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Marginal cost price rule for homogeneous cost functions

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Abstract. Mirman and Tauman (1982) show that axioms of cost sharing, additivity, rescaling invariance, monotonicity, and consistency uniquely determine a price rule on the class of continuously differentiable cost problems as the Aumann-Shapley price mechanism. Here we prove that standard versions of these axioms determine uniquely the marginal cost price rule on the class of homogeneous and convex cost functions, which are, in addition, continuously differentiable. This result persists even if the cost functons are not required to be convex.

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Key words: marginal cost price rule; continuously differentiable, homogeneous and convex cost functions; axiomatic characterization.

1. Introduction

The Aumann-Shapley mechanism, introduced in Billera and Heath (1982) and Mirman and Tauman (1982), distributes costs of joint production of a finite number of infinitely divisible consumption goods by means of per unit costs of each good, or prices. Defined for the entire set of continuously differentiable (c.d.) cost functions, the mechanism is characterized by the following five axioms:

(1) *cost sharing* – total revenue equals total cost;
(2) *additivity* – if the cost consists of several components (e.g., costs related to management and acquisition of raw materials), then price vectors can be determined separately for each component, and then added up;

(3) *monotonicity* – an increase in marginal costs of production leads to an increase in prices;
(4) *rescaling invariance* – changing the scale on which a good is measured yields the equivalent change in its price;
(5) *consistency* – breaking existing goods into "sub-goods" with different names does not affect the pricing.

In the study of long run production it is natural to assume that cost functions are nondecreasing, homogeneous of degree 1, and convex. While only linear cost functions of this type are c.d. at zero, quite rich and interesting class of functions is obtained by restricting the c.d. requirement to non-zero bundles of goods. The Aumann-Shapley mechanism, which can be naturally defined for this class of cost functions despite their non-differentiability at zero, is precisely the *marginal cost price rule*, which attributes to each of the goods its marginal cost of production.

The marginal cost price rule was uniquely determined on the class of homogeneous c.d.[1] cost functions by Mirman and Neyman (1983), but under a set of axioms that differs from (1)–(5). While versions of additivity, rescaling invariance, and consistency are present in this set, the cost sharing and monotonicity axioms are replaced by the assumption of *coincidence* of the price rule with the marginal cost price rule on linear cost functions, and a *continuity* requirement, imposed on the behavior of the price rule.

The imposition of the cost sharing axiom, (1), is never a liability, since only cost sharing rules provide a solution to the need of distributing costs. As for the axiom of monotonicity, (3), it appears to be more natural and has stronger economic appeal than the continuity with respect to the bounded variation norm on cost functions. Also, continuity (of bounded variation norm 1) implies monotonicity in all cost sharing price rules, which makes the latter a weaker, and therefore even more attractive, axiom. It is thus highly desirable to derive the uniqueness of the marginal cost price rule on the class of homogeneous cost functions using the standard axioms (1)–(5), without having to replace (1) and (3) with coincidence and continuity.

In this paper we show that the marginal cost price rule is uniquely determined by the standard axioms, (1)–(5) above, on the class of homogeneous and convex c.d. cost functions (Theorem 7), as well as on the class of c.d. cost functions which are merely homogeneous (see Remark 4). We make a slightly stronger differentiability assumption than Mirman and Neyman (1983), by (essentially) assuming bounds on partial derivatives.

In addition to determining a common price rule under economically sound axioms, the present result is essential if one wishes to characterize price rules for homogeneous and convex cost functions, which are not necessarily differentiable. Defined by means of one-sided derivatives, the marginal cost price rule fails to be cost sharing for such cost functions. Nevertheless, there is a unique price rule satisfying (1)–(5) even on this domain (see Haimanko (2001), p. 461). This rule attributes to the goods their expected marginal costs, following a random infinitesimal perturbation at the production level. This characterization relies on an obvious implication of our result, that any price rule adhering to (1)–(5) coincides with the marginal cost price rule on homogeneous c.d. cost functions.

