

# THE EVOLUTION OF EXCHANGE\*

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## Abstract

Stochastic stability is applied to the problem of exchange. We analyze the stochastic stability of two dynamic trading processes in a simple housing market. In both models traders meet in pairs at random and exchange their houses when trade is mutually beneficial, but occasionally they make mistakes. The models differ in the probability of mistakes. When all mistakes are equally likely, the set of stochastically stable allocations contains the set of efficient allocations. When more serious mistakes are less likely, the stochastically stable states are those allocations, always efficient, with the lowest envy-level.

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# 1 Introduction

Evolutionary game theory has proposed new interesting concepts and tools of analysis. One of these concepts (see, for example, Freidlin and Wentzell (1984) and Foster and Young (1990)) is known as *stochastic stability*. This differs from the notions of local stability in dynamical systems or evolutionary stability in theoretical biology, which require from a population to be immune to isolated random shocks or mutations. In contrast, stochastic stability requires immunity against persistent random shocks.

Stochastic stability has been successfully applied to learning processes in normal form games by Kandori, Mailath and Rob (1993), Young (1993a) and Ellison (1993), and in extensive form games by Hart (1999). In addition, Young (1993b) uses the same methodology in a cooperative bargaining problem, and Vega-Redondo (1997) in an oligopoly of firms. Recently, Jackson and Watts (1999) study the stochastic stability of networks. As Young (1998) stresses and the list of above papers demonstrates, the stochastic stability approach can be applied to the analysis of a wide variety of social interactions.

In this paper we are interested in applying stochastic stability to the general problem of economic exchange of goods. As a first step of inquiry, we concentrate on the simple housing market introduced by Shapley and Scarf (1974). This simple environment describes pure barter of indivisible goods yet important issues concerning efficiency, envy and decentralization can be analyzed. Specifically, a housing market consists of  $n$  traders, each of whom is characterized by the only house he owns and by his preference relation over the set of houses. In order to apply stochastic stability, we endow the housing market with a simple perturbed stochastic dynamic process. The unperturbed process can be described as follows. At each period a pair of traders is matched randomly and they trade their endowments if and only if trade is mutually beneficial (therefore, myopia is a component of their behavior). In addition, this process is perturbed. The perturbation consists of allowing a small probability of trade when it is not mutually beneficial. We shall refer to this event as a mistake.

As argued by Bergin and Lipman (1996), the conclusions of the analysis are sensitive to the particular perturbation chosen. We analyze two perturbed processes that seem very natural. In the first, all mistakes made by a given agent are equally likely. In the second, more serious mistakes are less likely than less serious ones.

Within the first dynamic model, we show that the efficient allocations are always stochastically stable. Although there are economies where the inclusion is strict, we find several interesting classes of housing problems where the set of stochastically stable states and the set of efficient allocations coincide.<sup>5</sup> The fact that every efficient allocation is stochastically stable relies on the following interesting property of efficient allocations. For any two efficient allocations it is possible to move from one to the other by means of a sequence of bilateral trades, without ever passing through an inefficient allocation. This “connectedness” property of efficient allocations allows us to prove also that it is always possible to move from any allocation, not necessarily efficient, to any efficient allocation by means of a sequence of bilateral trades, at each of which at most one trader makes a mistake. More mistakes are needed, however, to end up at an allocation that is not stochastically stable.

As for the second perturbed process, we show that stochastic stability always yields a subset of efficient allocations. Indeed, the stochastically stable states are exactly those allocations where the envy level in the economy is minimized. The intuition behind this result relies on the fact that there is a tight connection between the difference of the envy-levels of any two “pairwise connected” allocations and the difference between the seriousness of the mistakes made by the agents when moving from one allocation to the other. In addition, the limit distribution represents a random allocation that is ordinally efficient (Bogomolnaia and Moulin (1999)), i.e., it is not first-order stochastically dominated by any other random allocation.

Therefore, in an exchange procedure subject to persistent small probability mistakes, pairwise trade is not in conflict with Pareto efficiency. If all mistakes have the same probability, the economy spends a positive fraction of time in the long run on each efficient allocation (and under some extra conditions, only on those). In the absence of those extra conditions, inefficiencies are also selected in the long run. When agents are more careful of avoiding serious mistakes, the economy spends a positive proportion of time in the long run only on those allocations, always efficient, where the level of envy is the lowest possible.

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<sup>5</sup>For deterministic pairwise (and  $t$ -wise) trading processes that yield Pareto efficient allocations in a pure exchange economy, see Feldman (1973) and Goldman and Starr (1982). Agents in these processes trade in pairs (or in groups of size  $t$ ) if there are myopic gains from trade. However, they do not make mistakes in our sense.

We find it appealing that such concepts, of a strong normative flavor, receive support from this novel approach to the problem.

The plan of the paper is as follows. Section 2 presents the model and introduces preliminaries. Section 3 studies the first perturbed process and contains a subsection devoted to sufficient conditions guaranteeing that all stochastically stable states are efficient. The second process is the subject of Section 4. Section 5 concludes.

## 2 The Model and Preliminaries

A *house allocation problem* is a triple  $\langle N, H, (\succeq_i)_{i \in N} \rangle$  where  $N$  is a finite set of individuals,  $H$  is a finite set of houses with  $|H| = |N|$ , and for each individual  $i \in N$ ,  $\succeq_i$  is a complete, transitive and antisymmetric preference relation over  $H$ . The *size* of the problem is the number of agents in it.

Let  $P$  be a house allocation problem. An *allocation* in  $P$  is a one to one function  $x : N \rightarrow H$  that assigns one house to each agent.

An allocation  $x$  is *efficient* if there is no allocation  $y$  such that  $y_i \succeq_i x_i$  for all  $i \in N$  and  $y_i \succ_i x_i$  for some  $i \in N$ . We denote the set of efficient allocations in  $P$  by  $\mathcal{E}(P)$ .

Let  $x$  be an allocation in  $P$ . We say that individual  $i$  *envies individual  $j$  at  $x$*  whenever  $x_j \succ_i x_i$ . Define the *envy-graph of allocation  $x$*  to be the directed graph whose vertices are the agents in the housing problem and there is an edge from agent  $i$  to agent  $j$  if and only if  $i$  envies  $j$ . The *envy-level* of allocation  $x$  is defined to be the number of edges in its envy graph. We denote the envy-level at  $x$  by  $e(x)$ . It is clear that allocation  $x$  is efficient if and only if the corresponding envy-graph is acyclic.

