distribution of types.

[9]Note that we do not exclude finite sets of messages.

$x^a(m)$ is the amount of money that agent a has to pay if she gets the object and, $y^a(m)$ is the amount of money that agent a has to pay if she does not get the object.

We assume that

- τ, x, y are Borel measurable, and x and y are bounded on bounded subsets of M.

Naturally a non participant agent neither wins nor pays. Hence, we assume

- $\tau^a(m) = x^a(m) = y^a(m) = 0$, whenever $m^a = e^a$.

Every auction mechanism $C = C(N, M, \tau, x, y)$, along side a valuation structure $I = (N, T, F)$ defines a Bayesian game, $A = A(C, I)$, which we call an *auction game*.

A *strategy* of agent a is a

- bounded Borel measurable function $b^a : T^a \to M^a$.

For $a \in N$ we denote $T^{-a} = \times_{i \in N, i \neq a} T^i$. For $t \in T$ we denote by t^{-a} the projection of t on T^{-a}.

Let $b = (b^a)_{a \in N}$ be a fixed strategy profile in the auction game A. Consider a fixed agent $a \in N$. Let $Q^a_A(m^a|t^a)$ and $U^a_A(m^a|t^a)$ be the probability that a is the winner and the expected utility of a respectively, when a sends the message m^a, given that her type is t^a and all the other players use their strategies in b. More precisely,

$$Q^a_A(m^a|t^a) = E_{T^{-a}} \left\{ \tau^a(m^a, b^{-a}) \right\}$$

and

$$U^a_A(m^a|t^a) = E_{T^{-a}} \left\{ u^a \left(t^a - x^a(m^a, b^{-a}) \right) \tau^a(m^a, b^{-a}) + u^a \left(-y^a(m^a, b^{-a}) \right) \left[1 - \tau^a(m^a, b^{-a}) \right] \right\},$$

where $b^{-a}(t^{-a})$ is the vector $(b^j(t^j), j \neq a)$.

Recall that b is an equilibrium strategy profile if for every agent a,

$$U^a_A(b(t^a)|t^a) \geq U^a_A(m^a|t^a)$$

for every $t^a \in T^a$ *and* $m^a \in M^a$.

The expected utility function U^a_A is decomposed to an expected gain function $U^a_{A,g}$ and an expected loss function $U^a_{A,l}$, where

$$U^a_{A,g}(m^a|t^a) = E_{T^{-a}} \left\{ u^a \left(t^a - x^a(m^a, b^{-a}) \right) \tau^a(m^a, b^{-a}) \right\}$$

and

$$U^a_{A,l}(m^a|t^a) = E_{T^{-a}}\left\{u^a\left(-y^a(m^a,b^{-a})\right)\left[1-\tau^a(m^a,b^{-a})\right]\right\}.$$

When b is a fixed equilibrium strategy profile in A, we denote $U^a_A(b^a(t^a))|t^a)$ by $U(t)$, and $Q^a_A(b^a(t^a)|t^a)$ by $Q(t)$.

3 Myerson's utility equivalence theorem

Myerson (1981) proved that risk neutral agent is indifferent up to a constant between any two auction mechanisms which have the same probability of winning function, Q. We will prove that such a result holds only for risk neutral agents.

Theorem (Myerson 1981) Let a be a fixed risk-neutral agent. Let $T^a = [\alpha^a, \beta^a]$. Then the following holds: Let A and B be two auction games in which the set of types of a is T^a, and let b and d be fixed equilibrium profiles in A and B respectively. If

$$Q^a_A(t) = Q^a_B(t) \quad \text{for Borel almost every } t \in T^a,$$

then,

$$U^a_A(t) - U^a_B(t) = U^a_A(s) - U^a_B(s), \tag{3.1}$$

for every $t, s \in T^a$.

We proceed to show that Myerson's equivalence principle holds only for risk neutral agents.

Theorem 1 *Let a be an agent with a utility function u, and let $T^a = [\alpha^a, \beta^a]$, $\alpha^a < \beta^a$. If the the following condition holds, a is risk-neutral:*

Let A and B be two auction games, in which the set of types of a is T^a, and let b and d be fixed equilibrium profiles in A and B respectively. If

$$Q^a_A(t) = Q^a_B(t) \quad \textit{for Borel almost every } t \in T^a,$$

then (3.1) holds.

