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THE COMMON PRIOR ASSUMPTION IN BELIEF SPACES: AN EXAMPLE

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The Common Prior Assumption in Belief Spaces: An Example

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Summary: With four persons there is an example of a probability space where 1) the space is generated by hierarchies of knowledge concerning a single proposition, 2) the subjective beliefs of the four persons are continuous regular conditional probability distributions of a common prior probability distribution (continuous with respect to the weak topology), and 3) for every subset that the four persons know in common there is no common prior probability distribution. Furthermore, for every measurable set, every person, and at every point in the space, the subjective belief in this measurable set is one of the quantities 0, 1/2, or 1. This example presents problems for understanding games of incomplete information through common priors.

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1 Introduction

Although the results of this paper belong to logic, combinatorics and probability theory, they were motivated by games of incomplete information. A game of incomplete information is a game where the players have different knowledge of some facts that are relevant to the outcome of the game. Card games are games of incomplete information; chess is a game of complete information. With games of incomplete information, one's belief concerning the belief of a fellow player can be more important than one's own knowledge of the facts. Witness the use of the bluff in poker; the certain knowledge of the bluffing player that she has a loosing hand is secondary to her opponents' belief that her hand is likely to be strong.

An equilibrium of a game is a set of strategies, one for each player, such that no player does better by choosing a different strategy, given that the other players do not change their strategies. In games of incomplete information a strategy of a player is a function of what the player believes. For games of incomplete information, there are two main approaches on how one should evaluate the strategy choices of the players -- descriptively, according to their subjective beliefs, and proscriptively, according to a probability distribution determined objectively by the game. Desirable is a synthesis, a way for the players' subjective beliefs and evaluations of strategies to be generated by a common prior probability distribution. Such a synthesis is the central idea behind the pioneering paper by J. Harsanyi (1967-8); here games of incomplete information are understood similarly to those of complete information, ultimately through the existence of a common prior. A common prior allows for a global understanding and evaluation that has profound implications for the possible local situations of a game.

In a game of cards, the distribution of hands determined by a perfect shuffle is common knowledge among the players. There is a bijection between the subjective beliefs of the players and the hands that could be distributed – a shuffle of the cards determines the hands and a hand of a given player determines for that player a Bayesian conditional probability distribution on the possible hands of the other players. But what if such a bijection does not exist? How does one model a game where a player does not know how the other players will form their beliefs?

Belief spaces model the uncertainty of players, concerning both the parameters relevant to the payoffs of the game and the beliefs of the other players. Belief spaces are self referential in nature – a point in a belief space is described by a parameter and for each player a subjective probability distribution on the belief space itself.

There are different ways to define belief spaces, and this paper concerns itself with the Mertens-Zamir (Mertens and Zamir, 1985) definition. We look at Mertens-Zamir belief spaces because they contain a nice relationship between subjective beliefs and common priors. A Mertens-Zamir belief space is a compact space such that at every point and for every player there is a subjective probability distribution on this space that is Borel and regular; furthermore for any fixed player this distribution changes continuously with respect to the weak topology. A priori, no common prior is assumed. Furthermore there is an underlying compact perameter set and a continuous map from the space to this parameter set. Compactness and regularity permit, for every such distribution, the existence of a "support" set, the smallest compact set of measure one. For each player a self consistency condition is required – if a point y is in the support of the subjective probability distribution of this player at the point x, then at both x and y this player has the same subjective probability distribution. The support sets add a combinatorial aspect to the players' beliefs. A compact subset Y is called a belief subspace when at every point y in Y every player's support set for the point y is contained in Y. A common prior of a belief subspace is a probability distribution on this belief subspace such that the subjective beliefs inside this subspace are conditional probability distributions of the common prior.

Define a *cell* of a Mertens-Zamir belief space to a minimal set C with the property that at every point y in C every player's support set for the point y is contained in C (without the requirement that C must be compact).

Our result is an example, involving four players, of a belief space (according to the Mertens-Zamir definition) where there is a common prior, the beliefs of the players are continuous, however there is no common prior defined on any of the cells of the space. Furthermore, our space is *strongly non-redundant* with respect to a finite parameter set; (see Section 2.1). Non-redundancy means informally that any two distinct points differ at some level of a hierarchy of belief generated initially by the given perameter set; strong non-redundancy means the same with respect to hierarchies of knowledge. (An example of part of a knowledge hierarchy would be the following: Adam knows that the proposition p is true, Adam knows that Eve does not know whether p is true.) Because the axioms of the Mertens-Zamir spaces are strong, our example would apply to most other formulations of belief spaces.

The example is constructed from a Kripke structure for the S5 multi-person modal logic. For two players and one primitive proposition, we examine a special closed subset S of the space of maximally consistent lists of formulas. This subset S is that where at every point both players consider at most two points to be possible. S is homeomorphic to a Cantor set. We consider the infinite dyadic group acting on the Cantor set, generated by two involutions. We introduce two new players with partitions corresponding, respectively, to the two-set orbits of these involutions. The orbit of any point generated by the whole group is a dense subset of S. The meet partition corresponding to the common knowledge of the four players contains only cells that are countable and dense in S. To construct the subjective beliefs of the players, if a player considers two points possible, she gives 1/2 - 1/2 probabilities to these two points; otherwise all weight is given to the only point considered possible. The partition structures for all four players are generated by measure preserving involutions that allow the 1/2 −1/2 assignment to be a conditional probability.

What could go wrong with such a construction? Starting with a Polish space, a Borel probability distribution on this space as a common prior, and a Borel field for each player, one can construct regular conditional probability distributions for each player (Dudley, 1989; Theorem 10.2.2), but there is no guarantee that there are such distributions with continuous versions, (such an example is given in the last section. A *regular* conditional probability distribution is a family of conditional probabilities, one for each Borel subset to be evaluated, however perceived also as a function from the space Ω in question to the space of probability distributions on Ω.) Given that one can construct regular conditional probability distributions for the players that are continuous with respect to the weak topology, there may exist new common priors defined on many of the cells held in common knowledge. (Always there will be regular conditional probability distributions of the original common prior with respect to a sigma algebra generated by the cells, but the question is whether they are compatible with those corresponding to the players. Our result demonstrates a non-commutativity in the operation of creating a regular conditional probability distribution.) On the other hand, if one starts with subjective beliefs that satisfy the desired properties there is no reason in general why they should build a common prior, let alone confirm the one already chosen.

Furthermore, without the non-redundancy conditions, it is much easier (and not interesting) to find a belief space with the other desired properties. After identifying points that are equivalent with respect to all hierarchies of beliefs, the space could lose the other desired properties. Such would be the case for our example if one were to discard the first two of the players and keep the last two players that correspond to the generators of the infinite dyadic group – the Cantor set would collapse to two points. In our example, the first two are the important players, because the beliefs of these players make the space strongly non-redundant. Finding a Kripke structure for two players such that their knowledge partitions could be the orbits of measure preserving involutions *and* such that every point represents a unique knowledge hierarchy was the major difficulty.

In understanding games of incomplete information modeled by belief spaces one must consider the minimal subsets of the belief space that are relevant to such a game. The smallest subset known in common (a cell) is relevant to equilibrium behavior. The smallest subset on which there exists a common prior distribution is relevant to the mutual consistency of the players and the definition of the game. Our results shows that these two sets can be very different.

We consider our example to be a problem for the Hansanyi approach to understand games of incomplete information through common priors. Given a game modeled by our example, one cannot understand the game through common priors, because the common knowledge of the players is beyond that of any common prior. On the other hand, there is nothing mutually inconsistent about the beliefs of the players that could to be corrected through a process toward a better common understanding of the game. Our example suffers not from a lack but from an excess of common knowledge.