[1] On non-zero bundles.

The proof of our result differs markedly from the traditional arguments used to establish uniqueness of price rules on c.d. cost functions, including proofs of Mirman and Tauman (1982) and Mirman and Neyman (1983). The first step of our proof asserts that the price of good i under a given price rule is a linear function of the i^{th} derivative of the underlying cost function. Next, additivity, monotonicity and cost sharing axioms imply that the functionals are given as integrals of the corresponding derivatives with respect to uniquely determined probability measures (Corollary 9). Consistency and rescaling invariance axioms imply that all these measures are equal (Corollary 11), and cost sharing axiom finally shows that the measure is concentrated on a single point – the level of production.

2. Definitions and statement of the result

In what follows R_+^k will be the set of all vectors with nonnegative coordinates in the k-dimensional Euclidean space R^k, and R_{++}^k will be the subset of vectors with strictly positive coordinates. A *cost problem* is a pair (f, a), where, for some positive integer $k, a \in R_{++}^k$ and f is a nondecreasing real valued function on R_+^k with $f(0) = 0$. Here a is interpreted as the vector of quantities of goods $1, \dots, k$ that are produced, and f – as a *cost function*, so that $f(x)$ is viewed as the cost of producing the bundle $x = (x_1, \dots, x_k)$ of goods. Given a set (class) F of cost functions, F^k will denote the subset of $f \in F$ which are functions of k variables.

A *price rule* on a class F of cost problems is a function $\psi : \bigcup_{k=1}^{\infty}(F^k \times R_{++}^k) \to \bigcup_{k=1}^{\infty} R^k$ such that $\psi(f, a) \in R^k$ for every $f \in F^k$ and $a \in R_{++}^k$. The j^{th} coordinate of $\psi(f, a)$ is denoted by $\psi_j(f, a)$.

Definition 1. *A price rule is cost sharing if*

$$\sum_{j=1}^{k} \psi_j(f, a) a_j = f(a) \tag{1}$$

for every $k \geq 1$ and $f \in F^k$ and $a \in R_{++}^k$.

Definition 2. *A price rule is additive if*

$$\psi(f_1 + f_2, a) = \psi(f_1, a) + \psi(f_2, a) \tag{2}$$

for $f_1, f_2 \in F^k$ such that $f_1 + f_2$ is also in F^k, and $a \in R_{++}^k$.

Let $m \geq k \geq 1$ and let $\pi = (S_1, \dots, S_k)$ be an ordered partition of $\{1, 2, \dots, m\}$. We define $\pi^* : R^m \to R^k$ by $\pi^*(x)_i := \sum_{j \in S_i} x_j$ for every $x \in R^m$ and $i = 1, \dots, k$.

Definition 3. *A price rule ψ is consistent if for every $m \geq k \geq 1$, $b \in R_{++}^m$, ordered partition $\pi = (S_1, \dots, S_k)$ of $\{1, 2, \dots, m\}$, and f such that both f and $f \circ \pi^*$ are in F,*

$$\psi_j(f \circ \pi^*, b) = \psi_i(f, \pi^*(b)) \tag{3}$$

for every $1 \leq i \leq k$, *and* $j \in S_i$.

This axiom of consistency is a stronger version of consistency in the sense of Mirman and Tauman (1982).

For each $x, y \in R^k$ denote $x * y = (x_1 y_1, \ldots, x_k y_k)$ and for each $\alpha \in R^k_{++}$ denote $\alpha^{-1} = \left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_k}\right)$. Also, for each function f on R^k_+ define $(\alpha * f)(x) = f(\alpha * x)$.

Definition 4. *A price rule is rescaling invariant if*

$$\psi(\alpha * f, \alpha^{-1} * a) = \alpha * \psi(f, a) \tag{4}$$

for all $k \geq 1$, $\alpha, a \in R^k_{++}$ *and* f *such that* $f, \alpha * f \in F^k$.

