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# Constitutional implementation

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**Abstract.** We consider the problem of implementing a social choice correspondence $H$ in Nash equilibrium when the constitution of the society is given by an effectivity function $E$. It is assumed that the effectivity function of $H, E^H$, is a sub-correspondence of $E$. We found necessary and efficient conditions for a game form $\Gamma$ to implement $H$ (in Nash equilibria), and to satisfy, at the same time, that $E^\Gamma$, the effectivity function of $\Gamma$, is a sub-correspondence of $E^H$ (which guarantees that $\Gamma$ is compatible with $E$). We also find sufficient conditions for the coincidence of the set of winning coalitions of $E^\Gamma$ and $E^H$, and for $E^\Gamma = E^H$. All our results are sharp as is shown by suitable examples.

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**Key words:** Implementation, Nash equilibrium, effectivity function

## 1 Introduction

Since Hurwicz (1972) and Maskin (1998) the literature on complete information implementation mainly focused on sufficient and necessary conditions for implementation under different equilibrium concepts. It is fair to say that there has been much less discussion concerning the reasonability and the attractiveness of different mechanisms that implement the same social choice correspondence (SCC). The fact that a certain game form implements an SCC under, say, Nash equilibrium does not necessarily imply that this game form can serve as an acceptable mechanism for implementation. Indeed, if all players tacitly coordinate on a certain Nash equilibrium of the game, then the planner can be sure to attain one of the socially desirable outcomes, but if some players choose non-equilibrium strategies or if players fail to coordinate on the same equilibrium the implemented outcome can turn out to be completely remote from any recommended outcome. As we know from the theory of repeated games, off-equilibrium behavior is even more relevant when the mechanism is executed repeatedly, which is characteristic to many real life situations. It is therefore important that the game form possesses desirable properties that come in addition to the property of implementability and that do not rely on the fact that players use equilibrium strategies.

In this paper we suggest one such property to which we refer as *Constitutional Implementation.*

By a constitution we refer to the set of rules that specify the distribution of power within the society. In the context of social choice we would like to think of a coherent constitution as defining the opportunities of each group of agents to force social outcomes on the rest of the society; or alternatively to block certain outcomes from being selected. Thus the right of speech, for example, is an outcome, which under liberal constitutions each individual in the society is effective for. That is, each individual alone has a legal option by which he can exercise his right to express opinion. In contrast, the nomination of a person to become the head of state according to most liberal constitutions is only within the power of a majority of the voters. Our condition of constitutional implementation roughly requires that the implementing game form will induce the same distribution of power as that of the implemented SCC, which we assume to be compatible with some pre-specified constitution. Thus the mechanism that the planner uses to implement the desirable outcome should not violate the constitution which we take to be more primitive than the SCC itself.

For a given set of alternatives $A$ and a given society $N$ we will use the notion of effectivity functions to represent constitutions. Specifically, an effectivity function $E$ maps subsets of agents to collections of subsets of alternatives. The claim that a subset of alternatives, say $B \subset A$ is in $E(S)$ (i.e., is $S$ is effective for $B$) for some set of agents $S$ should be understood as saying that the coherent constitution grants the members of $S$ (as a group) the ability to force the social outcome to be in $B$, or alternatively to veto the outcomes in $A \setminus B$.

We will define three notions of constitutional implementation which are based on the relation between the effectivity function derived from the implementing game form and the one derived from the implemented SCC. The main objective of this paper is to identify necessary and sufficient conditions on SCCs that admit constitutional implementation.

Following notations and basic definitions in Sect. 2 we start in Sect. 3 with an example of an SCC that violates constitutional implementation in a rather dramatic sense. This SCC has the property that one player is a dummy player, i.e., the selected set of outcomes does not depend on the preference that this player submits. Yet, while this SCC is Nash implementable, there exist no implementing game form in which this player is dummy. We will argue in the sequel that it is the notion of Nash equilibrium implementation, which is responsible for this paradoxical phenomenon. Indeed, implementation via Strong Equilibria or Coalition Proof Equilibria cannot give rise to such a paradox (see Moulin and Peleg 1982, and Peleg 1984b)

In Sect. 4 we give formal definitions of the three notions of constitutional implementation. We say that a game form is a weakly constitutional implementation of an SCC $H$ if in addition to implementing $H$ its effectivity function is a sub-correspondence of the effectivity function induced by the SCC $H$, i.e., if the game form never grants a coalition more power than it has in the SCC. It is a constitutional implementation of $H$ if it has exactly the same effectivity function that $H$ does. Finally, a third intermediate notion is the "Almost Constitutional Implementation" which requires that the game form is weakly constitutional and in addition the set of winning coalitions of the game form (i.e., those which can enforce every outcome) coincides with the set of winning coalitions derived by the SCC. Our notions of weakly and almost constitutional implementation are based on setting limits to the power of coalitions. This may appear counter intuitive at the first sight as rights are usually associated with alleviating limits and restrictions. Yet we feel that any legislative action, almost per definition, involves setting restrictions. A constitutional rule that grants individuals the right of speech or the right of association boils down to imposing restrictions on the actions of any majority that prevent the denial of these rights.

In Sect. 4 we identify necessary and sufficient conditions for almost constitutional implementation. In addition to Danilov's (1992) property of Strong Monotonicity that guarantees that an SCC $H$ is Nash implementable our condition imposes that according the effectivity function of $H$, for any player $i$ and any alternative $a$ in the range of $H$ the coalition $N \setminus \{i\}$ has to be effective for the lower contour of $a$ with respect to $i$'s preference relation. Indeed it is quite immediate to show that this condition must be possessed by the effectivity function of any game from that implements $H$. The interesting and elaborate part of our result is that the analog condition imposed on the SCC is sufficient for almost constitutional implementation. In Sect. 5 we use two examples to demonstrate the following facts: first we show that the sufficient conditions for almost constitutional implementation do not imply that all game forms are even weakly constitutional. Furthermore, we show that while these conditions guarantee that the set of winning coalitions of the SCC and the game form coincide under almost constitutional implementation the effectivity function of the game form is not necessarily a simple game.

Section 6 is devoted to some results on constitutional implementation. Our first result here shows that any 2-person SCC is constitutionally implementable. We also show that under non-dictatorship and unanimity Maskin's no veto power condition becomes a necessary condition for constitutional implementation. We use an example to demonstrate that the unanimity condition indeed plays an essential role. This result also provides us with a corollary that solves completely the case of 3-person SCC. We show that under the conditions mentioned above constitutional implementation implies that the effectivity function of the SCC is a simple majority game.

We conclude the paper with a short discussion and some straightforward extensions.

## 2 Definitions and notations

Let $A$ be a set of alternatives. $A$ may be finite or infinite. However, $|A| \geq 2$ if $A$ is finite. (If $D$ is a finite set, then $|D|$ is the number of members of $D$.)

A (linear) preference ordering on $A$ is a complete, transitive, and antisymmetric binary relation. We denote by $L = L(A)$ the set of all linear orders on $A$. If $S$ is a set, then $L^S = \{f | f : S \to L\}$.