To illustrate the problem, define a game Γ modeled on a belief space S with a common prior μ in the following way. Let X be the finite perameter space, $\psi : S \to X$ a continuous function, N the finite player set, and for each $j \in N$ $t^j : S \to \Delta(S)$ are the subjective beliefs of Player j. For each player j, there is a finite action set A^j with $n^j := |A^j|$. There are $|X|$ different $n^1 \times \ldots \times n^{|N|}$ matrices corresponding to the set X, every entry of every matrix is a vector payoff for the players in $\mathbf{R}^N$. Nature chooses a point in S according to the common prior μ, which means also that a parameter in X is chosen through the function ψ. The players choose actions in their respective A^j independently, and after the choices are made the payoff to the players

is the vector entry corresponding to their actions and nature's choice of the parameter in X. An equilibrium for a point z in the belief space is an $|N|$-set of functions $(f^j \mid j \in N)$, each f^j from the cell that contains z to $\Delta(A^j)$, the simplex of mixed strategies, with the following properties for every player $j \in N$
1) f^j is constant within all support sets of Player j,
2) for all $j' \neq j$ within the support set of $t^{j'}(z)$ the function f^j is $t^{j'}(z)$ measurable, and
3) within the support set of $t^j(z)$ Player j can do no better than $f^j(z) \in \Delta(A^j)$ in response to the other functions $f^{j'}, j' \neq j$, as evaluated by $t^j(z)$.
When the $|N|$-set of functions is an equilibrium for all points in a cell, then we call it a *cellular* equlibrium. A *global* equilibrium is a set of functions $(f^j : S \to \Delta(A^j) \mid j \in N)$, each f^j *measurable* with respect to the Borel field on S generated by Player j's continuous beliefs, such that no player can attain a higher expected payoff as evaluated by μ by choosing another such measurable function, (given that the strategies of the other players do not change).

A collection of cellular equilibria, one for each cell, will also define an equilibrium valid at *all* points of our belief space. In our example, every player at every point has a discrete subjective probability distribution (with support sets that are either singletons or two-sets), and therefore there are no measurability conditions to a cellular equilibrium. The collection of all cells adds no new conditions on the strategy functions, so that the only condition on a player's strategy function is that it must be constant within any of her finite support sets. Therefore there may be equilibria of the game Γ (perhaps preferable in some ways to the global equilibria), for which an evaluation of the payoffs according to the common prior may not be possible. Indeed, because the space of measurable functions available to a player with the L_1 norm (with respect to the common prior) is not compact, it is plausible that there exist no global equilibria for the game Γ, yet cellular equilibria could exist for all points of the space. To escape this potential crisis, we may prefer to perceive Γ as a collection of uncountably many independent games, one for each cell. Although we must understand equilibria independently for each cell, the game Γ cannot be broken down into uncountably many independent games, one for each cell, because there is no probability distribution for choosing a point once a cell has been chosen.

In the next section we introduce the necessary background for constructing our example: Mertens-Zamir belief spaces, the S5 modal logic of interactive epistemology, Kripke structures, and a canonical hierarchical construction of finite Kripke structures. In Section 3 we present the two-person Kripke structure on which our example is based. Lemma 3 is the central lemma of this paper; it shows that our hierarchical construction of the Kripke structure grows evenly by a fixed power of two. In Section 4 we define our four person example by building upon Section 3. In Section 5 we prove our main results.

In conclusion, Section 6, we consider the assumption in Mertens and Zamir (1985) that the subjective beliefs are continuous (with respect to the weak topology). We present a relatively simple example of continuously defined one-player uncertainty on a Cantor set with a non-atomic prior distribution such that positive mass is given to all non-empty open sets, yet there is no continuous regular conditional probability distribution. This example may demonstrate an external limitation of the Mertens-Zamir spaces to model multi-player uncertainty.

2 Background

In this paper we will assume that all belief spaces are Polish spaces; we can assume that there is no distinction between the Baire and Borel sets and measures and the regularity of all Borel measures can be assumed. $\Delta(S)$ stands for the space of regular Borel probability measures on S. Unless otherwise stated, the measurable sets of a topological space are the Borel sets generated by its topology. The topology for a $\Delta(S)$ will be the weak topology.

Due to the unique perspective on conditional probability explicit in belief spaces, a regular conditional probability distribution on a topological space S will be represented in an unorthodox way as a function from S to $\Delta(S)$. Given a probability distribution $\mu \in \Delta(S)$ and a Borel field $\mathcal{G}$, a function $f : S \to \Delta(S)$ is a regular conditional probability distribution for μ and $\mathcal{G}$ when for every fixed Borel set A the function $f(\cdot)(A) \to [0,1]$ is measurable and for every set $B \in \mathcal{G}$ we have $\int_B f(z) d\mu(z) = \mu(A \cap B)$.

We call a partition $\mathcal{P}$ of a topological space D upper (respectively lower) semi-continuous if the set valued correspondence that maps every $d \in D$ to the partition member of $\mathcal{P}$ containing d is an upper (respectively lower) semi-continuous correspondence. (We follow the definitions of Klein and

Thompson, 1984.) Upper and lower semi-continuity of a correspondence between Polish spaces implies Hausdorff continuity of the correspondence.

2.1 Mertens-Zamir Belief Spaces

A Belief Space (Mertens and Zamir, 1985) is a tuple $(S, X, \psi, N, (t^j : \mid j \in N))$, where X is a compact parameter set, S is a compact set, ψ is a continuous map from S to X, N is a finite set of players, for every $j \in N$ $t^j :\rightarrow \Delta(S)$ is a continous function (with respect to the weak topology), and for every player j and every pair of points $s, s' \in S$ if $s' \in$ support $(t^j(s))$ then $t^j(s) = t^j(s')$.

A belief morphism between two spaces $(S, X, \psi, N, (t^j : \mid j \in N))$ and $(\tilde{S}, \tilde{X}, \tilde{\psi}, N, (\tilde{t^j} : \mid j \in N))$ is defined to be a pair of functions $\phi : X \rightarrow \tilde{X}$ and $\phi' : S \rightarrow \tilde{S}$ such that $\phi \circ \psi = \tilde{\psi} \circ \phi' : S \rightarrow \tilde{X}$ and for every player $j \in N$ $\hat{\phi}' \circ t^j = \tilde{t^j} \circ \phi' : S \rightarrow \Delta(\tilde{S})$, where $\hat{\phi}' : \Delta(S) \rightarrow \Delta(\tilde{S})$ is induced canonically from ϕ', (Mertens and Zamir, 1985).

A subspace of a belief space $(S, X, \psi, N, (t^j : \mid j \in N))$ is defined to be a subset $C \subseteq S$ such that C is compact and for every $s \in C$ and every player $j \in N$ support $(t^j(s)) \subseteq C$. The subspace C is a belief space itself, with the functions $t^j|_S$ and $\psi|_S$ those inherited from the original belief space through restriction.

Starting with any compact set X and player set N, Mertens and Zamir (1985) construct a canonical belief space corresponding to X and N through belief hierarchies. Given a belief space $(S, X, \psi, N, (t^j : \mid j \in N))$, define $\mathcal{F}$ to be the smallest sigma algebra of subsets of S for which ψ is measurable with respect to $\mathcal{F}$ and for every player $j \in N$ and every $B \in \mathcal{F}$ $t^j(\cdot)(B) : S \rightarrow [0, 1]$ is measurable with respect to $\mathcal{F}$. They define a belief space to be *non-redundant* if the sigma algebra $\mathcal{F}$ separates all distinct pairs of points of S. They prove that any non-redundant belief space is isomorphic (with respect to belief space morphisms) to a subspace of their canonical belief space.