Definition 5. *A price rule* ψ *is monotone if*

$$\psi_i(f, a) \geq \psi_i(g, a) \tag{5}$$

whenever $f, g \in F_k$ *are such that* $(f - g)(x)$ *is a nondecreasing function.*

Denote by H^k_c the set of functions f on R^k_+ which are homogeneous of degree 1, convex, and continuously differentiable[2] on $R^k_+ - \{0\}$. Let $H_c = \bigcup_{k=1}^{\infty} H^k_c$.

Remark 1: Any consistent and rescaling invariant price rule ψ on H_c also satisfies dummy axiom – if $f \in H^k_c$ is such that $\frac{df}{dx_j}$ vanishes on $R^k_+ - \{0\}$, then for every $a \in R^k_{++}$

$$\psi_j(f, a) = 0. \tag{6}$$

Indeed, given such f, consider $f' \in H^{k+1}_c$, given by

$$f'(x_1, \ldots, x_k, x_{k+1}) = f\left(x_1, \ldots, x_{j-1}, \frac{1}{4}x_j + \frac{3}{4}x_{k+1}, x_{j+1}, \ldots, x_k\right). \tag{7}$$

From symmetry of ψ (which is an easy implication of the consistency axiom),

$$\psi_j(f', (a, a_j)) = \psi_{k+1}(f', (a, a_j)). \tag{8}$$

On the other hand, by the rescaling invariance and consistency,

$$\psi_j(f', (a, a_j)) = \frac{1}{4}\psi_j(f, a), \quad \textit{and} \quad \psi_{k+1}(f', (a, a_j)) = \frac{3}{4}\psi_j(f, a). \tag{9}$$

To reconcile (9) and (8), equality (6) must hold.

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[2] Function f is differentiable on a closed subset A of R^k if it can be extended to a continuously differentiable function on an open neighborhood of A.

Remark 2: Any additive and monotone price rule on H_c is homogenous: if $c > 0$ then $\psi(cf, a) = c\psi(f, a)$ for any $f \in H_c^k$, $a \in R_{++}^k$.

Definition 6. *For $f \in H_c^k$ and $a \in R_+^k$ denote by $\nabla f(a)$ the gradient of f at a. The marginal cost price rule φ^m is given for every cost problem (f, a), $f \in H_c^k$, by*

$$\varphi^m(f, a) = \nabla f(a).$$

It is easy to check that φ^m is cost sharing, additive, consistent, rescaling invariant and monotone. The following theorem shows that the rule φ^m is uniquely characterized by these properties.

Theorem 7. *The marginal cost price rule φ^m is the only cost sharing, additive, consistent, rescaling invariant and monotone price rule on H_c.*

The same axioms uniquely determine the marginal cost price rule on the class of all homogeneous and continuously differentiable cost functions. We address this issue in Remark 4.

3. Proof of the theorem

Let us start by briefly describing the proof. We use a given a price rule satisfying the axioms to induce a family of positive linear functionals on a certain space of continuous functions. Specifically, for a constant set $\{1, \ldots, k\}$ of goods we prove that the price of good i according to the price rule is determined by i^{th} partial derivatives of the cost function, restricted to points of the $k-1$-dimensional simplex. This gives rise to a positive linear functional on a subspace of continuous functions on the simplex. Such a functional is characterized by a unique measure on the simplex, as established in Lemma 8 and Corollary 9 with the aid of monotonicity, additivity, and dummy axioms. In Lemma 10 the consistency and rescaling invariance are used to shows that the $k-1$-dimensional measures are, in fact, restrictions of a single k-dimensional measure induced by the same price rule, and so they are equal (Corollary 11). This, however, implies that the measure(s) is determined uniquely, as the Dirac measure supported on the center of the simplex; this fact is established by exhibiting a cost function for which the sum of goods' shares would not add up to the total cost were the measure to have a larger support, contradicting the cost sharing axiom.

For any integer $k \geq 2$, denote $\mathbf{1}_k = (1, \ldots, 1) \in R^k$, and let $x \cdot y$ be the scalar product of $x, y \in R^k$. Let Δ^k be the simplex $\{x \in R_+^k \mid x \cdot \mathbf{1}_k = 1\}$, and denote by $C(\Delta^k)$ the space of continuous functions on Δ^k.