Proof: We will consider auctions in which the set of agents is $N = \{a\}$. Let z be a real number. Let k be a positive integer satisfying

$$u(-z) + (k-1)u(1) \geq 0. \tag{3.2}$$

let $A_{z,k}$ be the following direct auction mechanism;
For every $m \in M^a \ \{e^a\}$,
$\tau^a(m) = \frac{1}{k}$, $x^a(m) = z$, and $y^a(m) = -1$.
Obviously, $A_{z,k}$ generates a truth telling auction game, in which $Q^a_{A_{z,k}}(t^a) = \frac{1}{k}$ for every $t^a \in T^a$.
Note that (3.2) is satisfied for $z = 0$. Hence, the auction games $A_{z,k}$ and $A_{0,k}$ satisfy $Q^a_{A_{z,k}} = Q^a_{A_{0,k}}$.
Therefore,

$$u(t-z) - u(t) = u(s-z) - u(s), \quad \text{for all } t, s \in T^a. \tag{3.3}$$

Recall that (3.3) holds for every real number, z, and differentiate (3.3) with respect to z to get:
$u'(t-z) = u'(s-z)$ for every $t, s \in [\alpha^a, \beta^a]$, and for every $-\infty < z < \infty$.
Hence u' is a constant function. Therefore $u(x) = cx$, $c > 0$, for every x. ▮

A slight modification in the proof of Theorem 1 shows that this theorem holds also for a fixed set of agents. That is, given a set of agents if an agent is indifferent up to a constant between any two auctions which have the same probability to win function, then she must be a risk neutral agent.

4 The utility equivalence theorem

We introduce a generalized utility equivalence theorem which holds if and only if the agent is a $CART$ agent.

In order to prove it, we present important properties of the expected utility and probability to win in-equilibrium functions. Although those properties will be used for $CART$ agents only, we state and prove the results for an agent with an arbitrary attitude to risk.

Theorem 2 *Let A be an auction game, b be a fixed equilibrium profile in A and let a be a fixed agent with the utility function u.*

Then U_A^a is a Liptchitz non-decreasing function.
Moreover for Borel almost every t in $[\alpha^a, \beta^a]$,

$$[U_A^a(t)]' = E_{T^{-a}} \left\{ \left[u^a \left(t - x^a(b^a(t), b^{-a}) \right) \right]' \tau^a(b^a(t), b^{-a}) \right\}. \tag{4.1}$$

Proof: Consider an auction game A.
First we prove that U is a non-decreasing function.
Let s, t be in T^a, and note

$$U(s) - U(t) =$$

$$E_{T^{-a}} \left\{ u \left(s - x(b^a(s), b^{-a}) \right) \tau(b^a(s), b^{-a}) + u \left(-y(b^a(s), b^{-a}) \right) \left[1 - \tau(b^a(s), b^{-a}) \right] \right\} -$$
$$E_{T^{-a}} \left\{ u \left(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) + u \left(-y(b^a(t), b^{-a}) \right) \left[1 - \tau(b^a(t), b^{-a}) \right] \right\}. \tag{4.2}$$

Given s, to bid $b^a(s)$ is at least good as $b^a(t)$. Therefore, by plugging in (4.2) $b^a(t)$ instead of $b^a(s)$ we get:

$$U(s) - U(t) \geq$$
$$E_{T^{-a}} \left\{ \left(u(s - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) + u \left(-y(b^a(t), b^{-a}) \right) \left[1 - \tau(b^a(t), b^{-a}) \right] \right\} -$$
$$E_{T^{-a}} \left\{ \left(u(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) + u \left(-y(b^a(t), b^{-a}) \right) \left[1 - \tau(b^a(t), b^{-a}) \right] \right\}.$$

That is,

$$U(s) - U(t) \geq E_{T^{-a}} \left\{ \left[u \left(s - x(b^a(t), b^{-a}) \right) - u \left(t - x(b^a(t), b^{-a}) \right) \right] \tau(b^a(t), b^{-a}) \right\}. \tag{4.3}$$

For every $s > t$, we get $U(s) - U(t) \geq 0$ and therefore u is non-decreasing.
We show that U is a Liptchitz function:
Given t, to bid $b^a(t)$ is at least good as $b^a(s)$. Therefore, by plugging in (4.2) $b^a(s)$ instead of $b^a(t)$ we get, analogously to the way we got (4.3):

$$U(s) - U(t) \leq E_{T^{-a}} \left\{ \left[u \left(s - x(b^a(s), b^{-a}) \right) - u \left(t - x(b^a(s), b^{-a}) \right) \right] \tau(b^a(s), b^{-a}) \right\}. \tag{4.4}$$