Define $\tilde{\mathcal{F}}$ to be the smallest sigma algebra of subsets of S for which ψ is measurable with respect to $\tilde{\mathcal{F}}$ and for every player $j \in N$ and every $B \in \tilde{\mathcal{F}}$ the set $\{z \mid \text{support}(t^j(z)) \subseteq B\}$ is in $\tilde{\mathcal{F}}$. We define a belief space to be *strongly non-redundant* if the sigma algebra $\tilde{\mathcal{F}}$ separates all distinct pairs of points of S.

Of special interest to this paper is the definition of mutual consistency for Mertens-Zamir belief subspaces. For every player $j \in N$ and any subspace

$Y \subseteq S$ define $\mathcal{T}^j$ to be the smallest Borel field of subsets of Y such that the function $t^j|_Y$ is measurable. A probability distribution μ on the subspace Y is defined to be *consistent* if for every Borel subset $A \subseteq Y$ we have that $\mu(A) = \int_Y t^j(y)(A)d\mu(y)$. Mertens and Zamir (1985) showed that consistency is equivalent to the stronger statement that for every $B \in \mathcal{T}^j$ and Borel subset $A \subseteq Y$ we have $\mu(A \cap B) = \int_B t^j(y)(A)d\mu(y)$. A belief subspace C is defined to be consistent if there is a consistent μ on C such that the support of μ is equal to C. Starting at any point $y \in S$, we can define for every player $j \in N$ the sets $C_i^j(y)$ inductively by $C_1^j(y) :=$ support $(t^j(y))$ and for $i \geq 2$ $C_i^j(y) := C_{i-1}^j(y) \cup_{\tilde{y} \in C_{i-1}^j(y),\ k \in N}$ support $(t^k(\tilde{y}))$. Define the set $C^j(y)$ to be $\cup_{i=1}^{\infty} C_i^j(y)$. Given that a subspace Y is finite, Mertens and Zamir proved that the consistency of Y implies that
1) for all $y \in Y$ and pairs of players $j, k \in N$ $C^j(y) = C^k(y)$,
2) $C^j(y)$ for all $j \in N$ is the smallest belief subspace containing y, and
3) there is a uniquely determined consistent distribution for the set $C^j(y)$, namely that induced naturally from μ.
Our example is a refutation of 2) and 3) for infinite and consistent Y.

2.2 Formulas and Modal Logic

Let X be a finite set of primitive propositions and N the set of players. Construct the set $\mathcal{L}(X, N)$ of formulas using the finite sets X and N in the following way:
1) If $x \in X$ then $x \in \mathcal{L}(X, N)$,
2) If $g \in \mathcal{L}(X, N)$ then $(\neg g) \in \mathcal{L}(X, N)$,
3) If $g, h \in \mathcal{L}(X, N)$ then $(g \wedge h) \in \mathcal{L}(X, N)$,
4) If $g \in \mathcal{L}(X, N)$ then $k_j g \in \mathcal{L}(X, N)$ for every $j \in N$,
5) Only formulas constructed through application of the above four rules are members of $\mathcal{L}(X, N)$.
We write simply $\mathcal{L}$ if there is no ambiguity. We define $g \vee h$ to be $\neg(\neg g \wedge \neg h)$ and $g \rightarrow h$ to be $\neg g \vee h$. $E(f) = E^1(f)$ is defined to be $\wedge_{j \in N} k_j f$, $E^0(f) := f$, and for $i \geq 1$, $E^i(f) := E(E^{i-1}(f))$. $g \rightarrow h$ means that the truth of g implies the truth of h, and $\neg k_j(\neg f)$ means that j considers the truth of f to be possible.

A formula $f \in \mathcal{L}(X, N)$ is common knowledge in a subset of formulas $A \subseteq \mathcal{L}(X, N)$ if $E^n f \in A$ for every $n < \infty$.

Throughout this paper, the multi-agent epistemic logic $S5$ will be assumed. For a discussion of the $S5$ logic, see Cresswell and Hughes (1968); and for the multi-agent variation, see Halpern and Moses (1992) and also Bacharach, et al, (1997).

A set of formulas $\mathcal{A} \subseteq \mathcal{L}(X,N)$ is called *complete* if for every formula $f \in \mathcal{L}(X,N)$ either $f \in \mathcal{A}$ or $\neg f \in \mathcal{A}$. A set of formulas is called *consistent* if no finite subset of this set leads to a logical contradiction, meaning a deduction of f and $\neg f$ for some formula f. We define

$$\Omega(X,N) := \{S \subseteq \mathcal{L}(X,N) \mid S \text{ is complete and consistent}\}.$$

Any consistent set of formulas can be extended to a complete and consistent set of formulas, a property we call the *Extension Property*, proven by applying Lindenbaum's Lemma.

We define a topology for Ω, the same as in Samet (1990). For every $f \in \mathcal{L}$ define $\alpha(f) := \{z \in \Omega \mid f \in z\}$. Let $\{\alpha(f) \mid f \in \mathcal{L}\}$ be the base of open sets of Ω. (A topology is defined by the fact that $\alpha(f) \cap \alpha(g) = \alpha(f \wedge g)$). The topology of a subset A of Ω will be the relative topology for which the open sets of A are $\{A \cap O \mid O \text{ is an open set of } \Omega\}$. For any subset $D \subseteq \Omega$, $\overline{D}$ will stand for the closure of D. Notice that the Extension Property implies that Ω is compact.

For every agent $j \in N$ we define its *knowledge partition* $\mathcal{Q}^j(X,N)$ to be the partition of $\Omega(X,N)$ generated by the inverse images of the function $\beta^j : \Omega \to 2^{\mathcal{L}(X,N)}$, the set of subsets of $\mathcal{L}(X,N)$, defined by $\beta^j(z) := \{f \in \mathcal{L}(X,N) \mid k_j f \in z\}$. We will write $\mathcal{Q}^j$ if there is no ambiguity. A *possibility set* is defined to be a member of $\mathcal{Q}^j$ for some $j \in N$. Notice that every possibility set is compact.

Let $\mathcal{Q} := \wedge_j \mathcal{Q}^j$ be the meet partition of the $\mathcal{Q}^j$. A member of $\mathcal{Q}$ we call a "cell".

The following lemma is in Simon (1999), but all the components of the proof can be found in other papers (Lemma 4.1 of Halpern and Moses 1992, Aumann 1999):

Lemma A: For any cell C of $\Omega(X,N)$ $\{f \in \mathcal{L}(X,N) \mid f$ is common knowledge in z for some $z \in C\} = \{f \in \mathcal{L}(X,N) \mid f$ is common knowledge in z for all $z \in C\} = \{f \in \mathcal{L}(X,N) \mid f \in z$ for all $z \in C\}$.

Due to Lemma A, we have a map F from the meet partition $\mathcal{Q}$ to subsets of formulas defined by $F(C) := \{f \mid f$ is common knowledge in any (equivalently all) members of C}. We say that C is *centered* if and only if C is the only cell that corresponds to $F(C)$.

For any subset of formulas $T \subseteq \mathcal{L}$ define $\underline{Ck}(T) := \{f \in \mathcal{L} \mid$ there exists an $i < \infty$ and a finite set $T' \subseteq T$ with $(\wedge_{t \in T'} E^i(t)) \rightarrow f$ a tautology $\}$. We define $\mathcal{T}(X, N) = \{\underline{Ck}(T) \mid T \subseteq \mathcal{L}(X, N)\} \setminus \{\mathcal{L}(X, N)\}$, and we say that T generates $\underline{Ck}(T)$. If there is no ambiguity, we can write simply $\mathcal{T}$. $\underline{Ck}(T)$ is the set of formulas whose common knowledge is implied by the common knowledge of the formulas in T.