Denote by H^k the linear space of differences of functions in H_c^k. By homogeneity, every $f \in H^k$ is uniquely determined by the knowledge of its partial its derivatives $\frac{df}{dx_j}$, $j = 1, \ldots, k$, on Δ^k. Let V be the subspace $\{\nabla f(x) \mid f \in H^k\}$ of $[C(\Delta^k)]^k$, where $\nabla f(x)$ stands for the gradient of f at x.

Let ψ be a cost sharing, additive, consistent, rescaling invariant and monotone price rule on H_c. It induces a function on ψ_j^k on V, given for each $f \in H^k$ by

$$\psi_j^k(\nabla f) = \psi_j(f_1, \mathbf{1}_k) - \psi_j(f_2, \mathbf{1}_k), \tag{10}$$

where $f_1, f_2 \in H_c^k$ are such that $f = f_1 - f_2$. If $f = f_1 + f_4 = f_2 + f_3$ (where all f_i are in H_c^k) then additivity of ψ implies

$$\psi_j(f_1, \mathbf{1}_k) - \psi_j(f_2, \mathbf{1}_k) = \psi_j(f_3, \mathbf{1}_k) - \psi_j(f_4, \mathbf{1}_k), \tag{11}$$

and so $\psi_j^k(\nabla f)$ is well defined. From additivity and homogeneity of ψ it follows that ψ_j^k is a linear functional, which is also positive (that is, $\psi_j^k(\nabla f) \geq 0$ whenever $\nabla f \geq 0$) by monotonicity of ψ. Finally, given $c = (c_1, \dots, c_k) \in R^k$ there is $f \in H^k$ with $\nabla f \equiv c$, and so

$$\psi_j^k(c) = c_j, \tag{12}$$

by the cost sharing and the dummy axioms.

The space $[C(\Delta^k)]^k$ can be viewed as $C(\bigsqcup_{i=1}^k \Delta^k)$ (continuous functions on the union of k disjoint copies of Δ^k), V can be regarded as its subspace, and ψ_j^k – as a positive linear projection (i.e., nonnegative functions are mapped by ψ_j^k into R_+, and constant functions are mapped into their values). By the Kantorovitch Theorem (e.g., Theorem 83.15 in Zaanen (1983)), ψ_j^k can be extended to the entire $C(\bigsqcup_{i=1}^k \Delta^k)$ as a positive linear projection. It is then a sum of positive linear functionals $(\psi_k^j)_i$, each defined on the corresponding coordinate of $[C(\Delta^k)]^k$. Property (12) together with positivity now imply that $(\psi_j^k)_i$ are zero unless $i = j$. Therefore ψ_j^k is a function of j^{th} coordinate alone (which is in the space $C(\Delta^k)$). By Riez representation theorem

$$\psi_j^k(g) = \int g(x)\, d\lambda_j^k(x) \tag{13}$$

for some positive measure λ_j^k on Δ^k and all $g \in C(\Delta^k)$. Choosing $c = e_i$ (the i^{th} unit vector) in (12), for $i = 1, \dots, k$, we get $\psi_k^j(e_i) = \delta_{ij}$, and so λ_j^k is a probability measure.

The rest of the paper is dedicated to showing that λ_j^k is the Dirac measure on $\frac{1}{k}\mathbf{1}_k$.

Lemma 8. *For all k the space $\left\{\frac{df}{dx_j}|_{\Delta^k} \mid f \in H^k\right\}$ is dense in $C(\Delta^k)$ (in the maximum norm).*

Proof: We prove the lemma for $j = 1$ only, and other cases follow by permutation of coordinates. It suffices to show that $\left\{\frac{df}{dx_j}|_{R_+^k} \mid f \in H^k\right\}$ contains all twice continuously differentiable functions on $R_+^k - \{0\}$, which are homogeneous of degree 0.