Let $D$ be a set. We denote by $P(D)$ the set of all subsets of $D$, that is, $P(D) = \{D' | D' \subset D\}$. Also, $2^D = P(D) \setminus \{\emptyset\}$ is the set of all non-empty subsets of $D$.

Let $A$ be a set of alternatives and let $N = \{1, \ldots, n\}$ be a finite set of players. A *social choice correspondence* (SCC) is a function $H : L^N \to 2^A$. Let $H$ be an SCC. $H$ is *surjective* if for each $x \in A$ there exists $R^N \in L^N$ such that $H(R^N) = \{x\}$. $H$ satisfies *unanimity* if

$$[x \in A \quad \text{and } xR^i y \quad \text{for all } y \in A \quad \text{and } i \in N] \Rightarrow H(R^N) = \{x\}.$$

If $H$ is surjective, then a player $d \in N$ is a *dictator* for $H$ if

$$[x \in A, R^N \in L^N, \quad \text{and } xR^d y \quad \text{for all } y \in A] \Rightarrow H(R^N) = \{x\}.$$

If there is no dictator for $H$, then $H$ is called *non-dictatorial*. Let $a \in A$ and $R \in L$. We denote $L(a, R) = \{b \in A | aRb\}$. $H$ is *Maskin monotonic* if

$$[a \in H(R^N), Q^N \in L^N, \quad \text{and } L(a, R^i) \subset L(a, Q^i) \\ \text{for all } i \in N] \Rightarrow a \in H(Q^N).$$

Let $i \in N$ and $B \subseteq A$. An alternative $b \in B$ is *essential for* $i$ in the set $B$ with respect to $H$ if there exists $R^N \in L^N$ such that $b \in H(R^N)$ and $L(b, R^i) \subseteq B$. The set of all alternatives which are essential for $i$ in $B$ with respect to $H$ is denoted by $Ess_i(B, H)$. $H$ satisfies *strong monotonicity* if

$$[a \in H(R^N), Q^N \in L^N, \quad \text{and } Ess_i(L(a, R^i), H) \subseteq L(a, Q^i) \\ \text{for all } i \in N] \Rightarrow a \in H(Q^N).$$

Strong monotonicity was defined in Danilov (1992). We remark that strong monotonicity implies Maskin monotonicity. Finally, we need the following definition due to Maskin (see, e.g., Maskin 1985). $H$ satisfies *no veto power* if

$$[i \in N, a \in A, R^N \in L^N, \text{ and } aR^j b \text{ for all } j \neq i \text{ and } b \in A] \Rightarrow a \in H(R^N).$$

We now turn to define some basic properties of game forms. Let $A$ be a set of alternatives and let $N = \{1, \ldots, n\}$ be a set of players. A *game form* (GF) is an $(n+2)$-tuple $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$, where (i) $\Sigma^i$ is the (non-empty) set of *strategies* of player $i \in N$; (ii) $\pi : \Sigma^1 \times \cdots \times \Sigma^n \to A$ is the *outcome function*. Let $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ a GF. For $S \in 2^N$ we denote $\Sigma^S = \times_{i \in S} \Sigma^i$. Also, we denote $\Sigma^N = \Sigma$. Now let $R^N \in L^N$. The pair $(\Gamma, R^N)$ defines, in an obvious way, a game in strategic form. $\sigma \in \Sigma$ is a *Nash equilibrium* (NE) of $(\Gamma, R^N)$ if

$$\pi(\sigma) R^i \pi(\sigma^{N \setminus \{i\}}, \tau^i) \quad \text{for all } i \in N \quad \text{and } \tau^i \in \Sigma^i.$$

The set of all Nash equilibria of $(\Gamma, R^N)$ is denoted by $NE(\Gamma, R^N)$.

Now let, again, $A$ be a set of alternatives and $N = \{1, \ldots, n\}$ be a set of players. Furthermore, let $H : L^N \to 2^A$ be an SCC and $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be a GF.

$\Gamma$ *implements* $H$ *in* $NE$'s if $\pi(NE(\Gamma, R^N)) = H(R^N)$ for all $R^N \in L^N$. Let, again, $H : L^N \to 2^A$ be an SCC. $H$ is *implementable* in $NE$'s if there exists a GF $\Gamma$ that implements it in $NE$'s.

Let $A$ be a set of alternatives and let $N = \{1, \dots, n\}$ be a set of players. An *effectivity function* (EF) is a function $E : P(N) \to P(P(A))$ that satisfies the following conditions: (i) $E(N) = 2^A$; (ii) $E(\emptyset) = \emptyset$; (iii) $A \in E(S)$ for all $S \in 2^N$; and (iv) $\emptyset \notin E(S)$ for all $S \in 2^N$.

Let $E$ be an EF. $E$ is *superadditive* if it satisfies the following condition: If $S_i \in 2^N, B_i \in E(S_i), i = 1, 2$, and $S_1 \cap S_2 = \emptyset$, then $B_1 \cap B_2 \in E(S_1 \cup S_2)$ (in particular, $B_1 \cap B_2 \neq \emptyset$). $E$ is *maximal* if for all $S \in 2^N$ and $B \in 2^A$

$$B \notin E(S) \Rightarrow A \setminus B \in E(N \setminus S).$$

The core of $E$ with respect $R^N \in L^N$ is defined in the following way: Let $B \in 2^A, S \in 2^N$, and $x \in A \setminus B$. $B$ *dominates* $x$ *via* $S$ *at* $R^N$ if $B \in E(S)$ and $bR^i x$ for all $b \in B$ and $i \in S$. $x \in A$ is *dominated* at $R^N$ if there exist $B \in 2^A$ and $S \in 2^N$ such that $B$ dominates $x$ via $S$ at $R^N$. The *core* of $E$ with respect to $R^N, C(E, R^N)$, is the set of all undominated alternatives at $R^N$. $E$ is *stable* if $C(E, R^N) \neq \emptyset$ for all $R^N \in L^N$.

Let $H : L^N \to 2^A$ be a surjective SCC. With $H$ we associate an EF $E^H$ in the following way. Let $S \in 2^N$ and $B \in 2^A$. $S$ is *effective* for $B$ if there exists $R^S \in L^S$ such that $H(R^S, Q^{N\setminus S}) \subseteq B$ for all $Q^{N\setminus S} \in L^{N\setminus S}$. $E^H$ is now defined by $E^H(\emptyset) = \emptyset$, and

$$E^H(S) = \{B \in 2^A | S \quad \text{is effective for } B\}, \quad \text{for } S \in 2^N.$$

Clearly, $E^H$ is superadditive. (Effectivity functions of SCC's were defined in Moulin and Peleg (1982). $E^H$ is called there the $\alpha$-effectivity function of $H$.)

Let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a GF and assume that $\pi$ is surjective. The EF $E^\Gamma$, which is associated with $\Gamma$, is defined in the following way. Let $S \in 2^N$ and $B \in 2^A$. $S$ is *effective* for $B$ if there exists $\sigma^S \in \Sigma^S$ such that $\pi(\sigma^S, \mu^{N\setminus S}) \in B$ for all $\mu^{N\setminus S} \in \Sigma^{N\setminus S}$. $E^\Gamma$ is given by $E^\Gamma(\emptyset) = \emptyset$, and

$$E^\Gamma(S) = \{B \in 2^A | S \quad \text{is effective for } B\}, \quad \text{for } S \in 2^N.$$

($E^\Gamma$ is called the $\alpha$-effectivity function of $\Gamma$ in Moulin and Peleg (1982).)