For every set of formulas $T \subseteq \mathcal{L}$ define the set

$$\mathbf{Ck}(T) := \{z \in \Omega \mid \text{ every member of } T \text{ is common knowledge in } z\}.$$

For any $T \subseteq \mathcal{L}$, $\mathbf{Ck}(T)$ is a closed set, since the $\mathbf{Ck}(T)$ is the intersection of the sets $\alpha(E^l f)$ for all $l < \infty$ and all formulas f in T.

The following two lemmatta and Theorem 0 are proven in Simon (1999).

Lemma B: $\underline{Ck}$ is a closure operator on the subsets of $\mathcal{L}$, meaning that $\underline{Ck}(T) = T$ if T is already a member of $\mathcal{T}$. Furthermore, the image of $F : \mathcal{Q} \rightarrow 2^{\mathcal{L}}$ is a subset of $\mathcal{T}$.

Lemma C: If C is a cell and $S = F(C)$, then $\overline{C} = \mathbf{Ck}(S)$.

Theorem D: (from Theorem 1 of Simon 1999) A cell C is centered if and only if C contains an open set of $\overline{C}$; and if C is not centered then there exist uncountably many cells C' with $F(C') = F(C)$.

Our example is constructed from a $\mathbf{Ck}(T)$ such that T is maximal in $\mathcal{T}$, $\mathbf{Ck}(T)$ is topologically equivalent to a Cantor set, and there are uncountably many cells in $\mathbf{Ck}(T)$, all of which are un-centered and dense in $\mathbf{Ck}(T)$. Another example of this kind can be found in Simon, (1997).

2.3 Kripke structures

In this paper, a Kripke structure is a quintuple $\mathcal{K} = (S; N; (\mathcal{P}^j \mid j \in N); X; \psi)$ where N is a set of agents, for each $j \in N$ $\mathcal{P}^j$ is a partition of the set S, X is a set of primitive propositions, and $\psi : X \rightarrow 2^S$ is a map from X to the subsets of S, such that for every $x \in X$ the set $\psi(x)$ is interpreted to be the subset of S where x is true. (The usual definition of a Kripke structure is more general, but this more restricted usage applies to the $S5$ logic.) We

define a map $\alpha^{\mathcal{K}} : \mathcal{L}(X, N) \to 2^S$ inductively on the structure of the formulas in the following way:
Case 1 $f = x \in X$**:** $\alpha^{\mathcal{K}}(x) := \psi(x)$.
Case 2 $f = \neg g$**:** $\alpha^{\mathcal{K}}(f) := S \backslash \alpha^{\mathcal{K}}(g)$,
Case 3 $f = g \wedge h$**:** $\alpha^{\mathcal{K}}(f) := \alpha^{\mathcal{K}}(g) \cap \alpha^{\mathcal{K}}(h)$,
Case 4 $f = k_j(g)$**:** $\alpha^{\mathcal{K}}(f) := \{s \mid s \in P \in \mathcal{P}^j \Rightarrow P \subseteq \alpha^{\mathcal{K}}(g)\}$.
We define a map $\phi^{\mathcal{K}} : S \to \Omega(X, N)$ (see Fagin, Halpern, and Vardi 1991) by

$$\phi^{\mathcal{K}}(s) := \{f \in \mathcal{L}(X, N)) \mid \ s \in \alpha^{\mathcal{K}}(f)\}.$$

We are justified in using again the notation α for the following reason. Consider the map $\overline{\psi} : X \to 2^{\Omega}$ defined by $\overline{\psi}(x) := \{z \in \Omega \mid x \in z\}$. We have a Kripke structure $\Omega = (\Omega; N; \mathcal{Q}^1, \ldots, \mathcal{Q}^n; X; \overline{\psi})$. (Due to its canonical nature, we index this Kripke structure with Ω.)

Theorem E: For every $f \in \mathcal{L}(X, N)$, f is a theorem of the multi-agent S5 logic if and only if f is a tautology. Furthermore, $\phi^{\Omega}(z) = z$ for every $z \in \Omega$.

For a proof of the first part of this theorem, see Halpern and Moses (1992) and Cresswell and Hughes (1968), and for how the second part follows from the first part see Aumann (1999). We will call this result the "Completeness Theorem."

2.4 Canonical Finite Models

We define the *depth* of a formula inductively on the structure of the formulas. If $x \in X$, then depth $(x) := 0$. If $f = \neg g$ then depth $(f) :=$ depth (g); if $f = g \wedge h$ then depth (f):= max (depth (g) , depth (h)); and if $f = k_j(g)$ then depth (f):= depth (g) +1.

For every $0 \le i < \infty$ we define $\mathcal{L}_i := \{f \in \mathcal{L} \mid \text{depth } (f) \le i\}$ and define Ω_i to be the set of maximally consistent subsets of $\mathcal{L}_i$. If there may be ambiguity, we will write $\Omega_i(X, N)$. We must perceive an Ω_i in two ways, as a Kripke structure in its own right and as a canonical projective image of Ω inducing a partition of Ω through inverse images. We define $\pi_i : \Omega \to \Omega_i$ to be the canonical projection $\pi_i(z) := z \cap \mathcal{L}_i$. Due to an application of Lindenbaum's Lemma, the maps π_i are surjective.

For every $0 \le i < \infty$ we consider the Kripke structure $\Omega_i = (\Omega_i; X; \overline{\psi}_i; N; (\overline{\mathcal{F}}_i^j \mid j \in N))$, where $\overline{\psi}_i = \pi_i \circ \overline{\psi}$ and for $i > 0$ the partition $\overline{\mathcal{F}}_i^j$ of Ω_i, is induced by the inverse images of the function $\beta_i^j : \Omega_i \to 2^{\mathcal{L}_{i-1}(X,N)}$ defined by $\beta_i^j(w) := \{f \in \mathcal{L}_{i-1}(X,N) \mid k_j(f) \in w\}$. We define $\overline{\mathcal{F}}_0^j = \{\Omega_0\}$ for every $j \in N$.

Now we consider Ω_i again as a canonical projective image. $\mathcal{A}_i$ is defined to be the partition of Ω induced by the inverse images of π_i, $\mathcal{A}_i := \{\pi_i^{-1}(w) \mid w \in \Omega_i\}$. By the definition of Ω, the join partition $\vee_{i=1}^{\infty}\mathcal{A}_i$ is the discrete partition of Ω, meaning that it consists of singletons. Let $\mathcal{F}_i^j$ be the partition on Ω, coarser than $\mathcal{A}_i$, defined by $\mathcal{F}_i^j := \{\pi_i^{-1}(B) \mid\ B \in \overline{\mathcal{F}}_i^j\}$. From the definitions of the Ω_i and the $\mathcal{F}_i^j$ it follows that $\vee_{i=0}^{\infty}\mathcal{F}_i^j = \mathcal{Q}^j$.

An i- *atom* (or just atom) is a member of Ω_i.

Since X and N are finite, there are several important properties of the Kripke structures Ω_i, all of which are used in this paper.

(i) Ω_i is finite for every $0 \leq i < \infty$. (For a more general statement, see Lismont and Mongin 1995.)