Let g be a homogeneous of degree 0 function on R_+^k, twice continuously differentiable on $R_+^k - \{0\}$. Define for $x = (x_1, \dots, x_k) \in R_+^k$

$$f(x_1, \dots, x_k) = \int_0^{x_1} g(y, x_2, \dots, x_k)\, dy. \tag{14}$$

Observe that $\frac{df}{dx_1}(x) = g(x)$ for all $x \in R_+^k$, and so $\frac{df}{dx_1}|_{\Delta^k} = g|_{\Delta^k}$. Note also that f is homogeneous of degree 1 on R_+^k and twice continuously differentiable on $R_+^k - \{0\}$.

Now choose any twice differentiable h on R^k, which is strictly convex on Δ^k, and let h' be the unique function in H_c^k which coincides with h on Δ^k. Then for any $K > 0$ $f = (Kh' + f) - Kh'$, and $Kh' + f$ is convex for K large enough (its Hessian is positive definite), and so, for that K, both $Kh + f$ and Kh are in H_c^k. Therefore $f \in H^k$. ■

Lemma 8 and (13) now imply:

Corollary 9. *The measure λ_j^k is determined uniquely.*

Given $x \in R_+^k - \{0\}$ denote $\Upsilon^k(x) = \frac{1}{\sum_{j=1}^k x_j} x$ for $x \neq 0$, and let $\Upsilon^k(0) = \frac{1}{k}\mathbb{1}_k$. In other words, $\Upsilon^k(x)$ is a projection of $x \in R_+^k - \{0\}$ onto the simplex Δ^k. For $\alpha \in [0, 1]$, and $0 \leq j \leq k$, let $\Gamma_j^\alpha : \Delta^{k+1} \to R^k$ be the map given for $x = (x_1, \ldots, x_{k+1})$ by $\Gamma_j^\alpha(x)_l = x_l$ if $l \neq j$, and $\Gamma_j^\alpha(x)_j = \alpha x_{k+1} + (1 - \alpha)x_j$, and also denote $\Upsilon^0 = \Upsilon^k \circ \Gamma_j^0$ (Γ_j^0 is independent of j).

Remark 3: If $f \in H^k$, and for $\alpha \in (0, 1)$ the function $f_j^\alpha \in H^{k+1}$ is given by $f_j^\alpha(x) = f(\Gamma_j^\alpha(x))$, then

$$\psi_{k+1}^{k+1}(\nabla f_j^\alpha) = \alpha\psi_j^k(\nabla f). \tag{15}$$

This follows from the consistency and rescaling invariance of ψ.

Lemma 10. *If $1 \leq j \leq k$ then*

$$\lambda_j^k = \Upsilon^0 \circ \lambda_{k+1}^{k+1} \tag{16}$$

(i.e., for any Borel set $K \subset \Delta^k$, $\lambda_j^k(K) = \lambda_{k+1}^{k+1}((\Upsilon^0)^{-1}(K))$).

Proof: Let $1 \leq j \leq k$ and $f \in H^k$. Remark 3 yields for every n

$$\psi_j^k(\nabla f) = n\psi_{k+1}^{k+1}(\nabla f_j^{1/n}). \tag{17}$$

Since $f_j^{1/n} \in H^{k+1}$, this is equal to

$$n \int \frac{df_j^{1/n}}{dx_{k+1}}(x_1, \ldots, x_{k+1})\, d\lambda_{k+1}^{k+1}(x_1, \ldots, x_{k+1}) \tag{18}$$

$$= \int \frac{df}{dx_j}(\Gamma_j^{1/n}(x_1, \ldots, x_{k+1}))\, d\lambda_{k+1}^{k+1}(x_1, \ldots, x_{k+1}). \tag{19}$$