As u itself is a Liptchitz function on bounded intervals, and Q^a is bounded, there exists a constant $C > 0$ such that,

$$U(s) - U(t) \leq C(s - t),$$

and hence

$$|U(s) - U(t)| \leq C|s - t|, \quad \text{for all } s, t \in T^a.$$

We proceed to prove (4.1). As U is a Liptchitz function $U'(t)$ exists Borel almost everywhere in T^a. Let $U'(t)$ exists at t, $\alpha^a < t < \beta^a$.
Let $s > t$, by (4.3)

$$\frac{U(s) - U(t)}{s - t} \geq \frac{E_{T^{-a}} \left\{ \left[u\left(s - x(b^a(t), b^{-a})\right) - u\left(t - x(b^a(t), b^{-a})\right) \right] \tau(b^a(t), b^{-a}) \right\}}{s - t}. \tag{4.5}$$

The limit of the left-hand-side of (4.5) when $s \to t$ is $U'(t)$.
On the other hand by Lebesqe Converges Theorem the right-hand-side of (4.5) converges to $E_{T^{-a}} \left\{ u'\left(t - x(b^a(t), b^{-a})\right) \tau(b^a(t), b^{-a}) \right\}$. That is,

$$U'(t) \geq E_{T^{-a}} \left\{ u' \left(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) \right\}.$$

Similarly, by (4.4) we get:

$$U'(t) \leq E_{T^{-a}} \left\{ u' \left(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) \right\}.$$

Therefore

$$U'(t) = E_{T^{-a}} \left\{ u' \left(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) \right\}.$$

∎

In order to prove the utility equivalence, Myerson (1981) proved that for risk neutral agent, for every $t \in [\alpha^a, \beta^a]$,

$$U^a_A(t) = \int_{z=\alpha^a}^{t} u'(0) Q^a_A(z) dz + U^a_A(\alpha^a).$$

We generalizes this to $CART$ agents.

Theorem 3 *Let A be an auction game, b be a fixed equilibrium profile in A, and let a be a fixed $CART$ agent with the utility function u.*
Then,

$$U^a_A(t) = u'(0) \int_{z=\alpha^a}^{t} e^{u'(0)\Gamma(t-z)} [Q^a_A(z) - \Gamma U^a_{A,l}(z)] dz + U^a_A(\alpha^a) e^{u'(0)\Gamma(t-\alpha^a)}.$$

Proof: Recall that U is a Liptchitz function, and let $t \in T^a$, such that $U'(t)$ exists. By (4.1),

$$U'(t) = E_{T^{-a}} \left\{ u' \left(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) \right\}.$$

Moreover, for $CART$ agent, by (2.2), $u'(a) = u'(0)[1 + \Gamma u(a)]$.
Therefore

$$U'(t) = u'(0)[E_{T^{-a}} \left\{ \tau(b^a(t), b^{-a}) \right\} + \Gamma E_{T^{-a}} \left\{ u \left(t - x(b^a(t), b^{-a}) \right) \tau(b^a(t), b^{-a}) \right\}].$$

Hence,

$$U'(t) = u'(0)[Q(t) + \Gamma(U(t) - U_l(t))]. \tag{4.6}$$

Multiplying both sides of (4.6) by $e^{-u'(0)\Gamma t}$ and rewriting, yields

$$(U(t)e^{-u'(0)\Gamma t})' = u'(0)e^{-u'(0)\Gamma t}(Q(t) - \Gamma U_l(t)).$$

As $U(t)$ is a Liptchitz function, $U(t)e^{-u'(0)\Gamma t}$ is absolutely continuous in T^a, and it is the integral of its derivative.
Therefore,

$$U(t) = u'(0)e^{u'(0)\Gamma t} \int_{z=\alpha}^{t} e^{-u'(0)\Gamma z}(Q(z) - \Gamma U_l(z))dz + U(\alpha)e^{u'(0)\Gamma(t-\alpha)}. \tag{4.7}$$

∎

Note that, U depends only on u,Q , $U_l(.)$ and $U(\alpha)$.
In addition, for risk neutral agent, U depends only on Q and $U(\alpha)$.
The following theorem generalizes Myerson utility equivalence theorem to $CART$ agents:

Theorem 4 *Let a be a $CART$ agent with the utility function u, and let $T^a = [\alpha^a, \beta^a]$. Then there exists a positive function $h(t)$, $t \in T^a$ such that the following holds: Let A and B be two auction games, in which the set of types of a is T^a, and let b and d be fixed equilibrium profiles in A and B respectively. If*

$$Q^a_A(t) = Q^a_B(t) \quad \textit{for Borel almost every } t \in T^a, \tag{4.8}$$

and

$$U^a_{A,l}(t) = U^a_{B,l}(t) \quad \textit{for Borel almost every } t \in T^a, \tag{4.9}$$

then,

$$h(t)\left(U^a_A(t) - U^a_B(t)\right) = h(s)\left(U^a_A(s) - U^a_B(s)\right), \tag{4.10}$$

for every $t, s \in T^a$*.*
Moreover, when $u(x) = c(1 - e^{\lambda x})$*,* $\lambda < 0$*, or* $u(x) = c(e^{\lambda x} - 1)$*,* $\lambda > 0$*, one can choose* $h(t) = e^{\lambda(\alpha - t)}$ *for every* $t \in T^a$*.*
In addition, if a is risk neutral then (4.8) without (4.9) implies (4.10) with $h(t) = 1$ *for every* $t \in T^a$*.*

Proof: Let A and B be two auction games.
By Theorem 3:

$$U_A^a(t) = u'(0) \int_{z=\alpha^a}^{t} e^{u'(0)\Gamma(t-z)}[Q_A^a(z) - \Gamma U_{A,l}^a(z)]dz + U_A^a(\alpha^a)e^{u'(0)\Gamma(t-\alpha^a)}.$$

And,

$$U_B^a(t) = u'(0) \int_{z=\alpha^a}^{t} e^{u'(0)\Gamma(t-z)}[Q_B^a(z) - \Gamma U_{B,l}^a(z)]dz + U_B^a(\alpha^a)e^{u'(0)\Gamma(t-\alpha^a)}.$$

Therefore, if $Q_A^a(t) = Q_B^a(t)$ and $U_{A,l}^a(t) = U_{B,l}^a(t)$ for Borel almost every t, then,

$$U_A(t) - U_B(t) = e^{u'(0)\Gamma(t-\alpha)}(U_A^a(\alpha) - U_B^a(\alpha)) \quad \text{for every } t \in T^a.$$

Therefore, for $h(t) = e^{u'(0)\Gamma(\alpha-t)} = e^{\lambda(\alpha-t)}$, $t \in T^a$,

$$h(t)\,(U_A(t) - U_B(t)) = h(s)\,(U_A(s) - U_B(s)) \quad \text{for every } t, s \in T^a.$$

Finally note that if a is risk neutral, $\Gamma = 0$, and therefore $h(t) = 1$ and the right side of (4.7) depends on Q and $U(\alpha^a)$ only. ▮

One can conclude that $h(t)$ must has the form $h(t) = ce^{\lambda(\alpha-t)}$, for every $t \in T^a$, where $c > 0$.
The following is an important corollary of Theorem 4:

Corollary 1 *Let a be a fixed CART agent with the utility function u. Let A and B be two auction games with the same set of types for a, and let b and d be fixed equilibrium profiles in A and B respectively. Assume that* $Q_A^a(t) = Q_B^a(t)$*,* $U_{A,l}^a(t) = U_{B,l}^a(t)$ *for Borel almost every* $t \in T^a$ *, and* $U_A^a(\alpha^a) = U_B^a(\alpha^a)$*. Then,*

$$U_A^a(t^a) = U_B^a(t^a) \quad \textit{for every } t^a \in T^a. \tag{4.11}$$

We proceed to prove a converse to Theorem 4.

Theorem 5 *Let a be a fixed agent with the utility function u and a set of types T^a such that $\alpha^a < \beta^a$. Assume there exists a positive function $h(t^a)$, $t^a \in T^a$ such that the following holds: For every two auction games A and B, in which the set of types of a is T^a, and for every equilibrium profiles b and d in A and B respectively such that $Q^a_A(.) = Q^a_B(.)$ and $U^a_{A,l}(.) = U^a_{B,l}(.)$,*

$$h(t^a)\left(U^a_A(t^a) - U^a_B(t^a)\right) = h(s^a)\left(U_A(s^a) - U_B(s^a)\right) \quad \text{for every } t, s \in T^a. \tag{4.12}$$

Then a is a CART agent.