(ii) For every i-atom $w \in \Omega_i$ we can define a formula $f(w)$ of depth i or less such that $\alpha^{\Omega_i}(f(w)) = \{w\}$, meaning that the formula $f(w)$ is true with respect to Ω_i only at $w \in \Omega_i$. This follows from the finiteness of Ω_i, and implies that $\{A \mid A \in \mathcal{A}_i, i \geq 0\} = \{\pi_i^{-1}(\pi_i(w)) \mid w \in \Omega_i, i \geq 0\}$ form a basis for the open sets of Ω. For any subset $A \subseteq \Omega_i$ define $f(A) := \vee_{w \in A} f(w)$, a formula that is true with respect to Ω_i only in the subset A.

(iii) It is easy to extend an i-atom to an $i+1$-atom. Fix $0 \leq i < \infty$ and $w \in \Omega_i$. For every $j \in N$ define $\overline{F}_i^j$ by $w \in \overline{F}_i^j \in \overline{\mathcal{F}}_i^j$. If $(M_i^j \mid j \in N)$ are subsets of $(\overline{F}_i^j \mid j \in N)$, respectively, such that
1) $w \in M_i^j$ for every $j \in N$, and
2) for every $B \in \mathcal{A}_{i-1}$ $\quad \overline{F}_i^j \cap \pi_i(B) \neq \emptyset$ implies that $M_i^j \cap \pi_i(B) \neq \emptyset$,
then there is a unique $v \in \Omega_{i+1}$ such that $\pi_i \circ \pi_{i+1}^{-1}(v) = w$ and for every $u \in \Omega_i \quad \neg k_j \neg f(u) \in v$ if and only if $u \in M_i^j$. Furthermore, this is the only way to extend a member of Ω_i to a member of Ω_{i+1}; this is Lemma 4.2 of Fagin, Halpern, and Vardi (1991). For any $i \geq 0$ and $v \in \Omega_k$ with $k > i$ we define $M_i^j(v) := \{u \in \Omega_i \mid \neg k_j \neg f(u) \in v\}$. Notice that if $w \in F \in \overline{\mathcal{F}}_i^j$ then $M_{i-1}^j(w)$ is equal to $\pi_{i-1} \circ \pi_i^{-1}(F)$, which could be a proper subset of the member of $\overline{\mathcal{F}}_{i-1}^j$ that contains $\pi_{i-1} \circ \pi_i^{-1}(w)$.

(iv) For every formula $f \in \mathcal{L}_i$ and $l \geq i \quad \pi_l^{-1}(\alpha^{\Omega_l}(f)) = \alpha^{\Omega}(f)$. This follows from (iii) and the Completeness Theorem. (See also Lemma 2.5 in

Fagin, Halpern, and Vardi 1991.)

3 Two-Bounded Knowledge

Lemma 1: The partitions $\mathcal{Q}^j$ of Ω are upper and lower semi-continuous.

Proof: Assume that $z_1, z_2, \dots$ is a sequence of points in Ω converging to $z \in \Omega$, and for all $l \geq 1$ let $z_l \in F_l \in \mathcal{Q}^j$, and $z \in F \in \mathcal{Q}^j$.

Let w be any k-atom such that $\pi_k^{-1}(w) \cap F \neq \emptyset$. By Property (iv), $\neg k_j \neg(f(w)) \in z$, and therefore there is a number l' such that $l \geq l'$ implies that $\neg k_j \neg(f(w)) \in z_l$. This means also that $F_l \cap \pi_k^{-1}(w) \neq \emptyset$ for all $l \geq l'$, the lower semi-continuity of $\mathcal{Q}^j$.

Let y be a cluster point of the intersection of an infinite subset of the F_l and let w_k be the k-atom with $\pi_k(w_k)$ containing y. Given that $z_l \in \pi_{k+1}^{-1}(\pi_{k+1}(z))$ for all $l \geq l'$, we have infinitely many $l^* \geq l'$ such that $\neg k_j \neg f(w_k) \in z_{l^*}$ This implies by Property (iv) that $\neg k_j \neg f(w_k) \in z$. This true for all k implies that $y \in F$, the upper semi-continuity of $\mathcal{Q}^j$. □

For all $i < \infty$ define the subset of *k-bounded* i-atoms $\overline{A}_i^k \subseteq \Omega_i$ inductively in the following way:
$\overline{A}_0^k = \Omega_0$,
for every $i > 0$ $w \in \overline{A}_i^k$ if and only if for every $j \in N$ and $\pi_i^{-1}(w) \in F \in \mathcal{F}_i^j$ it follows that $\pi_{i-1} \circ \pi_i^{-1}(F)$ is a subset of $\overline{A}_{i-1}^k$ and $|\pi_{i-1} \circ \pi_i^{-1}(F)| \leq k$. Following the definition of k-bounded atoms, define a member F of $\mathcal{F}_i^j$ to be k-bounded if and only if $\pi_{i-1} \circ \pi_i^{-1}(F)$ is a subset of $\overline{A}_{i-1}^k$ and $|\pi_{i-1} \circ \pi_i^{-1}(F)| \leq k$. Define $T_k \subseteq \mathcal{L}$ to be the set of formulas $\{f(\overline{A}_i^k) \mid i < \infty\}$.

Lemma 2: $\mathbf{Ck}(T_k)$ is the union of all cells C satisfying $|F| \leq k$ for all possibility sets F contained in C.

Proof: Since $\vee_{i=0}^{\infty} \mathcal{A}_i$ is the discrete partition of Ω, if $z \in F \in \mathcal{Q}^j$ and $|F| > k$ then $|\pi_{i-1}(F)| > k$ for some $i < \infty$. Let $F^* \in \mathcal{F}_i^j$ contain z. Since $\pi_{i-1}(F^*) = \pi_{i-1}(F)$, $f(\overline{A}_i^k)$ won't be true at z, and T_k cannot be common knowledge in the cell containing z.

On the other hand, if T_k is not common knowledge in a cell C, then by Lemma A there is some $z \in C$ and some $i < \infty$ such that $f(\overline{A}_i^k)$ is not true at z. By Property (iv) this implies that the i-atom $w \in \Omega_i$ satisfying $w = \pi_i(z)$

is not a k-bounded atom. By induction there is an $l \leq i$, a $v \in \Omega_l$ and an $F \in \mathcal{F}_l^j$ such that $v \in \pi_l(F) \in \overline{\mathcal{F}}_l^j$, $\pi_l^{-1}(v) \cap C \neq \emptyset$ and $|\pi_{l-1}(F)| > k$. By Property (iv), for any $z' \in \pi_l^{-1}(v)$ with $z' \in F^* \in \mathcal{Q}^j$ we have $|F^*| > k$. $\square$

Lemma 3: Assume that $|N| = 2$.

(a) If $i \geq 1$, $F \in \mathcal{F}_i^j$ is two-bounded and $v \in \pi_{i-1}(F)$ then there are $2^{|X|}$ two-bounded members of Ω_i in $\pi_i \circ \pi_{i-1}^{-1}(v) \cap \pi_i(F)$.

(b) For every two-bounded $w \in \overline{F} \in \overline{\mathcal{F}}_i^j$ there are $2^{|X|}$ two-bounded members of $\mathcal{F}_{i+1}^j$ contained in $\pi_i^{-1}(\overline{F})$ with non-empty intersection with $\pi_i^{-1}(w)$.

(c) There are $2^{|X|}$ members of $\overline{A}_0^2 = \Omega_0$ and for every i and every $w \in \overline{A}_i^2$ there are $4^{|X|}$ members v of $\overline{A}_{i+1}^2$ such that $\pi_i \circ \pi_{i+1}^{-1}(v) = w$.

Proof: We proceed to prove (a) and (b) together by induction on i. For $i = 0$ we need to prove only (b). For every $w \in \Omega_0$ there are exactly $2^{|X|} - 1$ two-subsets of Ω_0 containing w and one one-subset of Ω_0 containing w, namely $\{w\}$.

