A Note on an Axiomatization of the Core of Market Games*

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Abstract

As shown by Peleg, the core of market games is characterized by nonemptiness, individual rationality, superadditivity, the weak reduced game property, the converse reduced game property, and weak symmetry. It was not known whether weak symmetry was logically independent. With the help of a certain transitive 4-person TU game it is shown that weak symmetry is redundant in this result. Hence the core on market games is axiomatized by the remaining five properties, if the universe of players contains at least four members.

In this note we solve an open problem of the theory of cooperative games which arises in a natural way in the context of the characterization of the core of market games due to Peleg ([1],[2]). Theorem 2 of [2] shows that the core of market games is characterized by *nonemptiness* (NE), *individual rationality* (IR), *superadditivity* (SUPA), the *weak reduced game property* (WRGP), the *converse reduced game property* (CRGP), and weak symmetry (WS). (Precise definitions of properties are recalled below.) The assertion without the assumption of WS was formulated in [1], but it turned out that WS, which is a weak variant of *anonymity*, was needed in addition in the proof of the uniqueness part. The problem whether WS is logically independent was mentioned in [2]. With the help of a "cyclic" 4-person game (its symmetry group is generated by a cyclic permutation) we show that the core of market games is characterized by NE, IR, SUPA, WRGP, and CRGP (see Theorem 3). Thus, WS is redundant.

We adopt the notation of [1]. Let U be a set of players satisfying $|U| \geq 4$, let us say $M := \{1,2,3,4\}$ is contained in U. A (cooperative TU) *game* is a pair (N,v) such that $\emptyset \neq N \subseteq U$ is finite and $v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$. For any game (N,v) let

$$X^*(N,v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} \text{ and } X(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

denote the set of *feasible* and *Pareto optimal* feasible payoffs (*preimputations*), respectively. The *core* of (N,v) is given by

$$\mathcal{C}(N,v) = \{x \in X^*(N,v) \mid x(S) \geq v(S)\ \forall S \subseteq N\}.$$

*The first author was partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

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A game is *balanced*, if its core is nonempty, and *totally balanced*, if every subgame is balanced. A *solution* σ on a set Γ of games associates with each game $(N,v) \in \Gamma$ a subset of $X^*(N,v)$. Let θ denote the set of all totally balanced games.

Let σ be a solution on a set Γ of games. σ satisfies *nonemptiness* (NE), if $\sigma(N,v) \neq \emptyset$ for every $(N,v) \in \Gamma$. σ is *covariant under strategic equivalence* (COV), if for $(N,v),(N,w) \in \Gamma$ with $w = \alpha v + \beta_*$ for some $\alpha > 0, \beta \in I\!R^N$ the equation $\sigma(N,w) = \alpha\sigma(N,v) + \beta$ holds. Here, (N,β_*) is the *inessential* (*additive*) game given by $\beta_*(S) = \sum_{i\in S}\beta_i$. In this case the games v and w are called *strategically equivalent.* σ is *individually rational* (IR), if $x_i \geq v(\{i\})$ holds for every $(N,v) \in \Gamma$, for every $x \in \sigma(N,v)$, and for every $i \in N$. σ is *superadditive* (SUPA), if $\sigma(N,v) + \sigma(N,w) \subseteq \sigma(N,v+w)$, when $(N,v),(N,w),(N,v+w) \in \Gamma$. Let (N,v) be a game, $x \in X^*(N,v)$, and $\emptyset \neq S \subseteq N$. The *reduced game* $(S,v_{x,S})$ with respect to S and x is defined by $v_{x,S}(\emptyset) = 0$, $v_{x,S}(S) = v(N) - x(N \setminus S)$ and $v_{x,S}(T) = \max_{Q\subseteq N\setminus S} v(T\cup Q) - x(Q)$ for every $T \neq \emptyset, S$. σ satisfies the *weak reduced game property* (WRGP), if $(S,v_{x,S}) \in \Gamma$ and $x^S \in \sigma(S,v_{x,S})$ for every $(N,v) \in \Gamma$, for every $S \subseteq N$ with $1 \leq |S| \leq 2$, and every $x \in \sigma(N,v)$. Here x^S denotes the the restriction of x to S. σ satisfies the *converse reduced game property* (CRGP), if for every $(N,v) \in \Gamma$ with $|N| \geq 2$ the following condition is satisfied for every $x \in X(N,v)$: If, for every $S \subseteq N$ with $|S| = 2$, $(S,v_{x,S}) \in \Gamma$ and $x^S \in \sigma(S,v_{x,S})$, then $x \in \sigma(N,v)$.