Finally, we recall some properties of simple games. A simple game is a pair $(N, W)$, where $N = \{1, \dots, n\}$ is a set of players, and $W \subset 2^N$ is a set of winning coalitions. Let $G = (N, W)$ be a simple game.

$G$ is *monotonic* if

$$[S \in W \quad \text{and } S \subseteq T \subseteq N] \Rightarrow T \in W.$$

$G$ is *proper* if

$$S \in W \Rightarrow N \setminus S \notin W \quad \text{for all } S \in 2^N.$$

(In the sequel we only deal with monotonic and proper games.) $G$ is *strong* if

$$S \notin W \Rightarrow N \setminus S \in W \quad \text{for all } S \in 2^N.$$

$G$ is *symmetric* if $G$ is an $(n,k)$ game, that is, there exists $\frac{n}{2} < k \leq n$ such that $W = \{S \subseteq N | \, |S| \geq k\}$. $G$ is *weak* if

$$V = \cap\{S | S \in W\} \neq \emptyset$$

$V$ is the set of *vetoers* of $G$. If $G$ is not weak, then the Nakamura number of $G$, $\nu(G)$, is given by

$$\nu(G) = \min\{|U| \, |U \subset W \quad \text{and } \cap \{S | S \in U\} = \emptyset\}$$

(see Nakamura 1979).

Let $G = (N, W)$ be a simple game, let $A$ be a set of alternatives, let $R^N \in L^N$, and let $x, y \in A, x \neq y$. *$x$ dominates $y$ at $R^N$* if there exists $S \in W$ such that $xR^i y$ for all $i \in S$. The *core* of $G$ with respect to $R^N, C(G, A, R^N) = C(R^N)$, is the set of undominated alternatives at $R^N$. If $G$ is not weak, then $C(G, A, R^N) \neq \emptyset$ for all $R^N \in L^N$ iff $\nu(G) > |A|$ (see, again, Nakamura 1979). (Obviously, if $G$ is weak and $A$ is finite, then $C(G, A, R^N) \neq \emptyset$ for all $R^N \in L^N$.)

Let $E : P(N) \to P(P(A))$ be an EF. The simple game $(N, W(E))$ which is associated with $E$ is given by

$$W(E) = \{S \in 2^N | E(S) = 2^A\}.$$

Let $H : L^N \to 2^A$ be an SCC. *The simple game which is associated with $H$* is $W(H) = W(E^H)$. Similarly, if $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ is a GF, then the *simple game which is associated with $\Gamma$* is $W(\Gamma) = W(E^\Gamma)$. We close this section with the following definition. Let $G = (N, W)$ be a simple game and let $N \in W$. The EF $E(G)$ which is associated with $G$ is given by

$$E(G)(S) = \begin{cases} 2^A, & S \in W; \\ \{A\}, & S \in 2^N \setminus W; \\ \emptyset, & S = \emptyset. \end{cases}$$

## 3 Examples

An equilibrium concept for games in strategic form is a function $e$ that associates with every GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$, and every $R^N \in L^N$, a subset $e(\Gamma, R^N)$ of $\Sigma$. A GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ implements an SCC $H : L^N \to 2^A$ in an equilibrium concept $e$ if $\pi(e(\Gamma, R^N)) = H(R^N)$ for all $R^N \in L^N$. A fundamental question in implementation theory is the following: Let $e$ be an equilibrium concept and let $H : L^N \to 2^A$ be implementable in $e$ (i.e., there exists a GF that implements $H$ in $e$). Is it possible to find a GF $\Gamma$ such that $(i)\Gamma$ implements $H$ in $e$; and (ii) $E^H = E^\Gamma$.

There are (at least) three reasons for considering the foregoing question.

(a) The set of voters in $H$, $N$, may be part of a larger society $N^*$. A constitution for $N^*$ may be specified by an EF $E^* : P(N^*) \to P(P(A^*))$, where $A^* \supset A$ (see Peleg 1998). We may usually assume that $H$ specifies the goals of some planner.

Hence $H$ is compatible with the constitution $E^*$, that is, $E(S) \subset E^*(S)$ for all $S \in 2^N$ (very often $E^* = E^H$). However, arbitrary GF's that implement $H$ in $e$ may not be compatible with $E^*$. The condition $E^H = E^\Gamma$ guarantees that the implementing GF $\Gamma$ is constitutional (i.e., compatible with $E^*$).

(b) The requirement $E^H = E^\Gamma$ may be part of the program of obtaining natural implementations of $H$ (see Saijo et al. 1999 where other properties are discussed).

(c) If $e$ is strong equilibrium or coalition-proof Nash equilibrium, then there is a positive answer for our question. Furthermore, if $H : L^N \to 2^A$ is an SCC and $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ is a GF that implements $H$ in strong equilibrium or coalition proof Nash equilibrium, then $E^H = E^\Gamma$ and $E^H$ is maximal (see Moulin and Peleg 1982 for strong implementation and Peleg (1984b) for coalition-proof Nash implementation; see also Moulin (1983, p. 174) for a similar result for implementation by backward induction). In this paper we investigate whether we can obtain similar results for implementation in NE. Our findings will enable us to compare NE with the foregoing concepts in terms of their constitutional compatibility (see (a)).

This section is devoted to two examples which explain and motivate the general results on constitutional implementation which will be presented in the following three sections. We start with the following observation. Henceforth, all GF's and SCC's are surjective and $n \geq 2$.

**Lemma 3.1.** *Let $H : L^N \to 2^A$ be an SCC and let the GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ implement $H$ is NE. Then*

$$[i \in N, R^N \in L^N, \quad \textit{and } a \in H(R^N)] \Rightarrow L(a, R^i) \in E^\Gamma(N \setminus \{i\}). \tag{3.1}$$

*Proof.* Let $\sigma \in \Sigma$ be an NE of $(\Gamma, R^N)$ such that $\pi(\sigma) = a$. Then $\pi(\sigma^{N\setminus\{i\}}, \tau^i) \in L(a, R^i)$ for all $\tau^i \in \Sigma^i$. Q.E.D.

Lemma 3.1 enables us to consider the following example.

*Example 3.2. The Dummy Paradox.* Let $A = \{a, b, c\}$ and let $N = \{1, 2, 3\}$. For $R \in L$ let $t(R)$ be the top alternative of $R$, that is, $t(R)Rx$ for all $x \in A$. Let $H : L^N \to 2^A$ be defined by $H(R^N) = \{t(R^2)\} \cup \{t(R^3)\}$. Then $H$ is unanimous and non-dictatorial. Clearly, 1 is a dummy with respect to $H$. This is reflected by $E^H$:

$$E^H(S) = \begin{cases} 2^A, & S \supseteq \{2,3\} \\ \{A\}, & S \neq \emptyset, S \not\supseteq \{2,3\} \\ \emptyset, & S = \emptyset \end{cases} \tag{3.2}$$

Also, $H$ is Maskin monotonic and satisfies no veto power. Therefore, by Maskin (1985), $H$ is implementable in NE. Let $\Gamma = (\Sigma^1, \Sigma^2, \Sigma^3; \pi; A)$ implement $H$ in NE's. By Lemma 3.1 and the superadditivity of $E^\Gamma$ we obtain:

$$E^\Gamma(S) = \begin{cases} 2^A, & |S| \geq 2 \\ \{A\}, & |S| = 1 \\ \emptyset, & S = \emptyset \end{cases} \tag{3.3}$$

Thus, $E^{\Gamma}$ is symmetric. Furthermore, (3.3) implies that 1 is not a dummy with respect to $\Gamma$. (Notice that $\Gamma$ may not be symmetric.) We conclude that $E^{\Gamma}$ strictly includes $E^H$ for every GF $\Gamma$ that implements $H$ is NE's.