Now assume that both claims are true for $i - 1 \geq 0$.

(a) Let F' be the two-bounded member of $\mathcal{F}_{i-1}^{j'}$ containing $\pi_{i-1}^{-1}(v)$ for $j' \neq j$. By the induction hypothesis and (b) there are $2^{|X|}$ different two-bounded members of $\mathcal{F}_i^{j'}$ contained in F' and intersecting $\pi_i^{-1}(v)$. By Property (iii) each one combined with F defines a two-bounded atom of Ω_i contained in $\pi_i(F)$ and extending v.

(b) Case 1; $|\pi_{i-1} \circ \pi_i^{-1}(\overline{F})| = 2$**:** Let F be $\pi_i^{-1}(\overline{F})$. Let v be the member of $\pi_{i-1}(F)$ such that $\pi_{i-1} \circ \pi_i^{-1}(w) \neq v$. By (a) there are $2^{|X|}$ different two-bounded i-atoms contained in $\pi_i(\pi_{i-1}^{-1}(v))$ that are also members of $\overline{F}$. This means that there are $2^{|X|}$ different two-subsets of $\overline{F}$ such that one member is in $\pi_i(\pi_{i-1}^{-1}(v))$ and the other is w.

(b) Case 2; $\pi_{i-1} \circ \pi_i^{-1}(\overline{F}) = \{u\}$ **with** $u \in \Omega_{i-1}$**:** By (a) there are $2^{|X|}$ two-bounded members of Ω_i in $\overline{F}$. There are $2^{|X|} - 1$ two-subsets of $\overline{F}$ containing $w \in \Omega_i$ and one one-subset containing w, namely $\{w\}$.

(c) follows directly from a,b, and Property (iii). $\square$

Additionally, Lemma 3 shows that $\mathbf{Ck}(T_2) \subseteq \Omega(X, N)$ for $|N| = 2$ is topologically equivalent to a Cantor set.

4 The Example

We restrict ourselves to the case of $|X| = 1$ and $X = \{x\}$, and define the set S to be $\mathbf{Ck}(T_2) \subseteq \Omega(\{x\}, \{1,2\})$. There are two inverse projections of 0-atoms in S and from Lemma 3, for every $i > 0$ and i-atom w with $\pi_i^{-1}(w) \cap \mathbf{Ck}(T_2) \neq \emptyset$ there are 4 different inverse projections of $i+1$-atoms in $\mathbf{Ck}(T_2)$ contained in $\pi_i^{-1}(w)$.

For every $i \geq 0$ and $j = 1, 2$ define a *half i-atom of player j* to be a pair u, v of distinct $i+1$-atoms such that $\pi_i \circ \pi_{i+1}^{-1}(u) = \pi_i \circ \pi_{i+1}^{-1}(v)$ and u and v share the same member of $\overline{\mathcal{F}}_{i+1}^j$.

The player set is $N = \{1, 2, 3, 4\}$. Before we introduce the two additional players 3 and 4, we look at a standard representation of the Cantor set and a group action on it.

Consider the Cantor set $D := \{0,1\} \times (a_1, a_2, \ldots)$ such that $a_i \in \{0,1\}$ for every $1 \leq i < \infty$. $(a_1, a_2, \ldots, a_l)$ will stand for the number $(a_1, a_2, \ldots)$ where $a_k = 0$ for all $k > l$. Consider the dyadic sum on $\{0,1\}^\infty$ with the carrying of numbers to the right, e.g. the dyadic sum of $(1,1,1)$ and $(1,0,1)$ is $(0,0,1,1)$. Define an addition on D in the following way: $(a_0; a_1, a_2, \ldots) + (b_0; b_1, b_2, \ldots) = (a_0 + b_0 \text{ modulo } 2\ ; c_1, c_2, \ldots)$ where $(c_1, c_2, \ldots)$ is the dyadic sum of $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$. Next consider the following two involutions $\sigma_1, \sigma_2 : D \to D$ defined by $\sigma_1(d) := -(d + (1;1))$ and $\sigma_2(d) := -(d + (1;0))$. One can consider σ_1 and σ_2 as the generators of the infinite dyhedral group acting on D, and for every $d \in D$ the orbit of this group action on d is dense in D.

We choose a homeomorphism γ from D to $\mathbf{Ck}(T_2) \subseteq \Omega(X, \{1,2\})$ to correspond to the structure of Ω. The zero coordinate of the dyadic expansion corresponds to the validity of the proposition x; for every $i > 0$ the $2i - 1$ coordinate corresponds to a half i- atom of Player 1 and the $2i$ coordinate to a half i-atom of Player 2, so that these pair of coordinates correspond to a i-atom in $\overline{A}_i^2$. We define two additional partitions $\mathcal{P}^3$ and $\mathcal{P}^4$ of $\mathbf{Ck}(T_2)$ corresponding to the knowledge of a third and fourth agent. Define $\mathcal{P}^3 := \{\{z, \gamma \circ \sigma_1 \circ \gamma^{-1}(z)\} | z \in S = \mathbf{Ck}(T_2)\}$, and likewise define $\mathcal{P}^4$ with σ_2. For $j = 1, 2$ define $\mathcal{P}^j$ to be the partition $\mathcal{Q}^j$ on $\Omega(\{x\}, \{1,2\})$ restricted to $S = \mathbf{Ck}(T_2)$. We have a Kripke structure $(S = \mathbf{Ck}(T_2); X; \overline{\psi}; N = \{1,2,3,4\}; \mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4)$. Since both σ_1 and σ_2 are continuous functions from D to D, all the partitions $\mathcal{P}^j$ are upper and lower semi-continuous.

For every Player j we define a function $\phi^j : S \to S$ by $\phi^j(z) = z$ if $\{z\} \in \mathcal{P}^j$ and $\phi^j(z) = y$ given that $\{y, z\} \in \mathcal{P}^j$ and $y \neq z$. Because the partitions $\mathcal{P}^j$ are upper and lower semi-continuous, the functions ϕ^j are continuous. Furthermore the functions are involutions, meaning that $\phi^j \circ \phi^j = id$.

We define the probability distribution μ on S to be that which gives a probability of 2^{-1-2i} to every inverse projection of an i-atom. μ is well defined by Lemma 3c. Notice that $1/2$ probability is given to the subset where x is true, and the same holds for the subset where x is not true. Next, for every $j \in N$ and every $z \in S$ we define a discrete subjective probability distribution $t^j(z) \in \Delta(S)$ such that if $z = \phi^j(z)$ then $t^j(z)(\{z\}) := 1$ and if $z \neq y = \phi^j(z)$ then $t^j(z)(\{z\}) := t^j(z)(\{y\}) := 1/2$. Because the partitions $\mathcal{P}^j$ are upper and lower semi-continuous, the functions $t^j : S \to \Delta(S)$ are continuous (with respect to the weak topology on $\Delta(S)$).

We define our belief space to be $(S = \mathbf{Ck}(T_2), \hat{X}, \hat{\psi}, N = \{1,2,3,4\}, (t^j \mid j \in N))$ where $\hat{X} := \{x, \neg x\}$ and $\hat{\psi}(z) = x$ if $x \in z$ and $\hat{\psi}(z) = \neg x$ if $x \notin z$.

5 Main Results

Lemma 4:

(a) For $j = 1, 2$ the functions ϕ^j map inverse projections of half i-atoms of Player j to inverse projections of half i-atoms of Player j, and do so bijectively. The same holds for $j = 3, 4$, with respect to inverse projections of atoms.

(b) The functions ϕ^j are measure preserving transformations with respect to μ.