Let  $\pi : \{1, \dots, n\} \rightarrow N$  be an ordering of the traders, i.e.,  $\pi(1)$  is the first trader,  $\pi(2)$  is the second trader and so on. We say that allocation  $x$  is the *outcome of the serial dictatorship mechanism with respect to  $\pi$*  or that  $x$  is induced by  $\pi$ , for short, if

- $x_{\pi(1)}$  is agent  $\pi(1)$ 's most preferred element in  $H$ , and
- for  $t \in \{2, \dots, n\}$ ,  $x_{\pi(t)}$  is agent  $\pi(t)$ 's most preferred element in  $H \setminus \{x_{\pi(1)}, \dots, x_{\pi(t-1)}\}$ .

It is known that allocation  $x$  is efficient if and only if it is the outcome of the serial dictatorship mechanism with respect to some ordering of the traders (see, for example Abdulkadiroglu and Sonmez (1998), Lemma 1).

We shall define a dynamic process according to which agents perform bilateral trades. These bilateral trades will allow us to transit from one allocation to another. Clearly, it is not always possible to go from one allocation to another by means of a single bilateral trade. When it is possible, we say that the allocations are pairwise connected. More formally, we say that allocations  $x$  and  $y$  are *pairwise connected* if there is a pair  $i$  and  $j$  of agents such that  $x_i = y_j$ ,  $x_j = y_i$  and  $x_k = y_k$  for all  $k \notin \{i, j\}$ . A  $(x, y)$ -*path* is a finite sequence of allocations  $(z^0, z^1, \dots, z^k)$  such that  $z^0 = x$ ,  $z^k = y$  and for  $t = 0, 1, \dots, k - 1$ ,  $z^t$  and  $z^{t+1}$  are pairwise connected.

The following proposition shows that the set of efficient allocations is “connected”. This result, of interest in its own right, will be instrumental in the sequel.

**Proposition 1** Let  $P = \langle N, H, (\succeq_i)_{i \in N} \rangle$  be a house allocation problem and let  $x$  and  $y$  be two efficient allocations in  $P$ . Then, there is an  $(x, y)$ -path consisting exclusively of efficient allocations.

**Proof :** The proof is by induction on the size of the problem. If the problem consists of one agent, there is nothing to prove because the only allocation is efficient. Assume that the claim holds for all problems of size  $K$ , let  $P = \langle N, H, (\succeq_i)_{i \in N} \rangle$  be a problem of size  $K + 1$  and let  $x$  and  $y$  be two efficient allocations in it.

**Case 1:** There is an agent,  $k \in N$ , who gets his most preferred house both at  $x$  and  $y$ . Namely,  $x_k = y_k \succeq_k h$  for all  $h \in H$ . Then, there are orderings  $\pi$  and  $\sigma$  of the traders, both with trader  $k$  as their first element, which induce allocations  $x$  and  $y$ , respectively. Let  $N' = N \setminus \{k\}$ ,  $H' = H \setminus \{x_k\}$  and consider the subproblem  $P' = \langle N', H', (\succeq_i |_{H'})_{i \in N'} \rangle$ , where  $\succeq_i |_{H'}$  is the restriction of  $i$ 's preferences to  $H'$ . The restricted allocations  $x|_{N'}$  and  $y|_{N'}$  are efficient in  $P'$  since they are induced by the orderings  $\pi$  and  $\sigma$  respectively, restricted to the agents in  $N'$ . Since  $P'$  is a problem of size  $K$ , by the induction hypothesis, there is a path  $(\hat{x}^0, \dots, \hat{x}^m)$  of efficient allocations in  $P'$  from  $x|_{N'}$  to  $y|_{N'}$ . Define now the allocations  $(x^0, \dots, x^m)$  in  $P$  by

$$x_i^t = \begin{cases} \hat{x}_i^t & \text{if } i \in N' \\ x_k & \text{if } i = k \end{cases}$$

for  $t = 0, \dots, m$ . The sequence  $(x^0, \dots, x^m)$  is a  $(x, y)$ -path of efficient allocations in  $P$

since they are induced by the orderings that induce  $(\hat{x}^0, \dots, \hat{x}^m)$ , respectively, after adding agent  $k$  as their first element.

**Case 2:** There is no agent that gets his most preferred house both at  $x$  and at  $y$ . In this case there are orderings  $\pi$  and  $\sigma$  of the traders which induce allocations  $x$  and  $y$ , respectively. Let  $\ell$  and  $k$  be the first agents in the orders  $\pi$  and  $\sigma$ , respectively. Namely  $\pi(1) = \ell$  and  $\sigma(1) = k$ . Clearly,  $\ell \neq k$ .

**Case 2.1:** Agent  $\ell$ 's and agent  $k$ 's respective top ranked houses differ. This implies that there is an allocation  $z$ , which is efficient in  $P$ , and at which both agent  $\ell$  and agent  $k$  get their respective most preferred houses. Since agent  $\ell$  gets his most preferred house both at the efficient allocation  $x$  and at the efficient allocation  $z$ , by case 1, there is a  $(x, z)$ -path of efficient allocations. But since  $k$  gets his most preferred house both at  $z$  and at  $y$ , by case 1 again there is a  $(z, y)$ -path of efficient allocations. Joining both paths, we conclude that there is a  $(x, y)$ -path of efficient allocations.

**Case 2.2:** Agents  $\ell$  and  $k$  have the same top ranked house. In this case,  $x$  awards agent  $\ell$  this house. Consider an ordering  $\mu$  of the agents in which agent  $\ell$  is first and agent  $k$  is last and let  $z$  be the efficient allocation induced by that ordering. Since agent  $\ell$  gets his most preferred house both at  $x$  and at  $z$ , by case 1, there is a  $(x, z)$ -path of efficient allocations. Let  $z'$  be the allocation that is obtained from  $z$  after agents  $\ell$  and  $k$  switch houses. Allocation  $z'$  is efficient because it is induced by the ordering that is obtained from  $\mu$  after  $\ell$  and  $k$  switch their places. Therefore,  $z$  and  $z'$  are two pairwise connected efficient allocations. Clearly,  $z'$  awards agent  $k$  his most preferred house. Therefore, by case 1 again, there is a  $(z', y)$ -path of efficient allocations. We have built then a path of efficient allocations that connects  $x$  with  $y$ .  $\square$

Given a house allocation problem we want to define a perturbed Markov process as in Young (1998). The states of the process are the allocations of the housing problem. In each period one pair of agents is selected at random and the system moves from one state to another when the matched agents trade. In the unperturbed Markov process, agents do not make mistakes in each meeting: they trade if and only if there are mutual gains from trade in the match. In the perturbed process, agents will make mistakes with a small probability. We denote a generic perturbed process by  $M^\epsilon$ .