For the second example we need an additional lemma.

**Lemma 3.3.** *Let $E : P(N) \to P(P(A))$ be a stable EF. If $n \geq 3$ then the core $C(E, R^N)$ is implementable in NE's.*

*Proof.* Let $H(R^N) = C(E, R^N)$ for all $R^N \in L^N$. We shall prove that $H$ is strongly monotonic. Let $i \in N$ and $B \in 2^A$. Clearly,

$$Ess_i(B, H) = \begin{cases} \emptyset, & A \setminus B \in E(\{i\}) \\ B, & A \setminus B \notin E(\{i\}) \end{cases}$$

Now let $R^N, Q^N \in L^N, a \in H(R^N)$, and

$$Ess_i(L(a, R^i), H) \subset L(a, Q^i) \quad \text{for all } i \in N.$$

As $a \in H(R^N), A \setminus L(a, R^i) \notin E(\{i\})$ for all $i \in N$. Hence

$$Ess_i(L(a, R^i), H) = L(a, R^i) \quad \text{for all } i \in N.$$

Therefore, $a \in H(Q^N)$ by the Maskin monotonicity of $H$. We conclude that $H$ is strongly monotonic. By Theorem 2 in Yamato (1992) $H$ is implementable in NE's. Q.E.D.

We may now proceed to the second example. We say that a simple game $(N, W)$ is non-dictatorial if $\{i\} \notin W$ for every $i \in N$. An EF $E : P(N) \to P(P(A))$ is non-dictatorial if $(N, W(E))$ is non-dictatorial.

*Example 3.4.* Let $E : P(N) \to P(P(A))$ be a maximal, stable, and non-dictatorial EF. We assume that $|A| \geq 3$ and $n \geq 3$. Further, let $H(R^N) = C(E, R^N)$ for $R^N \in L^N$. Clearly, $E^H = E$. Also, $H$ is Paretain (i.e., for every $R^N \in L^N$ and $a \in H(R^N)$, $a$ is Pareto optimal with respect to $R^N$), because $E(N) = 2^A$. By Lemma 3.3, $H$ is implementable in NE's. Let $\Gamma$ be a GF that implements $H$ in NE's. We claim that $E^{\Gamma} \neq E^H$. Indeed, assume on the contrary, that $E^{\Gamma} = E^H$. By Theorem 3.5 of Dutta (1984), $E^{\Gamma} = E(G)$ where $G$ is a strong simple game. However, $E(G)$ is not stable, because $|A| \geq 3$ and $G$ is non-dictatorial. Thus, we have reached the desired contradiction.

## 4 Almost constitutional implementability

We start with the main definition of our paper. It is motivated by the discussion at the beginning of Sect. 3.

**Definition 4.1.** *Let $H : L^N \to 2^A$ be a surjective SCC and let the GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ implement $H$ in NE's. Then*

*(i)* *$\Gamma$ is a constitutional implementation of $H$ if $E^\Gamma = E^H$;*
*(ii)* *$\Gamma$ is a weak constitutional implementation of $H$ if $E^\Gamma \subseteq E^H$ (i.e., $E^\Gamma(S) \subseteq E^H(S)$ for all $S \in 2^N$);*
*(iii)* *$\Gamma$ is almost a constitutional implementation of $H$ if (1) $E^\Gamma \subseteq E^H$, and (2) $W(\Gamma) = W(H)$.*

Clearly, $\Gamma$ is a constitutional implementation of $H$, if $\Gamma$ and $H$ are compatible with the same constitutions. $\Gamma$ is a weak constitutional implementation $H$ if the compatibility of $H$ with a certain constitution implies the compatibility of $\Gamma$. Almost constitutional implementability implies weak constitutional implementability and, in addition, the coincidence of the sets of winning coalitions with respect to $\Gamma$ and $H$.

*Remark 4.2.* The SCC of Example 3.2 has no weak constitutional implementation. Example 3.4 introduces a large set of "nice" SCC's which do not possess constitutional implementations. As we shall see many cores of effectivity functions have *almost* constitutional implementations. Indeed, we shall now prove an existence theorem for almost constitutional implementations. We start with the following observation.

**Lemma 4.3.** *Let $H : L^N \to 2^A$ be a surjective SCC. If $H$ has a weak constitutional implementation $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$, then $H$ is strongly monotonic and*

$$[i \in N, R^N \in L^N, \quad \text{and } a \in H(R^N)] \Rightarrow L(a, R^i) \in E^H(N \setminus \{i\}). \tag{4.1}$$

*Furthermore, (4.1) is equivalent to*

$$[i \in N, R^N \in L^N, \quad \text{and } a \in H(R^N)] \Rightarrow Ess_i(L(a, R^i), H) \in E^H(N \setminus \{i\}). \tag{4.2}$$

*Proof.* $H$ is strongly monotonic by Theorem 1 of Yamato (1992). (4.1) follows form (3.1) and $E^\Gamma \subseteq E^H$. Clearly, (4.2) $\Rightarrow$ (4.1). We now prove, (4.1) $\Rightarrow$ (4.2). Let $i \in N, R^N \in L^N$, and $a \in H(R^N)$. Choose $R_0^i \in L$ such that

$$A \setminus L(a, R^i) R_0^i \{L(a, R^i) \setminus Ess_i(L(a, R^i), H)\} R_0^i Ess_i(L(a, R^i), H)$$

By (4.1) there exists $R_0^{N\setminus\{i\}} \in L^{N\setminus\{i\}}$ such that $H(R_0^{N\setminus\{i\}}, R_0^i) \subset L(a, R^i)$. Let $b \in H(R_0^{N\setminus\{i\}}, R_0^i)$. Then $b \in Ess_i(L(a, R^i), H)$ by definition. By (4.1), $L(b, R_0^i) \in E^H(N \setminus \{i\})$. As $L(b, R_0^i) \subset Ess_i(L(a, R^i), H)$, the proof is complete. Q.E.D.

It is worth noting that condition (4.1) is satisfied by $H$ if $H$ satisfies the following version of strategy-proofness for correspondences: for all $R^N \in L^N, i \in N$, and $R_*^i \in L$, there exists no $a \in H(R^N)$ and $b \in H(R_*^i, R^{N\setminus\{i\}})$, $b \neq a$ such that $bR^i a$. By Lemma 4.3 (4.1) is a necessary condition for weak constitutional implementability. Our next theorem, the main result of this paper, shows that (4.1) is a sufficient condition for the existence of an almost constitutional implementation.