Proof: We consider the auctions $A = A_{z,k}$ and $B = A_{0,k}$ defined in the proof of Theorem 1. Recall that, $U_{A_{z,k}}(t) = \frac{1}{k}u(t-z) + [1-\frac{1}{k}]u(1)$ and $U_{A_{0,k}}(t) = \frac{1}{k}u(t) + [1-\frac{1}{k}]u(1)$.
By (4.12),

$$h(t)\left((u(t-z) - u(t)\right) = h(s)\left(u(s-z) - u(s)\right) \tag{4.13}$$

for every $s, t \in T^a = [\alpha^a, \beta^a]$, and $-\infty < z < \infty$. Twicely differentiating both sides of (4.13) with respect to z yields

$$h(t)u'(t-z) = h(s)u'(s-z)$$

and

$$h(t)u''(t-z) = h(s)u''(s-z).$$

Therefore, $\frac{u''}{u'}$ is a constant function, and therefore $u \in CART$. ▮
A slight modification in the proof of Theorem 5, as in Theorem 1, shows that this theorem holds also for a fixed set of agents.

5 Random Participation

In the common definition of auction mechanisms, like the one given in the previous sections, the whole model is commonly known to all agents. That means, that every agent $a \in N$ consider participation in the auction. That is, agent a knows the rules of the auctions, the set of agents, and the valuations' distribution. She may choose not to participate, but this would be a strategic decision[10].

[10]in most of auction theory it is further assumed that every agent must participate. In these models, the agent always have a safe bid that guarantees to her a non-negative expected utility.

In this section we follow Monderer and Tennenholtz (2000b), and we model a situation in which not all agents consider participation. This model incorporates an assumption, introduced by McAfee and McMillan(1987), to the effect that a agent need not know how many other buyers will participate. We assume that for each agent there exists a $\{0,1\}$ random variable $\tilde{\delta}^a$ such that if $\tilde{\delta}^a = 1$ this agent considers participation, and if $\tilde{\delta}^a = 0$ she does not. Let $\tilde{t}^a$ be the random variable that determines a's valuation. Hence an agent type is a pair (δ^a, t^a). We do not assume that $\tilde{\delta}^a$ and $\tilde{t}^a$ are stochastically independent. After all, it is likely that they are positively correlated. Therefore we noted the probability of $\tilde{\delta}^a = 1$ given $\tilde{t}^a = t^a$ by $p^a(t^a)$, and the probability of $\tilde{t}^a \leq t^a$ given $\tilde{\delta}^a = 1$ by $G^a(t^a)$. However we continue to assume the independent-private-value model in the sense that the random variables $(\tilde{\delta}^a, \tilde{t}^a)$, $a \in N$ are independent. A strategy of agent a is a Borel measurable function $b^a : T^a \to M^a$, which represents the bid if $\delta^a = 1$, recall that otherwise $(\delta^a = 0)$ she bids e^a. Every auction mechanism, A, (together with the valuation and participation structure) define a Bayesian game, which we call *random participation auction game*. Let $b = (b^a)_{a \in N}$ be a fixed strategy profile in this game. Consider a fixed agent $a \in N$ which considers to send the message m^a, given that her type is t^a. Consider also, that all the other players use their strategies in b according to their valuations and participation parameters, which are in T^{-a} and $\Delta^{-a} = \times_{i \in N, i \neq a} \Delta^a$ respectively. As in the previous sections, let $Q^a_A(m^a|t^a)$ and $U^a_A(m^a|t^a)$ be the probability that a is the winner and the expected utility of a respectively.
More precisely,

$$Q^a_A(m|t^a) = E_{(\Delta^{-a}, T^{-a})} \left\{ \tau^a(m, b^{-a}) \right\}$$

and

$$U^a_A(m|t^a) = E_{(\Delta^{-a}, T^{-a})} \left\{ u^a \left(t^a - x^a(m, b^{-a}) \right) \tau^a(m, b^{-a}) + u^a \left(-y^a(m, b^{-a}) \right) \left[1 - \tau^a(m, b^{-a}) \right] \right\}.$$

Where b^{-a} is the bids vector of the other agents.
It is assumed in the above formulas that $b^j(t^j, 0) = e^j$. Recall that b is an equilibrium strategy profile if for every agent a, and for every $t^a \in T^a$ such that $p^a(t^a) > 0$,

$$U^a_A(b^a(t^a)|t^a) \geq U^a_A(m|t^a)$$

for every $m \in M^a$.
In this paper we deal only with equilibrium profiles that satisfy a sort of subgame perfection condition.

We required the equilibrium condition to hold for **every** $t^a \in T^a$.
All theorems proved in the previous sections continue to hold (with some obvious and natural modifications) in the model of random participation. We omit the obvious proofs.

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