SIMPLE RATIO PROPHET INEQUALITIES
FOR A MORTAL WITH MULTIPLE CHOICES

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Abstract

Let $X_i \geq 0$ be independent, $i = 1, \ldots, n$, with known distributions and let $X_n^* = \max(X_1, \ldots, X_n)$. The classical 'ratio prophet inequality' compares the return to a prophet, which is $\mathrm{E}X_n^*$, to that of a mortal, who observes the X_is sequentially, and must resort to a stopping rule t. The mortal's return is $V(X_1, \ldots, X_n) = \max \mathrm{E}X_t$, where the maximum is over all stopping rules. The classical inequality states that $\mathrm{E}X_n^* < 2V(X_1, \ldots, X_n)$. In the present paper the mortal is given $k \geq 1$ chances to choose. If he uses stopping rules $t_1, \ldots, t_k$ his return is $\mathrm{E}(\max(X_{t_1}, \ldots, X_{t_k}))$. Let $t(b)$ be the 'simple threshold stopping rule' defined to be the smallest i for which $X_i \geq b$, or n if there is no such i. We show that there always exists a proper choice of k thresholds, such that $\mathrm{E}X_n^* \leq ((k+1)/k)\mathrm{E}(\max(X_{t_1}, \ldots, X_{t_k}))$, where t_i is of the form $t(b_i)$ with some added randomization. Actually the thresholds can be taken to be the $j/(k+1)$ percentile points of the distribution of X_n^*, $j = 1, \ldots, k$, and hence only knowledge of the distribution of X_n^* is needed.

Keywords: Optimal stopping rules; ratio prophet inequality; simple threshold rule; multiple stopping options

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1. Introduction

'Prophet inequalities' compare, for any sequence of random variables $(X_1, \ldots, X_n)$ belonging to a given class, the value that a 'prophet' can achieve from the random variables with the value an ordinary mortal, henceforth termed 'statistician', presented with the X_is sequentially, can obtain from the same. Some details and a summary of known results are given below.

Throughout the present note the class of random variables to be considered is the class of independent, non-negative random variables with finite expectations, where the sequence is not identically zero. Denote this class by $\mathcal{C}$. It is assumed until we state otherwise that the distributions of the Xs are known. The objective, shared by the statistician and the prophet alike, is to select as large an X value as possible. The statistician is restricted to the use of a stopping rule. When faced with X_i (if he has not stopped before time i), his decision to stop or continue can only depend on what he has seen up to and including time i, and possibly on some external randomization. No recall is allowed. He must stop by time n. Let T_n denote the set of all stopping rules for the sequence $(X_1, \ldots, X_n) \in \mathcal{C}$. The optimal value to the statistician is

defined by

$$V^n(X_1, \ldots, X_n) = \sup_{t \in T_n} \mathrm{E} X_t.$$

In contrast to the statistician, the prophet has complete foresight of the entire sequence (or complete recall) and can thus pick the largest X-value. His return will therefore be

$$\mathrm{E}(\max(X_1, \ldots, X_n)) = \mathrm{E} X_n^*,$$

where we let $X_n^* = \max(X_1, \ldots, X_n) = X_1 \vee \cdots \vee X_n$. The first, and by now classical, (ratio) prophet inequality states that for any $n \geq 2$ and any $(X_1, \ldots, X_n)$ in $\mathcal{C}$

$$\mathrm{E}(X_n^*) < 2V^n(X_1, \ldots, X_n) \tag{1.1}$$

and the constant 2 in (1.1) cannot be improved upon. See for example Krengel and Sucheston (1978) and Hill and Kertz (1981). A similar result with weak inequality holds for an infinite horizon, where it is assumed that $\mathrm{E}(\sup X_i) < \infty$.

In the present note we consider the situation where the statistician is given $k \geq 2$ chances to choose and gets the *largest* of the k X-values he has chosen. We consider only $n > k$, since otherwise the problem is trivial. In this setting the value to the statistician is therefore

$$V_k^n(X_1, \ldots, X_n) = \sup_{1 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq n} \mathrm{E}(X_{t_1} \vee \cdots \vee X_{t_k}), \tag{1.2}$$

where the choice of X_{t_j} should naturally take the values of $X_{t_1}, \ldots, X_{t_{j-1}}$ into account. In principle, $V_k^n(X_1, \ldots, X_n)$ can be computed by a 'multiple backward induction' scheme, although its actual evaluation is a formidable task, even for the rather simple case of two choices and moderate n.