**Theorem 4.4.** *Let $H : L^N \to 2^A$ be a surjective and strongly monotonic SCC. Furthermore, let $n \geq 3$. If $H$ satisfies (4.1), then $H$ has an almost constitutional implementation.*

*Proof.* First we introduce some notations. We denote

$$gr(H) = \{(R^N, a) | R^N \in L^N \quad \text{and } a \in H(R^N)\},$$

and $Z_+ = \{0, 1, 2, \dots\}$. Further, for $i \in N$ let $W^i = \{S \in W(H) | i \in S\}$. $N \in W(H)$ because $H$ is surjective. Hence, $W^i \neq \emptyset$ for all $i \in N$.

We now define a GF $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ by the following rules.
(i) $\Sigma^i = gr(H) \times Z_+ \times W^i$ for every $i \in N$. Thus, if $\sigma^i \in \Sigma^i$, then $\sigma^i = (R_i^N, a^i, t^i, S^i)$, where $R_i^N \in L^N, a^i \in H(R_i^N), t^i \in Z_+$, and $S^i \in W^i$.

It remains to define the outcome function $\pi$. Let $\sigma = (\sigma^1, \dots, \sigma^n) \in \Sigma$.
(ii) If $\sigma^i = (R^N, a, 0, N)$ for every $i \in N$, then $\pi(\sigma) = a$.
(iii) If $\sigma^i = (R^N, a, 0, N)$ for all $i \neq j$, then $\pi(\sigma) = a^j$ if $a^j \in Ess_j(L(a, R^j), H)$, and $\pi(\sigma) = a$ otherwise. (Here $R^j$ is the $j$-th component of $R^N$.)
(iv) If there exists $S \in W(H), S \neq N$ such that $\sigma^i = (R^N, a, 0, S)$ for every $i \in S$, then there is a unique player $j \notin S$ such that $(t^j, j)$ is the lexicographic maximum of the pairs $(t^k, k), k \notin S$. We define $\pi(\sigma) = a^j$ if $a^j \in L(a, R^j)$, and $\pi(\sigma) = a$ otherwise. (Again, $R^j$ is the $j$-th component of $R^N$.)
(v) In all other cases let $j$ be the unique player such that $(t^j, j)$ is the lexicographic maximum of $(t^k, k), k \in N$, and define $\pi(\sigma) = a^j$.

We claim that $\Gamma$ is an almost constitutional implementation of $H$. The proof consists of the following steps.

*Step 1.* Let $R^N \in L^N$ and $a \in H(R^N)$. Define $\sigma \in \Sigma$ by $\sigma^i = (R^N, a, 0, N)$ for every $i \in N$. By (iii) (of the definition of $\Gamma$) $\sigma$ is an NE of $(\Gamma, R^N)$, and $\pi(\sigma) = a$ (by (ii)).

*Step 2.* Let $R^N \in L^N$ and let $\sigma \in \Sigma$ be an NE of $(\Gamma, R^N)$. We must show that $\pi(\sigma) \in H(R^N)$. Several cases are distinguished.
(2.1) $\sigma^i = (\underset{\sim}{R}^N, a, 0, N)$ for every $i \in N$. By (iii) and our assumption that $\sigma$ is an NE

$$Ess_i(L(a, \underset{\sim}{R}^j), H) \subseteq L(a, R^j) \quad \text{for every } j \in N.$$

As $a \in H(\underset{\sim}{R}^N)$ and $H$ is strongly monotonic, we obtain that $a \in H(R^N)$. Furthermore, $\pi(\sigma) = a$ by (ii).
(2.2) $\sigma^i = (\underset{\sim}{R}^N, a, 0, N)$ for all $i \neq j$. Let $b = \pi(\sigma)$. Then $b \in Ess_j(L(a, \underset{\sim}{R}^j), H)$ by (iii). Also,

$$Ess_j(L(a, \underset{\sim}{R}^j), H) \subseteq L(b, R^j)$$

because $\sigma$ is an NE. Now by $n \geq 3$ and (v)

$$A = L(b, R^i) \quad \text{for all } i \neq j.$$

Hence, by the Lemma in Yamato (1992), $b \in H(R^N)$.
(2.3) $\sigma^i = (\underset{\sim}{R}^N, a, 0, S)$ for all $i \in S$, where $S \in W(H)$ and $S \neq N$. By (v) and $n \geq 3$, for all $i \in S, A = L(b, R^i)$, where $\pi(\sigma) = b$. As $S \in W(H)$ and $H$ is Maskin monotonic, $b \in H(R^N)$.

(2.4) In all remaining possibilities we obtain, by (v) and $n \geq 3$, $A = L(b, R^i)$ for all $i \in N$, where $b = \pi(\sigma)$. As $H$ is surjective and Maskin monotonic, $b \in H(R^N)$.

*Step 3.* $W(\Gamma) = W(H)$.

We first show that $W(\Gamma) \supset W(H)$. Clearly, $N \in W(\Gamma) \cap W(H)$. Thus, let $S \in W(H)$, $S \neq N$, and let $a \in A$. There exists $R^S \in L^S$ such that $H(R^S, Q^{N \setminus S}) = \{a\}$ for all $Q^{N \setminus S} \in L^{N \setminus S}$. Choose $\underset{\sim}{Q}^{N \setminus S} \in L^{N \setminus S}$ such that $b\underset{\sim}{Q}^i a$ for all $i \in N \setminus S$ and $b \in A$. Now define $\sigma^i = ((R^S, \underset{\sim}{Q}^{N \setminus S}), a, 0, S)$ for all $i \in S$. By (iv), $\pi(\sigma^S, \mu^{N \setminus S}) = a$ for all $\mu^{N \setminus S} \in \Sigma^{N \setminus S}$. Hence, $\{a\} \in E^{\Gamma}(S)$, and $S \in W(\Gamma)$.

Now let $S \in W(\Gamma)$. Assume, on the contrary, that $S \notin W(H)$. Then $|S| = n-1$ must be true. (If $|S| < n-1$ and $S \notin W(H)$, then $E^{\Gamma}(S) = \{A\}$ by (v)). Let $S = N \setminus \{i\}$. As $E^{\Gamma}(S) = 2^A$, for every $B \in 2^A$ there exist $a \in B$ and $R^N \in L^N$ such that $a \in H(R^N)$ and $Ess_i(L(a, R^i), H) \subset B$. Therefore, by (4.1) and Lemma 4.3, $E^H(S) = 2^A$. As $S \notin W(H)$, the desired contradiction has been obtained.

*Step 4.* $E^{\Gamma} \subseteq E^H$.

In view of the proof of Step 3 we only have to prove that $E^{\Gamma}(N \setminus \{i\}) \subseteq E^H(N \setminus \{i\})$ for every $i \in N$. However, the foregoing inclusions follow from Step 3, (4.1), (iii), and Lemma 4.3. Q.E.D.

**Corollary 4.5.** *Let $H : L^N \to 2^A$ be a surjective SCC and let $n \geq 3$. If $H$ has a weak constitutional implementation, then $H$ has an almost constitutional implementation.*

Corollary 4.5 is an immediate consequence of Lemma 4.3 and Theorem 4.4.