It is often the case that the unperturbed Markov process has many stationary distri-

butions. On the other hand for all  $\epsilon \in (0, 1)$ , the perturbed process  $M^\epsilon$  is ergodic, which implies that it has a unique stationary distribution. Denote the unique stationary distribution of  $M^\epsilon$  by  $\mu^\epsilon$ . It is known that this stationary distribution represents the proportion of time that the system will spend in each of its states in the long run. In order to define the stochastically stable states, we check the behavior of the stationary distribution  $\mu^\epsilon$  as  $\epsilon$  goes to 0. It is also known that  $\lim_{\epsilon \rightarrow 0} \mu^\epsilon$  exists and further it is one of the stationary distributions of the unperturbed process. The *stochastically stable states* of the system  $M^\epsilon$  are defined to be those states that are assigned positive probability by this limit distribution.

### 3 All Mistakes Are Equally Likely

Given a house allocation problem, consider first the following stochastic process, whose states are the allocations of the problem. In each period one pair of agents is selected at random. All pairs are equally likely to be chosen. Consider a pair of individuals, say  $i$  and  $j$ . The probability that they trade depends on the degree of advantageousness of the trade. If this is mutually beneficial, then it takes place with high probability, say 1. If the trade is not mutually beneficial, then it takes place with a very low probability. Specifically, assume that if the trade is advantageous for only one trader, it takes place with probability  $\epsilon$  and if the trade is disadvantageous for both traders, it takes place with probability  $\epsilon^2$ . We denote this process by  $M_1^\epsilon$ .

Define the unperturbed process  $M^0$  to be the one just specified, but where the probability of making mistakes is zero. It can be checked that a state of the unperturbed process  $M^0$  is absorbing if and only if there is no pair of agents that envy each other. As shown in the next proposition, it turns out that the absorbing states constitute the only recurrent classes of the process.

**Proposition 2** The recurrent classes of the unperturbed process  $M^0$  are the singletons containing the absorbing states.

**Proof :** It is clear that a singleton containing an absorbing state is a recurrent class. Conversely, assume that  $x$  is an allocation where there are at least two agents  $i$  and  $j$  that envy each other. Then, with positive probability they will meet and trade. As a result,

allocation  $x'$  arises, where  $x'_i = x_j \succ_i x_i$ ,  $x'_j = x_i \succ_j x_j$ , and  $x_k = x'_k$  for every  $k \neq i, j$ . Thus, the envy-level at  $x$  is higher than the envy-level at  $x'$ . Since, by the law of motion of the unperturbed process, the envy-level cannot increase, once the process arrives at  $x'$ , there is zero probability that it will return to  $x$ , which shows that  $x$  does not belong to a recurrent class.  $\square$

Note that each efficient allocation is an absorbing state of the unperturbed Markov process  $M^0$ , but in principle so are many other inefficient allocations. Note also that if the problem has more than one absorbing state, then  $M^0$  has many stationary distributions.

Let  $P$  be a house allocation problem. We are interested in the stochastically stable states  $\mathcal{S}_1(P)$  of the perturbed Markov process  $M_1^\epsilon$  defined above. In order to calculate them, we will use the characterization of the stochastically stable states provided by Young (1993a), based on the techniques developed by Freidlin and Wentzell (1984).

For any two allocations  $x$  and  $y$ , define the *resistance* of the transition  $x \rightarrow y$  as follows: if  $x$  and  $y$  are pairwise connected, then the resistance is the number of agents (0, 1, or 2) that find the bilateral trade unprofitable. Otherwise define the resistance to be  $\infty$ . Similarly, let  $\xi = (z^1, \dots, z^k)$  be an  $(x, y)$ -path. The resistance of the path  $\xi$  is the sum of the resistances of its transitions.

Let  $Z^0 = \{z^1, \dots, z^q\}$  be the set of absorbing states of the unperturbed process and consider the complete directed graph with vertex set  $Z^0$ , which is denoted by  $\Gamma$ . We want to define the resistance of each one of the edges in this graph. For this, let  $z^i$  and  $z^j$  be two elements of  $Z^0$ . The resistance of the edge  $(z^i, z^j)$  in  $\Gamma$  is the minimum resistance over all the resistances of the  $(z^i, z^j)$ -paths.

Let  $z^i$  be an absorbing state. A  $z^i$ -tree is a tree with vertex set  $Z^0$  such that from every vertex different from  $z^i$ , there is a unique directed path in the tree to  $z^i$ . The resistance of the  $z^i$ -tree is the sum of the resistances of the edges that compose it.

The *stochastic potential* of the absorbing state  $z^i$  is the minimum resistance over all the  $z^i$ -trees. Young (1993a) showed that the set of stochastically stable states of the process consists of those states with minimum stochastic potential.

The following lemma will be useful in later proofs.



**Lemma 1** Let  $x$  be a stochastically stable state and let  $y$  be an absorbing state such that the edge  $(x, y)$  has resistance 1. Then,  $y$  is a stochastically stable state.

**Proof :** Let  $T$  be an  $x$ -tree with minimum resistance over all the  $x$ -trees. Let  $s(y)$  denote the immediate successor of  $y$  in the unique path that connects  $y$  to  $x$ . Consider the tree  $T'$  that is built by deleting from  $T$  the edge  $(y, s(y))$  and adding the edge  $(x, y)$ . It can be seen that  $T'$  is a  $y$ -tree. Indeed, if there was a directed path in  $T$  from  $z$  to  $y$ , the same path connects  $z$  to  $y$  in  $T'$ . And if there was a directed path in  $T$  from  $z$  to  $x$  that did not go through  $y$ , now the path that is obtained from that path by adding the edge  $(x, y)$ , is a directed path in  $T'$  that connects  $z$  to  $y$ . The tree  $T'$  is a  $y$ -tree that is obtained from  $T$  by adding an edge of resistance 1 and deleting one edge of resistance greater or equal 1. Therefore, the resistance of  $T'$  is no greater than the resistance of  $T$ . But since  $T$  is an  $x$ -tree with minimum resistance over all the  $x$ -trees and since  $x$  is a stochastically stable state, the resistance of  $T'$  equals the resistance of  $T$  and therefore  $y$  is a stochastically stable state.  $\square$

**Corollary 1** If there is an efficient allocation that is stochastically stable in  $M_1^\epsilon$ , then so are all efficient allocations.