By the bounded convergence theorem, as $n \to \infty$ the equality

$$\psi_j^k(\nabla f) = \int \frac{df}{dx_j}(\Gamma_j^{1/n}(x_1, \ldots, x_{k+1}))\, d\lambda_{k+1}^{k+1}(x_1, \ldots, x_{k+1}) \tag{20}$$

turns into

$$\psi_j^k(\nabla f) = \int_{(x_1,\ldots,x_{k+1})\neq e_{k+1}} \frac{df}{dx_j}(x_1,\ldots,x_k)\, d\lambda_{k+1}^{k+1}(x_1,\ldots,x_{k+1}) + \frac{df}{dx_j}(e_j)\lambda_{k+1}^{k+1}(\{e_{k+1}\}), \tag{21}$$

where e_i stands for the i^{th} unit vector.

For a compact and strictly convex $C \subset \Delta^k$ it is known that the support function f_C of C, given by

$$f_C(x) = \max_{c\in C} x \cdot c, \tag{22}$$

is continuously differentiable on $R_+^k - D^k$ (D^k is the "diagonal" $\{t\mathbb{1}_k \mid t \geq 0\}$), with $\nabla f_C(x)$ being the (unique) point $C(x)$ in C at which the maximum in the definition of f_C is attained. Therefore

$$\sum_{j=1}^{k} \frac{df_C}{dx_j}(x) = \sum_{j=1}^{k} C(x)_j = 1 \tag{23}$$

for every $x \in R_+^k - D^k$.

The function f_C can be approximated pointwise on R_+^k by a sequence of homogeneous of degree 1 and convex functions, continuously differentiable on $R_+^k - \{0\}$. For a given compact and strictly convex $C \subset \Delta^k$ denote by f_C^n such an approximating sequence. By Theorem 24.5 of Rockafellar (1970), ∇f_C^n converges pointwise to ∇f_C on $R_+^k - D^k$.

By the cost sharing axiom,

$$\sum_{j=1}^{k} \psi_j^k(\nabla f) = f(\mathbb{1}_k) \tag{24}$$

for any $f \in H_c^k$. Now take the sum over $j = 1,\ldots,k$ of equalities (21) for $f = f_C^n$. By (24), we get

$$f_C^n(\mathbb{1}_k) = \int_{(x_1,\ldots,x_{k+1})\neq e_{k+1}} \left[\sum_{j=1}^{k} \frac{df_C^n}{dx_j}(x_1,\ldots,x_k)\right] d\lambda_{k+1}^{k+1}(x_1,\ldots,x_{k+1}) + \lambda_{k+1}^{k+1}(\{e_{k+1}\}) \sum_{j=1}^{k} \frac{df_C^n}{dx_j}(e_j). \tag{25}$$

When $(x_1,\ldots,x_k) \in D^k$,

$$\sum_{j=1}^{k} \frac{df_C^n}{dx_j}(x_1,\ldots,x_k) = f_C^n(\mathbb{1}_k), \tag{26}$$

which converges to $f_C(\mathbb{1}_k) = 1$ as $n \to \infty$. If $(x_1,\ldots,x_k) \notin D^k$, then by (23)

$$\lim_{n\to\infty}\sum_{j=1}^{k}\frac{df_C^n}{dx_j}(x_1,\ldots,x_k)=\sum_{j=1}^{k}\frac{df_C}{dx_j}(x_1,\ldots,x_k)=1. \tag{27}$$

Therefore, after taking the limit of both sides of (25) as $n\to\infty$, the bounded convergence theorem yields

$$1=\lambda_{k+1}^{k+1}(\{x\,|\,x\neq e_{k+1}\})+\lambda_{k+1}^{k+1}(\{e_{k+1}\})\sum_{j=1}^{k}C(e_j)_j. \tag{28}$$