**Corollary 4.6.** *Let $H : L^N \to 2^A$ be an SCC and let $n \geq 3$. If $H$ is Maskin monotonic and*

$$N \setminus \{i\} \in W(H) \quad \textit{for every } i \in N, \tag{4.3}$$

*then $H$ has an almost constitutional implementation.*

*Proof.* By (4.3), $N \in W(H)$. Hence, $H$ is surjective. Also, by Maskin monotonicity and (4.3), $H$ satisfies no veto power. Therefore, $H$ is strongly monotonic. As (4.3) implies (4.1), the proof follows from Theorem 4.4. Q.E.D.

**Corollary 4.7.** *Let $G = (N, W)$ be a proper and monotonic simple game. If $G$ has no vetoers and $2 \leq |A| < \nu(G)$, then $C(G, A, \cdot)$ has a constitutional implementation.*

*Proof.* Let $H(R^N) = C(G, A, R^N)$ for every $R^N \in L^N$. Then $n \geq 3$, because $G$ has no vetoers. Also, by the same reason, (4.3) is satisfied. By Corollary 4.6 $H$ has an almost constitutional implementation $\Gamma$. Now $W(H) = W$ and $E^H = E(G)$. As $E^{\Gamma} \subseteq E^H$ and $W(\Gamma) = W(H)$, it follows that $E^H = E^{\Gamma}$. Q.E.D.

*Remark 4.8.* Let $G = (N, W)$ be a non-dictatorial weak game and let $2 \leq |A| < \infty$. Then $C(G, A, \cdot)$ has no weak constitutional implementation.

*Proof.* Let $H(R^N) = C(G, A, R^N)$ for every $R^N \in L^N$. If $i \in N$ is a vetoer, then $N \setminus \{i\}$ is blocking, because $i$ is not dictator. Nevertheless, $E^H(N \setminus \{i\}) = \{A\}$, because $i$ is a vetoer. Thus, $H$ violates (4.1). Q.E.D.,

Remark 4.8 addresses the constrains on constitutional implementation in the context of committees. In particular it implies that in the UN Security Council one cannot use constitutional implementation to implement the core correspondence because of permanent members' veto power.

## 5 Further examples

In this section we present two examples which are closely related to the results in Sect. 4. First we show that (4.1) does not imply that all implementations (in NE's) are weakly constitutional. We remark that Yamato's GF is weakly constitutional in this case (see Yamato 1992).

*Example 5.1.* Let $G = (4, 3)$, let $A = \{a, b, c\}$, and let $H(R^N) = C(G, A, R^N)$ for all $R^N \in L^N$. As $N \setminus \{i\} \in W(H)$, for every $i \in N$, $H$ satisfies (4.1). We now consider the following GF $\Gamma = (\Sigma^1, \Sigma^2, \Sigma^3, \Sigma^4; \pi; A)$:

(i) $\Sigma^1 = \Sigma^2 = gr(H) \times Z_+ \times A$, and $\Sigma^3 = \Sigma^4 = gr(H) \times Z_+$. Let $\sigma = (\sigma^1, \sigma^2, \sigma^3, \sigma^4) \in \Sigma$. $\pi$ is defined by the following rules. For $\sigma^i \in \Sigma^i$ let $\sigma^i_k$ be the $k$-th component of $\sigma^i$.

(ii) If $(\sigma^i_1, \sigma^i_2, \sigma^i_3) = (R^N, a, 0)$ for all $i \in N$, then $\pi(\sigma^N) = a$;

(iii) If $(\sigma^i_1, \sigma^i_2, \sigma^i_3) = (R^N, a, 0)$ for all $i \neq j$, then $\pi(\sigma^N) = a^j$ if $a^j \in L(a, R^j)$, and $\pi(\sigma^N) = a$ otherwise.

If (iii) is not satisfied, then let $(t^j, j)$ be the lexicographic maximum of $(\sigma^i_3, i), i \in N$. We now further distinguish the following possibilities.

(iv) $j \in \{1, 2\}$. Then $\pi(\sigma^N) = a^j$;

(v) If $j \in \{3, 4\}$ and $\sigma^1_4 = \sigma^2_4 = a$, then $\pi(\sigma^N) = a$;

(vi) If $j \in \{3, 4\}$ and $\sigma^1_4 \neq a$ or $\sigma^2_4 \neq a$, then $\pi(\sigma^N) = a^j$.

Notice that $H$ satisfies Maskin monotonicity and no veto power. Furthermore, our GF is identical to Maskin's (1985) except for our condition (v). As the reader may easily verify $\Gamma$ implements $H$ in NE. However, $\{a\} \in E^\Gamma(\{1, 2\})$ whereas $E^H(\{1, 2, \}) = \{A\}$.

Our next example shows that the EF of an almost constitutional implementation may not be derived from a simple game.

*Example 5.2.* Let $G = (5, 4)$, let $A = \{a, b, c\}$, and for $R \in L$ let $\beta(R) = y$, where $y \in A$ and $xRy$ for all $x \in A$. Now define an SCC $H$ by

$$H(R^N) = \{x \in C(G, A, R^N) | \; |\{i \in N | \beta(R^i) = x\}| < 3\}$$

for all $R^N \in L^N$. $H$ is Maskin monotonic and satisfies no veto power. Furthermore, $H$ is anonymous and neutral (see Sect. 2.3 in Peleg 1984a for the relevant definitions). Clearly

$$E^H(S) = \begin{cases} 2^A, & ,|S| \geq 4 \\ \{A \setminus \{x\} | x \in A\}, & |S| = 3 \\ \{A\} & ,|S| = 1,2 \\ \emptyset & ,S = \emptyset \end{cases}$$

We shall now prove that $H$ has a constitutional implementation and thereby obtain the counter-example.

Let $\Gamma = (\Sigma^1, \ldots, \Sigma^5; \pi; A)$ be defined by the following rules.

(i) $\Sigma^i = gr(H) \times Z_+ \times 2^N \times A$ for every $i \in N$.

Let $\sigma^i = (R_i^N, a^i, t^i, S^i, b^i), i \in N$, where $R_i^N \in L^N, a^i \in H(R_i^N), t^i \in Z_+, S^i \in 2^N$, and $b^i \in A$. We now specify $\pi$.

(ii) If $(R_i^N, a^i, t^i, S^i) = (R^N, a, 0, N)$ for all $i \in N$, then $\pi(\sigma) = a$.

(iii) If $(R_i^N, a^i, t^i, S^i) = (R^N, a, 0, N)$ for all $i \in N \setminus \{j\}$, then $\pi(\sigma) = a^j$ if $a^j \in L(a, R^j)$, and $\pi(\sigma) = a$ otherwise.

(iv) If there exists $S \subset N, |S| = 3$, such that $(R_i^N, a^i, t^i, S^i, b^i) = (R^N, a, 0, S, b)$ for all $i \in S$, then let $j$ be the unique player such $(t^j, j)$ is the lexicographic maximum of $(t^k, k), k \in N$. Furthermore, let $\pi(\sigma) = a^j$ if $j \in S$, and $\pi(\sigma) = a^j$ if $j \notin S$ and $a^j \neq b$, and, finally, $\pi(\sigma) = a$ if $j \notin S$ and $a^j = b$.