Stopping rules with multiple stopping options have been considered in the literature, mainly in connection with 'secretary problems'. In a general setup, and in relation to prophet inequalities, multiple stopping options are discussed in Kennedy (1987). The objective considered there is the maximization of the sum $\mathrm{E}(X_{t_1} + \cdots + X_{t_k})$, where the statistician is given k choices and the maximization is over all stopping rules $t_1 < \cdots < t_k$. Our objective is suitable for a situation where the 'statistician' wants to buy a house out of n available prospective houses. He inspects one house every day, and can put at most k houses 'on hold'. He then selects the better of the k houses that were reserved. Thus, the objective function of Kennedy (1987) is different from our (1.2), and so are the results.

Consider the class of 'simple threshold rules' defined for any $b > 0$ as follows:

$$t(b) = \inf\{i : X_i \geq b\} \wedge n.$$

Thus $t(b)$ picks the first X to equal or exceed the value b (if such an X exists). (The term 'simple' was introduced to distinguish this rule from threshold rules of the form $t = \inf\{i : X_i \geq a_i\} \wedge n$ for a given sequence of constants $(a_1, \ldots, a_n)$.) Samuel-Cahn (1984) shows that, if the statistician resorts to simple threshold rules only, a statement similar to (1.1) still holds, i.e. for any $(X_1, \ldots, X_n)$ in $\mathcal{C}$

$$\mathrm{E} X_n^* \leq 2 \sup_{b > 0} \mathrm{E} X_{t(b)} \tag{1.3}$$

and clearly here 2 is a best constant. In order to achieve (1.3) several choices of b are explicitly exhibited in Samuel-Cahn (1984). One possible choice is to let b be a median, m, of the distribution of X_n^*, and use $t(m)$ or $\tilde{t}(m)$, where

$$\tilde{t}(b) = \inf\{i : X_i > b\} \wedge n.$$

It is shown there that

$$EX_n^* \leq 2 \max\{EX_{t(m)}, EX_{\tilde{t}(m)}\}.$$

In the present paper we generalize this result: we show that with k choices and all $n > k$, there exist thresholds $b_k \leq b_{k-1} \leq \cdots \leq b_1$ such that

$$EX_n^* \leq \frac{k+1}{k} E[X_{t^*(b_k)} \vee \cdots \vee X_{t^*(b_1)}],$$

where $t^*(b)$ is similar to $t(b)$ with possibly some added randomization when $X = b$. Actually we show more. If the exact $1/(k+1), 2/(k+1), \ldots, k/(k+1)$ percentile points of X_n^* exist, and are denoted by $q_k, \ldots, q_1$, so that $P(X_n^* \geq q_i) = i/(k+1)$, then

$$EX_n^* \leq \frac{k+1}{k} E[X_{t(q_k)} \vee \cdots \vee X_{t(q_1)}]. \tag{1.4}$$

In the general setting, a result similar to (1.4) holds with some added randomization. Thus, with just two choices, even using a good simple suboptimal rule, the prophet cannot receive more than 1.5 times what the statistician can guarantee with such a rule, and with four choices such a statistician is guaranteed at least 80% of the prophet's value, no matter how large n.

For $k = 1$ the constant $2 = (k+1)/k$ is the best possible, as seen from (1.1). For $k \geq 2$ we have good reasons to believe that $(k+1)/k$ is greater than the 'best' constant. The usefulness of our result is that (i) the proofs are very simple, and that (ii) very simple, explicit rules are given which achieve the inequality. Note also that, in order to apply the above rule, knowledge of the distribution of X_n^* only is needed, and not full knowledge of the distribution of the individual X_is. This has been termed 'partial information' by Wittman (1995).

In Section 2 we define randomized threshold rules and prove some Lemmas, which will be used in Section 3, where we state and prove the main result.

2. Preliminaries

Let U be a uniformly $U(0, 1)$ distributed random variable, independent of the Xs. For a fixed threshold $b > 0$, define, for $0 \leq \alpha \leq 1$,

$$t_\alpha(b) = \inf\{i : X_i > b \text{ or } (X_i = b, U \leq \alpha)\} \wedge n. \tag{2.1}$$

Clearly, if $P(X_n^* = b) = 0$ then (almost surely) $t_\alpha(b) = t(b) = \tilde{t}(b)$ for all α. Otherwise $t(b) = t_1(b)$ and $\tilde{t}(b) = t_0(b)$, while generally, for $0 < \alpha < 1, t_\alpha(b)$ requires 'external randomization'.