**Proof :** Let  $x$  be an efficient allocation that is stochastically stable and let  $y$  be another efficient allocation. By Proposition 1, there is a  $(x, y)$ -path of efficient allocations. By the definition of  $M_1^\epsilon$ , every edge along this path has resistance 1. By Lemma 1, all the efficient allocations along this path, and in particular allocation  $y$ , are in  $\mathcal{S}_1(P)$ .  $\square$

Corollary 1 still leaves the door open to no efficient allocation being stochastically stable. The following theorem, our first main result, shows that this is not the case. Thus, all efficient allocations are visited a positive proportion of time by the process  $M_1^\epsilon$  in the long run.

**Theorem 1** Let  $P$  be a house allocation problem. The set of stochastically stable allocations  $\mathcal{S}_1(P)$  contains the set of efficient allocations  $\mathcal{E}(P)$ .

**Proof :** Given Corollary 1, it is enough to show that there is one efficient allocation that is stochastically stable. Pick a stochastically stable allocation  $x$ . If  $x$  is efficient, we are done. Therefore, assume  $x$  is not efficient. It suffices to show that there is a path from  $x$  to an efficient allocation such that each of its transitions has resistance less or equal to 1, because, by Lemma 1, the efficient allocation will be stochastically stable. The existence of the required path is an immediate consequence of the next lemma. Prior to it, some definitions are required.

For any allocation  $z$ , let  $A_1(z)$  be the set of agents that are allocated their most preferred house under allocation  $z$  and let  $B_1(z)$  be its complement:

$$\begin{aligned} A_1(z) &= \{i \in N : z_i \succeq_i z_j \quad \forall j \in N\} \\ B_1(z) &= N \setminus A_1(z). \end{aligned}$$

Define recursively the following sets of agents: for  $k = 1, 2, \dots$

$$\begin{aligned} A_{k+1}(z) &= \{i \in B_k(z) : z_i \succeq_i z_j \quad \forall j \in B_k(z)\} \\ B_{k+1}(z) &= B_k(z) \setminus A_{k+1}(z). \end{aligned}$$

Let  $A(z) = \cup_{k=1}^{\infty} A_k(z)$  and  $B(z) = N \setminus A(z)$ .

For any allocation  $z$ , it is immediate that the agents in  $A(z)$  do not belong to any cycle of the envy-graph of  $z$ . It is also clear that no agent in  $A(z)$  envies any agent in  $B(z)$ . Finally, note that  $B(z) = \emptyset$  if and only if  $z$  is efficient.

**Lemma 2** Let  $x$  be an absorbing state that is not efficient and let  $|B(x)| = m$ . Then, there is an absorbing state  $y$  such that the edge  $(x, y)$  has resistance 1 and for which  $|B(y)| < m$ .

**Proof :** Let  $x$  be an inefficient absorbing state with  $|B(x)| = m$ . (Note that  $m \geq 3$ ). Let  $i \in B(x)$ . Then there is an agent in  $B(x)$  who is envied by  $i$ . Let  $j$  be the agent who owns the  $\succeq_i$ -maximal house in the set of houses that belong to agents in  $B(x)$ . That is,  $x_j \succeq_i x_t$  for all  $t \in B(x)$ . Let  $x'$  be the allocation that is obtained from  $x$  after  $i$  and  $j$  trade. At this allocation, no agent in  $A(x) \cup \{i\}$  envies any agent in  $B(x) \setminus \{i\}$ . In fact,

$A(x) \cup \{i\} \subseteq A(x')$ , and therefore,  $|B(x')| < m$ . If  $x'$  is absorbing, then we are done: let  $y = x'$ . If  $x'$  is not absorbing, there is a  $(x', y)$ -path of resistance 0 from  $x'$  to some absorbing state  $y$ . Clearly,  $|B(y)| \leq |B(x')| < m$ .  $\square$

By repeated applications of Lemma 2, we can build a path from  $x$  to an absorbing state  $z$  with  $B(z) = \emptyset$ , which means that  $z$  is efficient. This completes the proof of the theorem.  $\square$

As a consequence of Lemma 2 we get the following characterization of the  $z$ -trees that attain the minimum stochastic potential.

**Lemma 3** Assume that a house allocation problem has  $k > 1$  absorbing states and let  $z$  be a stochastically stable allocation in  $\mathcal{S}_1(P)$ . Any  $z$ -tree whose resistance attains the stochastic potential of  $z$  is composed of  $k$  vertices and  $k - 1$  edges of resistance 1.

**Proof :** Any  $z$ -tree with the set of absorbing states as vertex set has  $k$  vertices and  $k - 1$  edges. Therefore we need to show that each of the edges of a  $z$ -tree that attains the stochastic potential has resistance 1. Let  $z$  be a stochastically stable allocation.

Case 1:  $z$  is efficient. We shall build a  $z$ -tree whose edges have resistance 1. For this, we first draw one outgoing edge from each absorbing state  $z'$  different from  $z$  as follows. If  $z'$  is an absorbing and inefficient state, then by Lemma 2, there is an absorbing state  $z''$  such that the transition  $z' \rightarrow z''$  has resistance 1 and  $|B(z'')| < |B(z')|$ . By adding the edge  $(z', z'')$  and possibly proceeding in the same way if  $z''$  is inefficient, we construct a directed graph such that each one of its connected components is a tree that has a unique efficient allocation. Further, this unique efficient allocation is the root of that tree. In order to complete the  $z$ -tree we need to connect the efficient allocations to  $z$  using one outgoing edge from each efficient allocation different from  $z$ . But by Proposition 1 this can be done.

Case 2:  $z$  is inefficient. By case 1 and Theorem 1, the minimum stochastic potential is  $k - 1$ . Since  $z$  stochastically stable, its stochastic potential is also  $k - 1$ . Further, any  $z$ -tree has  $k - 1$  edges with resistance greater or equal 1. Therefore, each edge of any  $z$ -tree that attains  $z$ 's stochastic potential has resistance 1.  $\square$

As we have established in Theorem 1, all efficient allocations are selected by stochastic stability when applied to the bilateral trading process  $M_1^e$ . It is important, though, to point out that there are house allocation problems where the inclusion reported in Theorem 1 is strict, as the following example shows.