(iv) In all other cases let $(t^j, j)$ be the lexicographic maximum of $(t^k, k), k \in N$, and let $\pi(\sigma) = a^j$.

As the reader may easily verify, $\Gamma$ implements $H$ in NE. We only will prove that $E^H = E^\Gamma$. Clearly, $E^\Gamma(S) = 2^A$ if $|S| \geq 4$, and $E^\Gamma(S) = \{A\}$ if $|S| = 1, 2$. Thus, let $S \subset N, |S| = 3$, and let $x \in A$. Choose $R^N \in L^N$ and $y \in A \setminus \{x\}$ such that $y \in H(R^N)$. If $\sigma^i = (R^N, y, 0, S, x)$ for all $i \in S$, then $\pi(\sigma^S, \mu^{N \setminus S}) \neq x$ for all $\mu^{N \setminus S} \in \Sigma^{N \setminus S}$. Thus, $E^\Gamma(S) = \{A \setminus \{\xi\} | \xi \in A\}$.

## 6 Some results on constitutional implementation

We shall prove in this section two results:

(i) Every implementation in NE of a two-person SCC is a constitutional implementation; and

(ii) if $H$ is a (non-dictatorial and unanimous) three-person SCC that has a constitutional implementation, then $E^H$ is derived from three-person strong simple game.

In the course of the proof of the second result we shall obtain some results on NE implementation of independent interest.

### *6.1 Two-person Nash implementation*

Let $N = \{1, 2\}$, let $2 \leq |A| < \infty$, and let $H : L^N \to 2^A$ be an SCC. If $\Gamma = (\Sigma^1, \Sigma^2; \pi; A)$ is a finite GF (i.e., $|\Sigma^i| < \infty, i = 1, 2$), that implements $H$ in NE's, then $E^\Gamma$ is maximal (see Gurvich 1989; Abdou 1995).

**Theorem 6.1.** *Let $N = \{1, 2\}$, let $2 \le |A| < \infty$, and let $H : L^N \to 2^A$ be an SCC. If the finite GF $\Gamma = (\Sigma^1, \Sigma^2; \pi; A)$ implements $H$ in NE's, then $E^\Gamma = E^H$.*

*Proof.* It is sufficient to prove that $E^H(\{i\}) \supseteq E^\Gamma(\{i\})$ for $i = 1, 2$. (Recall that we assume that both $H$ and $\Gamma$ are surjective.) Thus, let say, $B \in E^\Gamma(\{1\})$. We may assume that $B \ne A$. By definition, there exists $\sigma^1 \in \Sigma^1$ such that $\pi(\sigma^1, \mu^2) \in B$, for all $\mu^2 \in \Sigma^2$. Let $R^1 \in L(A)$ satisfy $BR^1A \setminus B$ (i.e., $xR^1y$ for all $x \in B$ and $y \in A \setminus B$). If $Q^2 \in L(A)$ and $\mu$ is an NE of $(\Gamma, (R^1, Q^2))$, then $\pi(\mu) \in B$. Hence, $H(R^1, Q^2) \subseteq B$ for all $Q^2 \in L(A)$. Therefore, $B \in E^H(\{1\})$. We have proved that $E^H(\{1\}) \supseteq E^\Gamma(\{1\})$. Similarly, $E^H(\{2\}) \supseteq E^\Gamma(\{2\})$. As $E^\Gamma$ is maximal and $E^H$ is superadditive, $E^\Gamma = E^H$. Q.E.D.

As a corollary of the proof of Theorem 6.1 we obtain the following result: Let $2 \le |A| < \infty$, and let $H : L^N \to 2^A$ be an SCC, and let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a finite GF that implements $H$ in NE's. Then $E^H(\{i\}) \supseteq E^\Gamma(\{i\})$ for all $i \in N$. This result is a special case of (the proof) of Theorem 4.1 in Peleg, Peters and Storcken (2001).

### *6.2 Constitutional implementation of unanimous SCCs*

We start with the following general result.

**Theorem 6.2.** *Let $H : L^N \to 2^A$ be non-dictatorial and satisfy the unanimity condition. If $H$ has a constitutional implementation, then $H$ is Maskin monotonic and*

$$N \setminus \{i\} \in W(H) \quad \textit{for every } i \in N. \tag{6.1}$$

The following lemma is used in the proof of Theorem 6.2.

**Lemma 6.3.** *Let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a GF. Assume that there exist two coalitions $S, T \subset N, S \cap T = \emptyset$, and an alternative $x \in A$ such that $A \setminus \{x\} \in E^\Gamma(S) \cap E^\Gamma(T)$. Then there exist $R^N \in L^N$ and an NE of $(\Gamma, R^N)$ whose outcome is Pareto dominated.*

*Proof of Theorem 6.2.* Let $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ be a constitutional implementation of $H$. Then, in particular, $\Gamma$ implements $H$ in NE's. Hence $H$ is Maskin monotonic. As $H$ is unanimous, $H$ must be Paretian. Thus, all the NE's of $\Gamma$ are Pareto optimal. We shall now prove the following claim:

$$\text{If } i \in N, \quad \text{then } E^H(\{i\}) = \{A\}. \tag{$*$}$$

Assume, on the contrary, that there exist $j \in N$ and $x \in A$ such that $A \setminus \{x\} \in E^H(\{j\})$. Then $A \setminus \{x\} \in E^\Gamma(\{j\})$. Let $\hat{R}^j \in L$ satisfy $t(\hat{R}^j) = x$ (i.e., $x\hat{R}^jy$ for all $y \in A$). Then $H(R^{N\setminus\{j\}}, \hat{R}^j) = \{x\}$ for all $R^{N\setminus\{j\}} \in L^{N\setminus\{j\}}$. Indeed, if $H(Q^{N\setminus\{j\}}, \hat{R}^j) \setminus \{x\} \ne \emptyset$ for some $Q^{N\setminus\{j\}}$, then, by (3.1), $A \setminus \{x\} \in E^\Gamma(N \setminus \{j\})$, contradicting Lemma 6.3. Thus, $\{x\} \in E^H(\{j\})$ which implies $A \setminus \{y\} \in E^H(\{j\})$ for all $y \in A$. Therefore, by the foregoing argument,

$$[\hat{R}^j \in L \quad \text{and } y = t(\hat{R}^j)] \Rightarrow H(R^{N\setminus\{j\}}, \hat{R}^j) = \{y\}$$

for all $R^{N\setminus\{j\}} \in L^{N\setminus\{j\}}$ and $y \in A$. Hence $j$ is a dictator for $H$,and the desired contradiction has been obtained.

We now prove (6.1). Assume, on the contrary, that there exist $i \in N$ and $B \in 2^A$ such that $B \notin E^H(N \setminus \{i\})$. Then $B \notin E^\Gamma(N \setminus \{i\})$. Hence, by (3.1), if $R^i \in L, a \in B$, and $L(a, R^i) = B$, then $a \notin H(Q^{N\setminus\{i\}}, R^i)$ for all $Q^{N\setminus\{i\}}$ in $L^{N\setminus\{i\}}$. Thus, $A \setminus \{a\} \in E^H(\{i\})$ contradicting $(*)$. Q.E.D.