Proof:

(a) For $j = 3, 4$, the claim follows from the definition of ϕ^j, so we can assume that $j \in \{1, 2\}$ is fixed. Let z be any point in S and let $F \in \mathcal{F}^j_{i+1}$ contain z. Let l be the first level such that $\neg k_j \neg f(v) \in z$ for two distinct $v \in \Omega_l$, with $l = \infty$ if this never happens. Given that $w = \pi_i(z)$ and $i \geq l$ consider the i-atom $w' \neq w$ such that $\neg k_j \neg f(w') \in z$. ϕ^j maps $F \cap \pi_i^{-1}(w)$ to $F \cap \pi_i^{-1}(w')$ and vice versa. The bijection follows from $\phi^j \circ \phi^j = id$. If $l < i$, then ϕ^j maps $F \cap \pi_i^{-1}(w)$ to itself, also bijectively from $\phi^j \circ \phi^j = id$.

(b) This follows from (a), (from Lemma 3) that μ gives equal weight to the inverse projections of all half i-atoms (repectively i-atoms), and that the half atoms of either player also build a base for the open sets. $\square$

For every Player j we define a Borel field $\mathcal{G}^j$ on S in the following way: If $j = 1, 2$ define a special collection of open sets: $\mathcal{O}^j := \{F \mid F \in \mathcal{F}^j_k,\ k \geq 0\}$; and if $j = 3, 4$, then define $\mathcal{O}^j$ to be $\{\pi_i^{-1}(\pi_i(z)) \cup \pi_i^{-1}(\pi_i(\phi^j(z))) \mid z \in S,\ i \geq 0\}$. For every j define $\mathcal{G}^j$ to be the Borel field generated by $\mathcal{O}^j$.

Lemma 5:

(a) If A is a non-empty open subset of S satisfying $y \in A \Rightarrow \phi^j(y) \in A$ then A is the countable union of members of $\mathcal{O}^j$ and therefore is in $\mathcal{G}^j$.

(b) For every j the Borel field $\mathcal{G}^j$ is that generated by the continuous function t^j, meaning that $\mathcal{G}^j$ is the smallest sigma algebra for which the function t^j is measurable.

Proof:

(a) Choose any $y \in A$. Let v and w be any two atoms such that $v \in \Omega_i$ is an i-atom with the (base) open set $\pi_i^{-1}(v)$ containing y and contained in A and w is a k-atom with the (base) open set $\pi_k^{-1}(w)$ containing $\phi^j(y)$ and contained in A. Let l equal $\max\{i, k\}$. If $j = 1, 2$ and $y, \phi^j(y) \in F \in \mathcal{F}^j_{l+1}$ we have that F is a member of $\mathcal{O}^j$ containing y and contained in A. If $j = 3, 4$ then $\pi_l^{-1}(\pi_l(y)) \cup \pi_l^{-1}(\pi_l(\phi^j(y)))$ is a member of $\mathcal{O}^j$ containing y and contained in A. Notice that there are only countably many members in $\mathcal{O}^j$.

(b) First we must show that t^j is measurable with respect to $\mathcal{G}^j$. Because the measurable sets of $\Delta(S)$ are the Borel sets of $\Delta(S)$, it suffices to show that the inverse image of any open set of $\Delta(S)$ is in $\mathcal{G}^j$. Since t^j is continuous, such an inverse image is an open set of S. Since $t^j(y) = t^j(\phi^j(y))$ for all $y \in S$, such an open set satisfies the condition of (a).

Second, to show that there is no smaller sigma algebra , let us suppose for the sake of contradiction that there exists a member G of $\mathcal{G}^j$ that is not an inverse image of $t^j(A)$ for any measurable subset $A \subseteq \Delta(S)$. Since G is a clopen subset of the Cantor set S, the function 1_G is continuous. Consider the open set of $\Delta(S)$ defined by $A := \{\lambda \mid \lambda(G) > 2/3\}$. $t^j(z)$ is in A if and only if $z \in G$, a contradiction which completes the proof. $\square$

For any partition $\mathcal{P}$ of a topological space such that all members of $\mathcal{P}$ are Borel sets, let $\mathcal{P}_*$ be the largest Borel field such that $x \in G \in \mathcal{P}_*$ and $x, y \in F \in \mathcal{P}$ imply that $y \in G$. With regard to our example, we have

$\mathcal{P}_*^j := \{B \mid B$ is Borel, $y \in B \Leftrightarrow \phi^j(y) \in B\}$ and $\mathcal{G}^j$ is a subset of $\mathcal{P}_*^j$.

Lemma 6: For all $j \in N$ the function $t^j : S \to \Delta(S)$ is a regular conditional probability distribution induced by μ and $\mathcal{P}_*^j$.

Proof:

First we must show for any $j \in N$, fixed Borel set A and value $r \in [0,1]$ that the set $\{z \mid t^j(z)(A) > r\}$ is in $\mathcal{P}_*^j$. We consider two cases: $r < 1/2$ and $r \geq 1/2$. If $r < 1/2$ then the above set is $A \cup \phi^j(A)$, which is in $\mathcal{P}_*^j$ because ϕ^j is a continuous and bijective involution. If $r \geq 1/2$ then the above set is $A \cap \phi^j(A)$, which is in $\mathcal{P}_*^j$ for the same reasons.

Second, we will show for every $B \in \mathcal{P}_*^j$ and Borel set A that $\mu(B \cap A) = \int_B t^j(z)(A)d\mu(z)$. Let B be any member of $\mathcal{P}_*^j$ and z any member of B. $t^j(z)(A) = 1$ if $z, \phi^j(z) \in A$, $t^j(z)(A) = 0$ if $z, \phi^j(z) \notin A$, and otherwise $t^j(z)(A) = 1/2$.

Let B_0 be the subset of B where $\phi^j(z) = z$,
B_1 the subset where either z or $\phi^j(z)$ is in A but not both,
B_2 the subset where $z \neq \phi^j(z)$ and both z and $\phi^j(z)$ are in A, and
B_3 the subset where $z \neq \phi^j(z)$ and neither z nor $\phi^j(z)$ is in A.
Since all sets are Borel, we can write $\int_B t^j(z)(A)d\mu(z)$ as $\int_{B_0} t^j(z)(A)d\mu(z) + \int_{B_1} t^j(z)(A)d\mu(z) + \int_{B_2} t^j(z)(A)d\mu(z) + \int_{B_3} t^j(z)(A)d\mu(z)$.

Case 0: Since $t^j(z)$ is the function 1_A in B_0, $\int_{B_0} t^j(z)(A)d\mu(z) = \mu(A \cap B_0)$.

Case 1: Notice that $z \in B_1$ if and only if $\phi^j(z) \in B_1$. Since $t^j(z)(A) + t^j(\phi^j(z))(A) = 1/2 + 1/2 = 1$ for all $z \in B_1$, $\int_{B_1}(t^j(z)(A) + t^j(\phi^j(z))(A))d\mu(z)$ is equal to $\int_{B_1} d\mu(z) = \mu(B_1)$, but also to $2\int_{B_1} t^j(z)(A)d\mu(z)$ from the measure preserving property of Lemma 4b and the fact that $t^j(z)(A)$ is a constant $1/2$ for all $z \in B_1$. Again from the measure preserving property we have that $\mu(A \cap B_1) = \mu(\phi^j(A \cap B_1))$. But $\phi^j(A \cap B_1)$ is exactly $B_1 \backslash A$, and therefore $2\mu(A \cap B_1) = \mu(B_1)$ and $\int_{B_1} t^j(z)(A)d\mu(z) = \mu(A \cap B_1)$, as desired.