**Example 1** Consider the following  $n$ -agent problem, where  $n \geq 4$ :

$$\begin{array}{cccccc}
 h_1 & \succeq_1 & h_n & \succeq_1 & h_2 & \dots \\
 h_1 & \succeq_2 & h_2 & \succeq_2 & \dots & \\
 h_2 & \succeq_3 & h_3 & \succeq_3 & \dots & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \\
 h_{n-1} & \succeq_n & h_n & \succeq_n & \dots & 
 \end{array}$$

Consider the following allocations.

Allocation	Agents						
	1	2	3	...	$k$	...	$n$
$z^0$	$h_1$	$h_2$	$h_3$	...	$h_k$	...	$h_n$
$z^1$	$h_2$	$h_1$	$h_3$	...	$h_k$	...	$h_n$

**Table 1: Two absorbing states**

It can be easily checked that allocation  $z^0$  is an efficient allocation: it is induced by the natural ordering of the players. Therefore, by Theorem 1,  $z^0$  is stochastically stable. On the other hand,  $z^1$  is an inefficient absorbing state: the only cycle in its envy-graph consists of all the agents except agent 2 (see Figure 1 for the case of  $n = 4$ ).

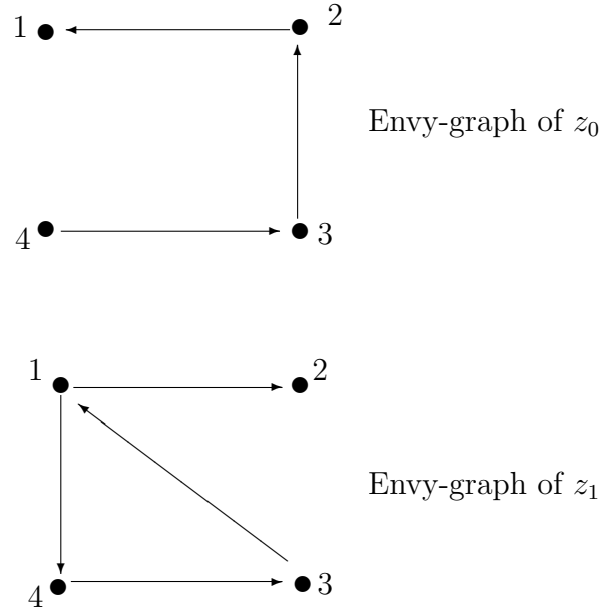


Figure 1

But since  $z^0$  and  $z^1$  are pairwise connected (with agents 1 and 2 trading), the transition  $z^0 \rightarrow z^1$  has resistance 1. Therefore, by Lemma 1  $z^1$  is stochastically stable.

As the previous example shows, there are house allocation problems where the set of stochastically stable allocations contains strictly the set of efficient allocations. In the next subsection, we investigate restrictions on the house allocation problems that assure that every stochastically stable allocation is efficient. It may be skipped without loss of understanding of the sequel.

### 3.1 When Stochastic Stability Implies Efficiency

This subsection identifies three sufficient conditions that render equality between  $\mathcal{S}_1(P)$  and  $\mathcal{E}(P)$ . It closes with a more general result that suggests a procedure to generate housing problems where the same property holds.

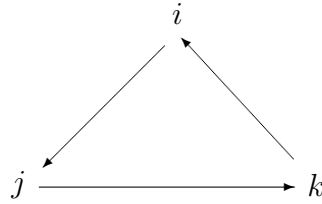
**Proposition 3** Let  $P$  be a house allocation problem with a unique efficient allocation. There is only one stochastically stable state in  $\mathcal{S}_1(P)$ , the efficient allocation.

**Proof :** Note that if there is a unique efficient allocation  $x$ , then at  $x$  each agent gets his top-ranked house. Therefore, any edge that exits  $x$  has resistance 2. By Lemma 3, no allocation other than  $x$  is stochastically stable.  $\square$

Thus, if preferences are sufficiently diverse so that each agent has a different top-ranked house, our pairwise procedure with mistakes selects only efficient allocations. A second simple class of problems with the same property is the following:

**Proposition 4** Let  $P$  be a problem with at most 3 agents. The set of stochastically stable allocations  $\mathcal{S}_1(P)$  is the set of efficient allocations  $\mathcal{E}(P)$ .

**Proof :** For problems with one or two agents, the statement is trivially true. Let  $P$  be a problem with 3 agents, and let  $z$  be an inefficient absorbing state in it. Therefore  $z$ 's envy graph must look as follows:



Namely, there are no two agents with the same top-ranked house. But this means that in this economy, there is a unique efficient allocation. By Proposition 3, it is the only stochastically stable one.  $\square$

In order to provide another class of problems where there is equality between  $\mathcal{S}_1(P)$  and  $\mathcal{E}(P)$ , we need the following definition.

**Definition 1** Let  $P = \langle N, H, (\succeq_i)_{i \in N} \rangle$  be a house allocation problem and let  $<$  be a complete order of the houses in  $H$ . We say that  $P$  has the single-peak property with respect to  $<$  if for every  $i \in N$  there is a house  $h(\succeq_i) \in H$  such that for all  $h, h' \in H$ ,

$$[h < h' \leq h(\succeq_i) \text{ or } h(\succeq_i) \leq h' < h] \quad \text{implies} \quad h' \succ_i h.$$

An example of a problem with the single-peak property is one where all agents have identical preferences, but one could think of many other examples. Now we state the following:

**Proposition 5** Let  $P = \langle N, H, (\succeq_i)_{i \in N} \rangle$  be a house allocation problem that has the single-peak property. Then, the set of stochastically stable allocations  $\mathcal{S}_1(P)$  is the set of efficient allocations  $\mathcal{E}(P)$ .