One can easily find closed and strictly convex $C_1, C_2\subset\Delta^k$ such that $C_1(e_j)=C_2(e_j)$ for $j\neq 1$, and $C_1(e_1)$ differs from $C_2(e_1)$ in the first coordinate (C_2 can be obtained from C_1 by modifying C_1 in a small neighborhood of $C_1(e_1)$). If $\lambda_{k+1}^{k+1}(\{e_{k+1}\})>0$, this contradicts (28). Therefore $\lambda_{k+1}^{k+1}(\{e_{k+1}\})=0$. From (28)

$$\lambda_{k+1}^{k+1}(\{x\,|\,x\neq e_{k+1}\})=1, \tag{29}$$

and thus (21) is transformed into

$$\psi_j^k(\nabla f)=\int\frac{df}{dx_j}(x_1,\ldots,x_k)\,d\lambda_{k+1}^{k+1}(x_1,\ldots,x_{k+1}), \tag{30}$$

and $(x_1,\ldots,x_k)\neq 0$ λ_{k+1}^{k+1}-almost everywhere. Thus it can be rewritten as

$$\psi_j^k(\nabla f)=\int\frac{df}{dx_j}(x)\,d\Upsilon^0\circ\lambda_{k+1}^{k+1}(x). \tag{31}$$

Since it holds for all $f\in H^k$,

$$\lambda_j^k=\Upsilon^0\circ\lambda_{k+1}^{k+1} \tag{32}$$

by Corollary 9. ■

Corollary 11. *The measure λ_j^k is independent of j.*

The measure λ_j^k will now be denoted by λ^k.
For every k consider the function $f_k: R_+^k\to R$, given by

$$f_k(x_1,\ldots,x_k)=\frac{\sum_{j=1}^k x_j^2}{\sum_{j=1}^k x_j}$$

for $(x_1,\ldots,x_k)\neq 0$, and $f_k(0)=0$. It is easy to see that $f_k\in H^k$ (the function is clearly convex on Δ^k). Since

$$\frac{df_k}{dx_i}(x)=\frac{2x_i\sum_{j=1}^k x_j-\sum_{j=1}^k x_j^2}{\left(\sum_{j=1}^k x_j\right)^2},$$

for every $x \in \Delta^k$

$$\sum_{j=1}^{k} \frac{df_k}{dx_j}(x) = 2 - k \sum_{j=1}^{k} x_j^2,$$

and so the maximum of $\sum_{j=1}^{k} \frac{df_k}{dx_j}(x)$ on Δ^k is attained at $x = \frac{1}{k}\mathbf{1}_k$. On the other hand, by the cost sharing axiom and (13)

$$\int \left[\sum_{j=1}^{k} \frac{df_k}{dx_j}(x) \right] d\lambda^k(x) = \sum_{j=1}^{k} \psi_j^k(\nabla f_k) = f_k(\mathbf{1}_k) = \sum_{j=1}^{k} \frac{df_k}{dx_j}\left(\frac{1}{k}\mathbf{1}_k\right),$$

which implies that the measure λ^k is supported on $\left\{\frac{1}{k}\mathbf{1}_k\right\}$.

By the definition of λ^k and ψ_j^k,

$$\psi(f, \mathbf{1}_k) = \nabla f(\mathbf{1}_k) = \varphi^m(f, \mathbf{1}_k) \tag{33}$$

for every $f \in H_c^k$. By rescaling invariance of both ψ and φ^m,

$$\psi(f, a) = \varphi^m(f, a) \tag{34}$$

for every $k \geq 2$, $f \in H_c^k$ and $a \in R_{++}^k$. Since (34) also holds for $k = 1$, by the cost sharing axiom satisfied by both ψ and φ^m, the two rules coincide on all cost problems with cost functions in H_c. This completes our proof.

Remark 4: Our proof can be used to characterize the marginal cost price rule on any class F of homogeneous of degree 1 and continuously differentiable cost functions, provided each F^k is a cone that contains H_c^k. In particular, we can prove Theorem 7 if the domain of price rules is the class of all homogeneous of degree 1 and continuously differentiable cost functions. Indeed, measures λ_j^k can be defined just as in the above proof. Then, confining the attention only to functions in H^k (which are contained in the linear space spanned by F^k), verbatim repetition of the above proof determines λ_j^k as the Dirac measure on $\frac{1}{k}\mathbf{1}_k$ for each k.

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