We now prove Lemma 6.3.

*Proof of Lemma 6.3.* Choose $\sigma^S \in \Sigma^S$ and $\sigma^T \in \Sigma^T$ such that $\pi(\sigma^S, \mu^{N\setminus S}) \neq x$ for all $\mu^{N\setminus S} \in \Sigma^{N\setminus S}$, and $\pi(\sigma^T, \mu^{N\setminus T}) \neq x$ for all $\mu^{N\setminus T} \in \Sigma^{N\setminus T}$. Further, let $\sigma^{N\setminus(S\cup T)} \in \Sigma^{N\setminus(S\cup T)}$, and denote $y = \pi(\sigma) = \pi(\sigma^S, \sigma^T, \sigma^{N\setminus(S\cup T)})$. Now choose a profile $R^N \in L^N$ that satisfies $xR^iyR^iA \setminus \{x, y\}$ for all $i \in N$. Then $\sigma = (\sigma^S, \sigma^T, \sigma^{N\setminus(S\cup T)})$ is an NE of $(\Gamma, R^N)$ and $y = \pi(\sigma)$ is not Pareto optimal. Q.E.D.

**Corollary 6.4.** *Let $N = \{1, 2, 3\}$ and let $H : L^N \to 2^A$ be a non-dictatorial and unanimous SCC. Then $H$ has a constitutional implementation if and only if $H$ is Maskin monotonic and $E^H = E(G)$, where $G = (3, 2)$.*

Corollary 6.4 is an immediate consequence of (6.1) and Theorem 4.4. The next corollary sheds new light on the no veto power assumption.

**Corollary 6.5.** *Let $H : L^N \to 2^A$ be a non-dictatorial and unanimous SCC. If $H$ has a constitutional implementation, then $H$ satisfies no veto power.*

*Proof.* As the reader may easily verify (6.1) and Maskin monotonicity imply no veto power. Q.E.D.

*Remark 6.6.* The SCC $H$ of Example 5.2 is non-dictatorial and unanimous. Nevertheless, by Example 5.2, it has a constitutional implementation whose EF is not derived from a simple game.

The next example shows that unanimity is a necessary condition for Theorem 6.2.

*Example 6.7.* Let $N = \{1, 2, 3\}$ and let $|A| = 4$. For $R \in L$ let $\beta(R) = y$, where $y \in A$ and $xRy$ for all $x \in A$. Define $H : L^N \to 2^A$ by

$$H(R^N) = \{x \in A | x \neq \beta(R^i) \quad \text{for } i = 1, 2, 3\}.$$

Then $H$ is surjective and Maskin monotonic (but not unanimous). Also

$$E^H(S) = \{B \subseteq A| \; |B| \geq 4 - |S|\} \quad \text{for } |S| = 1, 2, 3.$$

Thus, $H$ does not satisfy (6.1). Nevertheless, we claim that $H$ has a constitutional implementation.

Consider the following GF $\Gamma = (\Sigma^1, \Sigma^2, \Sigma^3; \pi; A)$. Let $\Sigma^i = gr(H) \times Z_+, i \in N$, where $Z_+ = \{0, 1, 2, \dots\}$. $\pi$ is defined by the following rules. Let $\sigma^i = (R_i^N, a^i, t^i)$ where $R_i^N \in L^N, a \in H(R_i^N), t^i \in Z_+$, for $i \in N$.

(i) If $(R_i^N, a^i, t^i) = (R^N, a, 0)$ for all $i \in N$, then $\pi(\sigma) = a$.

(ii) If $(R_i^N, a^i, t^i) = (R^N, a, 0)$ for all $i \neq j$, then $\pi(\sigma) = a^j$ if $a^j \in L(a, R^j)$, and $\pi(\sigma) = a$ otherwise.

(iii) In all other cases let $(t^j, j)$ be the lexicographic maximum of $(t^k, k), k \in N$, and let $\pi(\sigma) = a^j$.

As the reader may easily verify, $E^\Gamma = E^H$. Furthermore, $\Gamma$ implements $H$ in NE's.

## 7 Discussion

We shall now present some (straightforward) extensions of our results, and compare our construction in the proof of Theorem 4.4 with some earlier constructions of implementations in NE (see Maskin 1985 and Yamato 1992). We also shall comment on the importance of our work from the point of view of applications.

### *7.1 Refinements of Nash equilibrium*

An equilibrium concept $e$ is a *refinement* of NE if for every GF $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$, and for every $R^N \in L^N, e(\Gamma, R^N) \subseteq NE(\Gamma, R^N)$. Let $e$ be a refinement of NE and let $H : L^N \to 2^A$ be an SCC. If the GF $\Gamma = (\Sigma^1, \dots, \Sigma^n; \pi; A)$ implements $H$ in $e$, then, obviously, $\Gamma$ satisfies (3.1). We say that a GF $\Gamma$ is a weak constitutional implementation of $H$ in $e$ if: (i) $\Gamma$ implements $H$ in $e$; and (ii) $E^\Gamma \subseteq E^H$. It follows now that (4.1) is a necessary condition for weak constitutional implementability by any refinement of NE. Thus, the SCC of Example 3.2 is not weakly constitutionally implementable in any refinement of Nash.

For the rest of this subsection we focus on the subgame perfect equilibrium (SPE) concept. (Thus, we consider *extensive* GF's, and not only GF's). As every GF is, trivially an extensive GF, we obtain that every implementation in NE is an implementation in SPE. Therefore, Theorem 4.4 is generalized to SPE implementation. ((iii) of Definition 4.1 is generalized in a straightforward manner). However, strong monotonicity is not a natural condition for implementation in SPE's. Thus, the extension of our results to implementation in SPE's and in undominated Nash equilibria (see Jackson et al. 1994) remains as an open problem.

### *7.2 Weak preferences*

All our results remain true for weak preference orderings, that is, complete and transitive binary relations.

### *7.3 Previous constructions of NE implementations*

Our construction of NE implementation in the proof of Theorem 4.4 is comparable with that of Yamato (1992). Let $H$ be a strongly monotonic SCC that satisfies (4.1). Yamato's GF is a weak constitutional implementation of $H$ which may not be almost constitutional. We enlarge the message space of each player and thereby obtain an almost constitutional implementation.

### *7.4 Sen's liberal paradox and constitutional implementation*

Let $H : L^N \to 2^A$ be an SSC. $H$ satisfies *minimal liberalism ML* if there exist $i, j \in N, i \neq j$ such that $E^H(k) \neq \{A\}, k = i, j$ (see Peleg 1998). Peleg (1998) found tension between ML and Nash implemenation (See Theorem 5.1 ibid). Here we have the following corollary of Theorem 6.2.

**Corollary 7.1.** *Let H be a constitutionally implementable SCC. If H satisfies ML, then H violates unanimity.*

Corollary 7.1 expresses tension between ML and unanimity on the class of constitutionally implementable SCC's. Thus it generalizes the liberal paradox to our model. Also, it may explain why we concentrated on the notion of almost constitutional implementation. Indeed, there is no tension between ML and unanimity on the set of almost constitutionally implementatable SCC's.

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