**Proof :** Since the set of absorbing states is a superset of the set of stochastically stable states, which in turn contains the set of efficient allocations, it suffices to show that all absorbing states are efficient. Hence, consider an inefficient allocation  $z$ . We shall show that it is not absorbing. The envy graph of  $z$  has at least one cycle. Choose a cycle with a minimal number of agents. Without loss of generality assume that the minimal cycle is composed of  $N_K = \{1, 2, \dots, K\} \subseteq N$  and that agent 2 envies 1,  $\dots$ , agent  $K$  envies  $K - 1$ , and agent 1 envies  $K$ . Since there is no smaller subcycle, each trader in the cycle has its second best house out of the houses in  $z(N_K) = \{z_1, \dots, z_K\}$ . Assume that  $P$  has the single-peak property and let  $<$  be the corresponding ordering of  $H$ . Let  $z_k$  be the first house in  $z(N_K)$  according to  $<$ . Restricting attention to houses in  $z(N_K)$ , agent  $k$ 's top-ranked house is  $z_{k-1}$  (modulo  $K$ ) and his second best house is  $z_k$ . Therefore, we must have that  $z_{k-1}$  is  $z_k$ 's immediate successor in  $z(N_K)$  according to  $<$ . Also,  $z_k$  is  $k + 1$ 's (modulo  $K$ ) top-ranked house while  $z_{k+1}$  is his second best house. Consequently  $z_{k+1}$  must be  $z_k$ 's immediate successor in  $z(N_K)$ . Therefore  $z_{k-1} = z_{k+1}$  which implies that  $K = 2$ . Namely,  $N_K = \{1, 2\}$  and agents 1 and 2 envy each other at  $z$ . Hence,  $z$  is not absorbing.  $\square$

In order to get a more general result of constructive nature, consider the following definition.

**Definition 2** Let  $P^1 = \langle N^1, H^1, (\succeq_i)_{i \in N^1} \rangle$  and  $P^2 = \langle N^2, H^2, (\succeq_i)_{i \in N^2} \rangle$  be two house allocation problems such that  $N^1 \cap N^2 = \emptyset$  and  $H^1 \cap H^2 = \emptyset$ . Define  $P^1 * P^2$  to be the family of problems  $\langle N, H, (\succeq'_i)_{i \in N} \rangle$  such that:

- $N = N^1 \cup N^2$ ;

- $H = H^1 \cup H^2$ ;
- for each  $i \in N$  and for  $\ell, k \in \{1, 2\}$ , if  $i \in N^\ell$  then for all  $h \in H^\ell$  and  $h' \in H^k$ :

$$\begin{cases} h \succeq'_i h' & \text{if } k \neq \ell \\ h \succeq'_i h' \Leftrightarrow h \succeq_i h' & \text{if } k = \ell. \end{cases}$$

The essential feature of a problem in  $P^1 * P^2$  is that every agent in  $N^\ell$ , for  $\ell = 1, 2$ , prefers any house in  $H^\ell$  to any house in  $H^k$ , for  $k \neq \ell$  and that  $\succeq_i$  is the restriction of  $\succeq'_i$  to  $H^\ell$ . The following proposition suggests a technique to generate larger problems satisfying  $\mathcal{S}_1(P) = \mathcal{E}(P)$  by “composing” simpler problems where the same equality holds. For example, this can be done by combining housing problems, each satisfying one of the three sufficient conditions identified earlier in this subsection.

**Proposition 6** Let  $P^1 = \langle N^1, H^1, (\succeq_i)_{i \in N^1} \rangle$  and  $P^2 = \langle N^2, H^2, (\succeq_i)_{i \in N^2} \rangle$  be two house allocation problems such that  $N^1 \cap N^2 = \emptyset$  and  $H^1 \cap H^2 = \emptyset$ . Assume that for both problems, the set of stochastically stable allocations of  $M_1^\epsilon$  coincides with the set of efficient allocations. Then the same is true for every problem in  $P_1 * P_2$ .

**Proof :** Let  $P = \langle N, H, (\succeq'_i)_{i \in N} \rangle \in P^1 * P^2$ .

**Lemma 4** Let  $z = (z^1, z^2) = ((z_i)_{i \in N_1}, (z_j)_{j \in N_2})$  be an allocation in  $P$ . Then,  $z$  is efficient in  $P$  if and only if  $z^1 = (z_i)_{i \in N_1}$  and  $z^2 = (z_j)_{j \in N_2}$  are efficient allocations in  $P^1$  and  $P^2$ , respectively.

**Proof :** Note first that if  $z^1 \in \mathcal{E}(P^1)$  and  $z^2 \in \mathcal{E}(P^2)$  are efficient allocations induced by  $\pi$  and  $\sigma$  respectively, then  $z$  is induced by the ordering  $(\pi, \sigma)$  of  $N$  where the agents in  $N^1$ , ordered according to  $\pi$ , are followed by the agents in  $N^2$ , ordered according to  $\sigma$ . Secondly, if  $z$  is an efficient allocation in  $P$  induced by an ordering  $\tau$ , then by the way the preferences  $(\succeq'_i)_{i \in N}$  are defined (see Definition 2), it is clear that  $z(N^t) = H^t$  for  $t = 1, 2$ . That is,  $z^1$  and  $z^2$  are allocations in  $P^1$  and  $P^2$ , respectively. Moreover, they are efficient since they are induced by  $\tau$  restricted to each subproblem.  $\square$



We need to show that every stochastically stable allocation in  $P$  is also efficient in  $P$ . So let  $y = (y^1, y^2)$  be a stochastically stable allocation in  $P$  and let  $\Gamma_y$  be a  $y$ -tree that attains the minimum stochastic potential. Assume by contradiction that  $y$  is not efficient and let  $x = (x^1, x^2)$  be an efficient allocation in  $P$ . By lemma 4,  $x(N^1) = H^1$  and  $x(N^2) = H^2$ . Consider the unique  $(x, y)$ -path in  $\Gamma_y$ . We claim that at each allocation along this path every agent in  $N^1$  gets a house in  $H^1$  and every agent in  $N^2$  gets a house in  $H^2$ . For if there was an edge  $(z, z')$  in this path from allocation  $z$  to allocation  $z'$ , where  $z(N^k) = H^k$  for  $k = 1, 2$  but  $z'(N^k) \neq H^k$  for  $k = 1, 2$ , this edge would have a resistance greater than 1, which by Lemma 3 contradicts the fact that  $y$  is stochastically stable. As a result, the  $(x, y)$ -path induces a  $(x^1, y^1)$ -path and a  $(x^2, y^2)$ -path in the subproblems  $P^1$  and  $P^2$ , respectively. But these paths are composed by edges of resistance 1. Therefore, since  $x^1$  and  $x^2$  are stochastically stable in  $P^1$  and  $P^2$  respectively, by Lemma 1 allocations  $y^1$  and  $y^2$  are stochastically stable in  $P^1$  and  $P^2$ , respectively. But by assumption,  $\mathcal{E}(P^k) = \mathcal{S}(P^k)$  for  $k = 1, 2$  which implies that  $y^1$  and  $y^2$  are efficient allocations in  $P^1$  and  $P^2$ , respectively. By Lemma 4 then,  $y = (y^1, y^2)$  is efficient in  $P$ .  $\square$