Lemma 2.1. *Let* $A = \{(X_n^* > b) \cup (X_n^* = b, U \leq \alpha)\}$ *and let* $\bar{A}$ *be the complementary event. Then*

$$EX_{t_\alpha(b)} \geq E[X_{t_\alpha(b)}\mathbf{1}(A)] \geq bP(A) + E(X_n^* - b)^+ P(\bar{A}). \tag{2.2}$$

Proof. The first inequality in (2.2) follows since the X_is are non-negative. Now

$$E[X_{t_\alpha(b)}\mathbf{1}(A)] = bP(A) + \sum_{i=1}^n E[(X_i - b)^+ \mathbf{1}(t_\alpha(b) = i)]. \tag{2.3}$$

To complete the proof it suffices to show that the second term in the right-hand side of (2.3) exceeds or equals the corresponding term in (2.2). But

$$
\begin{aligned}
\sum_{i=1}^{n} \mathrm{E}[(X_i - b)^+ \mathbf{1}(t_\alpha(b) = i)] &= \sum_{i=1}^{n} \mathrm{E}[(X_i - b)^+ \mathbf{1}(t_\alpha(b) > i - 1)] \\
&= \sum_{i=1}^{n} \mathrm{E}(X_i - b)^+ \mathrm{P}(t_\alpha(b) > i - 1) \\
&\geq \mathrm{P}(\bar{A}) \sum_{i=1}^{n} \mathrm{E}(X_i - b)^+ \\
&\geq \mathrm{P}(\bar{A}) \mathrm{E}(X_n^* - b)^+.
\end{aligned}
\tag{2.4}
$$

The first equality in (2.4) follows since $(X_i - b)^+ \mathbf{1}(t_\alpha(b) = i) = (X_i - b)^+ \mathbf{1}(t_\alpha(b) > i - 1)$ for all i, the second equality follows from independence, and the first inequality follows since, for each i, one has $\mathrm{P}(t_\alpha(b) > i - 1) \geq \mathrm{P}(\bar{A})$.

Corollary 2.1. *Let b^* satisfy $\mathrm{P}(X_n^* < b^*) \leq \frac{1}{2}$ and $\mathrm{P}(X_n^* > b^*) \leq \frac{1}{2}$ and choose α so that $\mathrm{P}\{X_n^* > b^*\} + \alpha \mathrm{P}(X_n^* = b^*) = \frac{1}{2}$. Then $\mathrm{E}X_{t_{\alpha(b^*)}} \geq \mathrm{E}X_n^*/2$.*

Proof. By (2.2), $\mathrm{E}X_{t_\alpha(b^*)} \geq [b^* + \mathrm{E}(X_n^* - b^*)^+]/2$. Since for any b we have $b + (X_n^* - b)^+ \geq X_n^*$, the result follows by taking expectations.

We now generalize Lemma 2.1 to k simple threshold rules. Consider $b_k \leq \cdots \leq b_1$ (not necessarily all distinct), and corresponding uniformly $U(0, 1)$ distributed random variables $U_1, \ldots, U_k$. To avoid unnecessary complications, we assume that all b_js belong to the support of X_n^*. When $b_j = b_s$ we take $U_j \equiv U_s$, but for distinct b_js we assume that the U_js are independent. Similar notation and assumptions will be used throughout.

Lemma 2.2. *Let $b_k \leq \cdots \leq b_1$, $0 \leq \alpha(j) \leq 1$ and, if $b_j = b_s$ with $j < s$, assume $\alpha(j) \leq \alpha(s)$. Let $t_{\alpha(j)}(b_j)$ be defined through (2.1) with $U = U_j$ and $\alpha = \alpha(j)$. Let $A_j = \{(X_n^* > b_j) \cup (X_n^* = b_j \text{ and } U_j \leq \alpha(j))\}$ and $A_0 = \varnothing$. Then*

$$
\mathrm{E}\{[X_{t_{\alpha(k)}(b_k)} \vee \cdots \vee X_{t_{\alpha(1)}(b_1)}]\mathbf{1}(A_k)\}
$$
$$
\geq \sum_{j=1}^{k} b_j \mathrm{P}(A_j \cap \bar{A}_{j-1}) + \sum_{j=1}^{k} \mathrm{E}[(X_n^* - b_j)^+ \mid \bar{A}_{j-1}]\mathrm{P}(\bar{A}_j). \tag{2.5}
$$

Proof. Note that $A_k \supset A_{k-1} \supset \cdots \supset A_1$. We prove (2.5) by induction on k. For $k = 1$ this is Lemma 2.1. Suppose now that (2.5) holds for $k - 1$. Then

$$
\begin{aligned}
[X_{t_{\alpha(k)}(b_k)} \vee \cdots \vee X_{t_{\alpha(1)}(b_1)}]\mathbf{1}(A_k) \geq &X_{t_{\alpha(k)}(b_k)}\mathbf{1}(A_k \cap \bar{A}_{k-1}) \\
&+ [X_{t_{\alpha(k-1)}(b_{k-1})} \vee \cdots \vee X_{t_{\alpha(1)}(b_1)}]\mathbf{1}(A_{k-1}).
\end{aligned}
$$