Remark 1 *It is well-known (see, e.g., [1]) that the core satisfies* IR, SUPA, CRGP, *and* COV *on every set* Γ *of games. Moreover, it satisfies* NE *and* WRGP *on* θ.

The 4-person game (M,u), defined by

$$u(S) = \begin{cases} 0 & , \text{ if } S \in \{M,\{1,2\},\{2,3\},\{3,4\},\{4,1\},\emptyset\} \, , \\ -1 & , \text{ if } |S| = 3 \, , \\ -4 & , \text{ otherwise} \, , \end{cases}$$

will be used in the proof of Theorem 3. Note that the symmetry group of (M,u) is generated by the cyclic permutation, which maps 1 to 2, 2 to 3, 3 to 4, and 4 to 1; thus the game is *transitive.* (A game is called *transitive*, if its symmetry group is transitive.)

Remark 2 *(1) The core of* (M,u) *is the line segment with the extreme points* $(-1,1,-1,1)$ *and* $(1,-1,1,-1)$*, i.e.,*

$$\mathcal{C}(M,u) = \{(\gamma,-\gamma,\gamma,-\gamma) \in I\!R^M \mid -1 \leq \gamma \leq 1\}.$$

Indeed, every member $x^\gamma := (\gamma,-\gamma,\gamma,-\gamma) \in I\!R^M$ *with* $-1 \leq \gamma \leq 1$ *of this line segment belongs to the core. On the other hand, every core element assigns zero to* M *and, thus, to the members of the partitions* $\{\{1,2\},\{3,4\}\}$ *and* $\{\{2,3\},\{4,1\}\}$*. Therefore the core is contained in the line* $\{x^\gamma \mid \gamma \in I\!R\}$*. The facts* $x^\gamma(\{1,2,3\}) < -1 \; \forall\gamma < -1$ *and* $x^\gamma(\{2,3,4\}) < -1 \; \forall\gamma > 1$ *show that the core has the claimed shape.*

(2) Let $x = x^\gamma \in \mathcal{C}(M,u)$*. Then the reduced coalitional function* $w := u_{x,\{1,2\}}$ *is given by*

$$\begin{aligned} w(\{1\}) &= \max\nolimits_{Q\subseteq\{3,4\}} u(\{1\}\cup Q) - x(Q) &&= u(\{4,1\}) - x_4 &&= \gamma, \\ w(\{2\}) &= \max\nolimits_{Q\subseteq\{3,4\}} u(\{2\}\cup Q) - x(Q) &&= u(\{2,3\}) - x_3 &&= -\gamma, \\ w(\{1,2\}) &= u(M) - x(\{3,4\}) &&= 0, \quad \textit{and } w(\emptyset) &&= 0, \end{aligned}$$

thus $(\{1,2\},w)$ *is an additive game. Similarly it can be shown that all* 2*-person reduced games are additive. Note that the restriction of* x *to any* 2*-person coalition is the unique member of the core of the corresponding additive* 2*-person reduced game.*

(3) The restrictions of the vectors x^γ *for* $\gamma = 1$ *and* $\gamma = -1$*, respectively, to the coalitions* $\{1,2,4\},\{2,3,4\}$ *and* $\{1,2,3\},\{1,3,4\}$*, respectively, show that the* 3*-person subgames are balanced. All* 1*- and* 2*-person subgames are balanced as well. We conclude that* (M,u) *is totally balanced.*

Theorem 3 *There is a unique solution on* θ *that satisfies* NE, IR, SUPA, WRGP, *and* CRGP, *and it is the core.*

Proof: By Remark 1 the core satisfies the desired properties. Let σ be a solution that satisfies the desired properties. By Lemma 4.2 of [1] the solution is a subsolution of the core. Let $(N,v)\in\theta,\ x\in\mathcal{C}(N,v)$ and put $n := |N|$. It remains to show Lemma 4.10 of [1], i.e.

$$\mathcal{C}(N,v) \subseteq \sigma(N,v). \tag{1}$$

The cases $n = 1$ and $n \geq 3$ of the proof of Lemma 4.10 can be copied: If $n = 1$, then $x \in \sigma(N,v)$ by NE and IR. If $n \geq 3$ and (1) is shown for 2-person games, then $x^S \in \mathcal{C}(S, v_{x,S})$ and $(S, v_{x,S}) \in \theta$ for every $S \subseteq N$ with $|S| = 2$, because the core satisfies WRGP. Hence $x^S \in \sigma(S, v_{x,S})$ for every $S \subseteq N$ with $|S| = 2$. By CRGP, $x \in \sigma(N,v)$. It remains to prove (1) for $n = 2$. Without loss of generality we may assune $N = \{1,2\}$. Two cases can be distinguished.