## 4 Serious Mistakes Are Less Likely

We now turn to a second perturbed dynamic process, denoted by  $M_2^\epsilon$ , of the same unperturbed process  $M^0$ . In order to define the transition probabilities we need some notation. Consider the pair consisting of agents  $i$  and  $j$ , and let  $h_i$  and  $h_j$  be their houses before they trade. Define

$$n_i = |\{h \in H : h_i \succ_i h \succeq_i h_j\}|,$$

and

$$n_j = |\{h \in H : h_j \succ_j h \succeq_j h_i\}|.$$

Namely,  $n_i$  is the number of houses that agent  $i$  considers at least as good as house  $h_j$  and worse than  $h_i$ , and  $n_j$  is the number of houses that agent  $j$  considers at least as good as house  $h_i$  and worse than  $h_j$ . At a transaction between agents  $i$  and  $j$ , we shall say that agent  $i$  makes a mistake of order  $n_i$  and agent  $j$  of order  $n_j$ . Notice that if as a result of this trade, agent  $k$  gains ( $k = i, j$ ),  $n_k = 0$ .

The dynamic process is described as follows. As before, all pairs are chosen with equal probability. Once they are chosen, the probability that agents  $i$  and  $j$  trade is  $\epsilon^{n_i+n_j}$ . In words, if trade is mutually beneficial, it takes place with probability 1. If it is not, then it takes place with a probability that depends on the seriousness of the mistakes made. An individual is much more careful of not making bad mistakes.

Note again that the unperturbed process, i.e., the version of this process when  $\epsilon = 0$ , is the same unperturbed process  $M^0$  as before. Therefore, Proposition 2 continues to apply here.

Within the new perturbed dynamic process  $M_2^\epsilon$ , the resistance of a transition  $x \rightarrow y$  should be redefined as follows: If  $x$  and  $y$  are pairwise connected, then the resistance is  $n_i + n_j$  where  $i$  and  $j$  are the agents who trade houses. If  $x$  and  $y$  are not pairwise connected, then, as before, the resistance is defined to be  $\infty$ .

Note that if  $x$  and  $y$  are pairwise connected, then the resistance of the transition  $x \rightarrow y$  is the number of directed edges in the envy-graph of  $y$  that are not in the envy-graph of  $x$ : if, in going from  $x$  to  $y$ , agent  $i$  makes a mistake of order  $n_i$  and agent  $j$  of order  $n_j$ , there will be  $n_i + n_j$  additional directed edges in the envy-graph of  $y$  that were not present in the envy-graph of  $x$ .

The following theorem characterizes the set  $\mathcal{S}_2(P)$  of stochastically stable allocations of the process  $M_2^\epsilon$ .

**Theorem 2** Let  $P$  be a house allocation problem. The set of stochastically stable allocations  $\mathcal{S}_2(P)$  is the set of allocations with minimum envy level. Furthermore, it is a subset of the set of efficient allocations  $\mathcal{E}(P)$ .

**Proof :** Let  $x$  be stochastically stable in the process  $M_2^\epsilon$  and let  $y \neq x$  be an allocation with minimum envy level (if there is no such  $y$ , then  $x$  is the only allocation with minimum envy-level and we are done). Therefore  $e(y) \leq e(x)$ . Since  $x$  is stochastically stable, there is an  $x$ -tree with minimum stochastic potential. We shall construct a  $y$ -tree with a resistance no greater than that of the  $x$ -tree.

The  $y$ -tree consists of the following directed edges. If edge  $(z^k, z^{k'})$  is an edge in the  $x$ -tree that does not belong to the unique directed path in that tree that connects  $y$  to  $x$ , then  $(z^k, z^{k'})$  also belongs to the  $y$ -tree. If edge  $(z^k, z^{k'})$  does belong to the unique

$(y, x)$ -path in the  $x$ -tree, then  $(z^k, z^{k'})$  is deleted and replaced with  $(z^{k'}, z^k)$ , that is, the edge is reversed. Note that the graph so constructed is indeed a  $y$ -tree: it is a tree because it is a connected graph with the same number of edges as the  $x$ -tree; it is a  $y$ -tree because if a  $(z, y)$ -path was in the  $x$ -tree, it remains in the  $y$ -tree, whereas if it was not, there is now a  $(z, y)$ -path either via  $x$  or as a subpath of the  $(x, y)$  path just built.

We are going to show that the resistance of this  $y$ -tree, denoted by  $R(y)$ , is no greater than that of the  $x$ -tree, denoted by  $R(x)$ . The difference  $R(y) - R(x)$  is less than or equal to the difference between the resistance of the  $(x, y)$ -path in the  $y$ -tree and that of the  $(y, x)$ -path in the  $x$ -tree.

Let  $(z^0, z^1, \dots, z^K)$  be a  $(y, x)$ -path of minimum resistance (of course,  $z^0 = y$  and  $z^K = x$ ). Recall that for  $t = 0, \dots, K - 1$ ,  $z^t$  and  $z^{t+1}$  are pairwise connected. Denote the resistance of the edge  $(z^k, z^{k+1})$  by  $r(z^k, z^{k+1})$ : this number is the sum of mistake orders made by the pair of agents involved in the transition from  $z^k$  to  $z^{k+1}$ .

Similarly, consider the  $(x, y)$ -path  $(z^K, \dots, z^1, z^0)$  and denote the resistance of the edge  $(z^{k+1}, z^k)$  by  $r(z^{k+1}, z^k)$ .

By definition of resistance of a tree,

$$R(y) - R(x) \leq \sum_{k=0}^{K-1} [r(z^{k+1}, z^k) - r(z^k, z^{k+1})],$$

where the inequality follows because, in going from  $x$  to  $y$ , we are following the reverse path of going from  $y$  to  $x$  (while in principle there could exist a cheaper path).