Case 2: $t^j(z)(A) = t^j(\phi^j(z))(A) = 1$ for all $z \in B_2$ and therefore $\int_{B_2} t^j(z)(A)d\mu(z) = \int_{B_2} d\mu(z) = \mu(B_2) = \mu(A \cap B_2)$.

Case 3: If neither $z \in A$ nor $\phi^j(z) \in A$ then $t^j(z)(A) = t^j(\phi^j(z))(A) = 0$ and therefore $0 = \int_{B_3} t^j(z)(A)d\mu(z) = \mu(A \cap B_3)$. □

Theorem :

(a) For the above belief space $(S = \mathbf{Ck}(T_2); X; \hat{\psi}; N = \{1,2,3,4\}; (t^j \mid j \in N))$ there is no common prior for any of the subsets (cells) that the players know in common.

(b) The belief space is strongly non-redundant (and therefore maps bijectively to a subspace of the canonical Mertens-Zamir space).

Proof:

(a) Every member of $\wedge_{j=3,4}\mathcal{P}^j$ contained in $S = \mathbf{Ck}(T_2)$ is countable and dense in the Cantor set $S = \mathbf{Ck}(T_2)$, and therefore the same is true for $\wedge_{j=1,2,3,4}\mathcal{P}^j$. Because all the t^j give $1/2 - 1/2$ probability to any two distinct points comprising a member of $\mathcal{P}^j$, any common prior probability distribution on a $C \in \wedge_{j=1,2,3,4}\mathcal{P}^j$ contained in S must give equal probability to all the points of C. This is impossible.

(b) Recall the definition of $\tilde{\mathcal{F}}$ from Section 2.1. We show that $\mathcal{A}_i \subseteq \tilde{\mathcal{F}}$ for all i. Since the inverse projections of the atoms form the base of open sets of the Hausdorff topology on S, this would imply that $\tilde{\mathcal{F}}$ is the collection of Borel subsets.

We proceed by induction on the depth i. If i=0 then the two distinct 0-atoms correspond to the truth of falsity of $x \in X$. We assume the claim for all k less than or equal to $i-1$. Let w be any i-atom; we will prove that $\pi_i^{-1}(w) \in \mathcal{A}_i$ is also in $\tilde{\mathcal{F}}$. From the induction hypothesis $\{z \in S \mid t^j(z)(\pi_{i-1}^{-1}(u)) > 0\}$ and $\{z \in S \mid t^j(z)(\pi_{i-1}^{-1}(u)) = 0\}$ are in $\tilde{\mathcal{F}}$ for every $u \in \Omega_{i-1}$. Recall the sets $M_{i-1}^j(w) \subseteq \Omega_{i-1}$ from Property (iii). For both $j = 1,2$ consider the set O^j, a member of $\tilde{\mathcal{F}}$ (and open set of S), defined by

$$O^j := \pi_{i-1}^{-1}(\pi_{i-1} \circ \pi_i^{-1}(w)) \cap \bigcap_{u \in M_{i-1}^j(w)} \{z \in S \mid t^j(z)(\pi_{i-1}^{-1}(u)) > 0\} \cap$$

$$\bigcap_{u \in \Omega_{i-1} \setminus M_{i-1}^j(w)} \{z \in S \mid t^j(z)(\pi_{i-1}^{-1}(u)) = 0\}.$$

The intersection $O^1 \cap O^2$, a member of $\tilde{\mathcal{F}}$, is exactly the set $\pi_i^{-1}(w) \in \mathcal{A}_i$. $\square$

Corollary: There is a consistent belief subspace of a canonical Mertens-Zamir space such that for every subset held in common knowledge there is no common prior probability distribution on this set.

Remark: Although it is not necessary for this paper, one can prove for all $j \in N$ that the Borel field $\mathcal{G}^j$ is equal to $\mathcal{P}_*^j$.

6 Conclusion

There are many Mertens-Zamir belief subspaces that lack common priors; Mertens and Zamir (1985) provide a finite example. A special property of Mertens-Zamir spaces is the ability (of a player or an outside analyst) to determine uniquely a common prior, should one exist, on the smallest subspace containing a point. Such an ability of a player is dependent on her ability to consider the true state in the space to be possible, meaning that the true state y is in the support of her subjective probability distribution at y. This is a strong condition, given the interactive construction of belief spaces. Nevertheless, if we assume, for whatever reasons, that this condition is fullfilled (and it is fullfilled in the above example), the continuity of subjective beliefs is very useful to the process of determining the unique common prior, should one exist, when the players know in common a cell of measure zero that is dense in the smallest containing subspace. Relatively simple situations of player uncertainty that cannot be modeled through the continuity of subjective belief could reveal a weakness of the Mertens-Zamir construction.

When $\mathcal{P}$ is a partition with Borel members, recall the definition of $\mathcal{P}_*$ from the last section.

Claim: There is an example of a Polish space Ω with the following:
1) a Borel probability measure μ on Ω that is both non-atomic and gives positive measure to all non-empty open sets,
2) an upper and lower semi-continuous partition $\mathcal{P}$ of Ω,
3) no regular conditional probability distribution defined on the Borel field $\mathcal{P}_*$ that is continuous with respect to the weak topology.

Example: Consider two coins; when flipped one lands heads with 2/3 probability, the other tails with 2/3 probability. Nature has chosen one of the coins with even probability, and this choice is never changed over time. There is only one player, and his task is to determine which coin was chosen by nature. The player is allowed to perform an infinite sequence of flipping experiments.

Let H stand for the coin that lands 2/3 heads, and h for an experiment that results in heads. Let T and t be defined likewise. Let Ω be $\{H, T\} \times \{h, t\}^\infty$, where the first coordinate is the choice of nature. For any $x \in \Omega$, let x_0 be the coordinate with H or T, and for $i \geq 1$ let x_i be the result of the ith experiment. The topology on Ω will be the product topology. Let μ be the induced probability distribution on Ω. After an infinite sequence of flips, the player observes everything but the zero coordinate. Define the partition $\mathcal{P} := \{\{H,T\} \times a \mid a \in \{h,t\}^\infty\}$. Consider the Borel subset $A = \{x \in \Omega \mid \liminf_{i\to\infty} \frac{|\{j \mid 1\le j\le i,\ x_j=h\}|}{i} \ge 2/3\}$ and the set of pairs $\{(H,a),(T,a)\} \in \mathcal{P}$ such that both points are in A. The only possibility for a regular conditional probability distribution is to give probability one to $\Omega^H := \{H\} \times \{h,t\}^\infty$ at almost all such pairs of points in A. (Consider the formula $\int_A \mu(\Omega^H \mid x) d\mu(x) = \mu(A \cap \Omega^H)$, where $\mu(\cdot \mid \mathcal{P}_*)$ is a regular conditional probability distribution. $\mu(A \cap \Omega^H) = 1/2$ and $\mu(A) = 1/2$ force $\mu(\Omega^H \mid x) = 1$ for almost all $x \in A$.) The same holds for the corresponding set B defined by tails. Since A and B are both dense subsets of Ω and $\Omega^H = \{H\} \times \{h,t\}^\infty$ and $\Omega^T := \{T\} \times \{h,t\}^\infty$ are separated clopen subsets, there can be no continuous version of the regular conditional probability. Choosing any open basis set of the form $O = \{(H,x) \mid x_i = a_i\ \forall i = 1,2,\dots n\}$ for some finite sequence $a_0, a_1, \dots, a_n$, we have $\mu(O) > 0$, with $\mu(O) = \frac{1}{2}(\frac{1}{3})^n$ if $a_i = t$ for all i.

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7 References

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