(1) If (N,v) is additive, then the proof is finished by NE and IR (see Corollary 4.5 of [1]).

(2) If $(N,v)\in\theta$ is not additive, then $v(N) > v(\{1\}) + v(\{2\})$. Hence (N,v) is strategically equivalent to (N,w) defined by

$$w(S) = \begin{cases} -2 & \text{, if } |S| = 1\ , \\ 0 & \text{, if } S = \emptyset, N\ , \end{cases}$$

let us say $v = \alpha w + \beta_*$ for some $\alpha > 0,\ \beta \in I\!R^N$. Indeed, $\alpha = \frac{v(N)-v(\{1\})-v(\{2\})}{4}$ and $\beta = \left(\frac{v(N)+v(\{1\})-v(\{2\})}{2}, \frac{v(N)+v(\{2\})-v(\{1\})}{2}\right) \in I\!R^N$ yields $\alpha > 0$ and $v = \alpha w + \beta_*$. Put $y := \frac{x-\beta}{\alpha}$ and observe that $y \in \mathcal{C}(N,w)$ by COV of the core. We first prove $y \in \sigma(N,w)$. With $\gamma := y_1/2$ the vector y can be expressed as $y = (2\gamma, -2\gamma)$. Moreover, $-1 \leq \gamma \leq 1$ holds, because $y \in \mathcal{C}(N,w)$. Let π be the permutation of M which exchanges 1 and 2 and let $(M,\pi u)$ be the "permuted" game given by $(\pi u)(S) = u(\pi^{-1}(S))$. Remark 2 (1) yields $x^\gamma, x^{-\gamma} \in \mathcal{C}(M,u)$ and, similarly, $\pi x^{-\gamma} = (\gamma,-\gamma,-\gamma,\gamma) \in \mathcal{C}(M,\pi u)$. An application of CRGP shows that $x^\gamma \in \sigma(M,u)$ and $\pi x^{-\gamma} \in \sigma(M,\pi u)$ holds true by Remark 2 (2) and part (1) of the present proof. SUPA implies $z := (y,0,0) = x^\gamma + \pi x^{-\gamma} \in \sigma(M, u+\pi u)$. WRGP yields $y \in \sigma\left(N, (u+\pi u)_{z,N}\right)$. The reduced coalitional function $\widehat{u} := (u+\pi u)_{z,N}$ coincides with w. Indeed, $\widehat{u}(N) = (u+\pi u)(M) - z(\{3,4\}) = 0 = w(N)$ by definition of the reduced game. Moreover, the unique 2-person coalitions S with $u(S) = (\pi u)(S) = 0$ are $\{1,2\}$ and $\{3,4\}$, thus $(u+\pi u)(\{i,j\}) = -4$ holds true for $i \in N$ and $j \in M\setminus N$. This observation directly implies

$$\widehat{u}(\{i\}) = (u+\pi u)(\{i,3,4\}) - z(\{3,4\}) = -2 = w(\{i\}) \text{ for } i = 1,2,$$

thus $y = z^N \in \sigma(N, w)$.

In order to show $x \in \sigma(N, v)$ we can proceed similarly. Only the coalitional functions u and πu are replaced by $\alpha u + \left(\frac{1}{2}(\beta, 0, 0)\right)_*$ and $\alpha(\pi u) + \left(\frac{1}{2}(\beta, 0, 0)\right)_*$. The proof is finished by COV of the core. **q.e.d.**

Examples 5.1, 5.3. 5.4, 5.5 of [1] show that each of the axioms NE, SUPA, WRGP, CRGP is logically independent of the remaining axioms. Example 5.2 contains a misprint. The solution which only differs from the core inasmuch as it assigns to any 1-person game the set of all feasible payoffs, satisfies all axioms of Theorem 3 except IR.

References

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