Recall that, following the paths underlying the  $x$ -tree and their reverse paths for the  $y$ -tree constructed, the resistance  $r(z^k, z^{k+1})$  is exactly the number of directed edges in the envy-graph of  $z^{k+1}$  which are not in the envy-graph of  $z^k$ , and similarly for  $r(z^{k+1}, z^k)$ . Hence, we have that

$$e(z^k) - e(z^{k+1}) = r(z^{k+1}, z^k) - r(z^k, z^{k+1}).$$

Therefore,

$$R(y) - R(x) \leq \sum_{k=0}^{K-1} [e(z^k) - e(z^{k+1})] = e(y) - e(x) \leq 0.$$

But  $R(y) - R(x) \geq 0$  because  $x$  has minimum stochastic potential. Hence,  $R(y) - R(x) =$

$e(y) - e(x) = 0$ , from which it follows that:

- (i)  $y$  has minimum stochastic potential, and that
- (ii)  $x$  has minimum envy level.

To complete the proof, we need to show that if  $x$  has minimum envy level, it is efficient. If it were not efficient, its envy-graph would contain a cycle. Eliminating the cycle by letting its agents trade leads to an allocation with a lower envy level.  $\square$

**Remark:** As shown in Theorem 2, the support of the limit distribution of the process  $M_2^\epsilon$  consists of the set of allocations of minimum envy level, itself a subset of the set of efficient allocations. The limit distribution can be regarded as a random allocation: the distribution indicates the likelihood of being at each of the states in the long run. Although the support of this distribution consists solely of efficient allocations, one may ask whether the distribution itself is ordinally efficient in the sense defined by Bogomolnaia and Moulin (1999). That is, it may happen that another random allocation first-order stochastically dominates the limit distribution of the process; in this case, all agents whose preferences over lotteries satisfy monotonicity in first-order stochastic dominance would prefer the former random allocation to the latter. To understand the concept of ordinal efficiency, consider the following 4-agent example:

$$\begin{array}{cccccc}
 h_1 & \succeq_1 & h_2 & \succeq_1 & h_3 & \succeq_1 & h_4 \\
 h_1 & \succeq_2 & h_2 & \succeq_2 & h_3 & \succeq_2 & h_4 \\
 h_2 & \succeq_3 & h_1 & \succeq_3 & h_4 & \succeq_3 & h_3 \\
 h_2 & \succeq_4 & h_1 & \succeq_4 & h_4 & \succeq_4 & h_3
 \end{array}$$

Consider the random allocation that results from the following mechanism: each of the 24 possible serial dictatorship mechanisms is chosen with equal probability. Note that the support of the resulting random allocation is the set of efficient allocations. In this random allocation one can check that each agent receives his top ranked house with probability 5/12, his second ranked house with 1/12, his third ranked house with 5/12 and his fourth ranked house with probability 1/12. But this random allocation is not ordinally efficient, because it is first-order stochastically dominated by the random allocation that

assigns  $(h_1, h_3, h_2, h_4)$  and  $(h_3, h_1, h_4, h_2)$ , each with equal probability: under this random allocation, each agent receives his top ranked and third ranked houses with probability  $1/2$ .

It follows from Theorem 2 that the limit distribution of the process  $M_2^\epsilon$  is ordinally efficient. This is so because it assigns positive probability only to allocations with minimum envy level. That is, the resulting lottery minimizes the expected envy, i.e., maximizes the sum of the agents' expected utilities, where each agent is endowed with the von Neumann-Morgenstern utility function given by the negative of his envy.

## 5 Conclusion

The first welfare theorem states that competitive allocations are efficient. Further, it is also known that competitive allocations from equal division are envy-free. We can interpret these results as saying that when rational agents have equal endowments and trade with an anonymous market, the resulting outcome is efficient and envy-free. Our results show that, in some models of exchange, pairwise trade where agents make mistakes with small probability leads the economy to efficiency and minimum envy in the long run. It is good news, we believe, to find this very different underpinning of such well established concepts.

On the other hand, another leading solution concept in this problem is the competitive equilibrium (which always yields a unique allocation in this context, and coincides with the core). An important question, raised first by Vega-Redondo (1997), is whether one can find evolutionary support to Walrasian allocations. He succeeds in doing this in an imitation process applied to the Cournot oligopoly model, a game in normal form. In an exchange economy, though, the competitive equilibrium is obviously sensitive to the *initial endowments*. As a result, it is not clear how a solution with this property could be selected by a methodology based on ergodic processes, which are independent of *initial conditions*. Nonetheless, this is an important question that deserves further thought.

## References

- Abdulkadiroglu, A. and T. Sonmez (1998), "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems," *Econometrica* **66**, 689-701.
- Bergin, J. and B. Lipman (1996), "Evolution with State Dependent Mutations," *Econometrica* **64**, 943-956.
- Bogomolnaia, A. and H. Moulin (1999), "A New Solution to the Random Assignment Problem," mimeo, Rice University.
- Ellison, G. (1993), "Learning, Local Interaction and Coordination," *Econometrica* **61**, 1047-1071.
- Feldman, A. (1973), "Bilateral Trading Processes, Pairwise Optimality, and Pareto Optimality," *Review of Economic Studies* **40**, 463-473.
- Foster, D. P. and H. P. Young (1990), "Stochastic Evolutionary Game Dynamics," *Theoretical Population Biology* **38**, 219-232.
- Freidlin, M. and A. Wentzell (1984), *Random Perturbations of Dynamical Systems*, Berlin: Springer-Verlag.
- Goldman, M. and R. Starr (1982), "Pairwise,  $t$ -wise, and Pareto Optimality," *Econometrica* **50**, 593-606.
- Hart, S. (1999), "Evolutionary Dynamics and Backward Induction," discussion paper no. 195, Center for Rationality, Hebrew University of Jerusalem.
- Jackson, M. and A. Watts (1999), "On the Formation of Interaction Networks in Social Coordination Games," mimeo.
- Kandori, M., G. Mailath, and R. Rob (1993), "Learning, Mutations and Long Run Equilibria in Games," *Econometrica* **61**, 29-56.
- Shapley, L. and H. Scarf (1974), "On Cores and Indivisibility," *Journal of Mathematical Economics* **1**, 23-28.

Vega-Redondo, F. (1997), "The Evolution of Walrasian Behavior," *Econometrica* **65**, 375-384.

Young, H. P. (1993a), "The Evolution of Conventions," *Econometrica* **61**, 57-84.

Young, H. P. (1993b), "An Evolutionary Model of Bargaining," *Journal of Economic Theory* **59**, 145-168.

Young, H. P. (1998), *Individual Strategy and Social Structure: an Evolutionary Theory of Institutions*, Princeton university Press, Princeton, NJ.