By the induction hypothesis it suffices to prove that

$$
\mathrm{E}[X_{t_{\alpha(k)}(b_k)}\mathbf{1}(A_k \cap \bar{A}_{k-1})] \geq b_k \mathrm{P}(A_k \cap \bar{A}_{k-1}) + \mathrm{E}[(X_n^* - b_k)^+ \mid \bar{A}_{k-1}]\mathrm{P}(\bar{A}_k). \tag{2.6}
$$

Now,

$$E[X_{t_{\alpha(k)}(b_k)}\mathbf{1}(A_k \cap \bar{A}_{k-1})] = b_k P(A_k \cap \bar{A}_{k-1})$$
$$+ E\sum_{i=1}^{n}\{[X_i - b_k]^+ \mathbf{1}(t_{\alpha(k)}(b_k) = i)\mathbf{1}(\bar{A}_{k-1})\}. \qquad (2.7)$$

The right most term in (2.7) can also be written as

$$P(\bar{A}_{k-1})\sum_{i=1}^{n} E[(X_i - b_k)^+ \mathbf{1}(t_{\alpha(k)}(b_k) = i) \mid \bar{A}_{k-1}]. \qquad (2.8)$$

Now, by replacing expectation by conditional expectation in (2.4) (since the random variables X_i and $\mathbf{1}(t_{\alpha(k)}(b_k) > i - 1)$ are conditionally independent given $\bar{A}_{k-1}$), the evaluation of the sum of the conditional expectations yields that the value of (2.8) is greater than or equal to

$$P(\bar{A}_{k-1})P(\bar{A}_k \mid \bar{A}_{k-1})E[(X_n^* - b_k)^+ \mid \bar{A}_{k-1}] = P(\bar{A}_k)E[(X_n^* - b_k)^+ \mid \bar{A}_{k-1}],$$

which is the value in the last term of (2.6). This, together with (2.8) and (2.7), yields (2.6).

Remark 2.1. Note that use of $k \geq 2$ simple threshold rules will usually be far from optimal, for two (related) reasons: (i) In an optimal rule one should never stop with a value not exceeding the previously obtained (unless forced to stop at time n). Simple threshold rules do not adhere to this principle. (ii) When using $k \geq 2$ simple threshold rules one may pass up several stopping options altogether. This happens when for the smallest i for which $X_i \geq b_j$ the same X_i is also greater than one or more b_ss, with $b_s \geq b_j$.

3. Main result

Let b_j and $\alpha(j)$ be such that, for $j = 1, \ldots, k$,

$$P(X_n^* < b_j) \leq \frac{k - j + 1}{k + 1}, \qquad P(X_n^* > b_j) \leq \frac{j}{k + 1},$$
$$P(X_n^* > b_j) + \alpha(j)P(X_n^* = b_j) = \frac{j}{k + 1}. \qquad (3.1)$$

We call the constants thus determined the 'exact $j/(k + 1)$ percentiles' of X_n^*.

Theorem 3.1. *Let b_j and $\alpha(j)$ satisfy (3.1). Then for all $n > k$*

$$E[X_{t_{\alpha(1)}(b_1)} \vee \cdots \vee X_{t_{\alpha(k)}(b_k)}] \geq \frac{k}{k + 1}EX_n^*. \qquad (3.2)$$

Proof. Let Z be any non-negative random variable, with finite positive expectation. Let $b_k \leq \cdots \leq b_1$ belong to the support of Z, and $0 \leq \alpha(j) \leq 1$. Let $A_0 = \varnothing$ and $A_j = \{(Z > b_j) \cup (Z = b_j, U_j \leq \alpha(j))\}$, where for $b_j = b_s$ with $j < s$ we assume that $\alpha(j) \leq \alpha(s)$ and $U_j \equiv U_s$. Consider

$$\sum_{j=1}^{k} b_j P(A_j \cap \bar{A}_{j-1}) + \sum_{j=1}^{k} E[(Z - b_j)^+ \mid \bar{A}_{j-1}]P(\bar{A}_j). \qquad (3.3)$$

For $Z = X_n^*$ the value of (3.3) equals the right-hand side of (2.5). We show, by induction, that if one chooses $b_k, \ldots, b_1$ and $\alpha(k), \ldots, \alpha(1)$ to be the exact $j/(k+1)$ percentiles of Z, then the value of (3.3) is greater than or equal to $(k/(k+1))EZ$. The theorem clearly follows from this and Lemma 2.2.

For $k = 1$ the proof is similar to that of Corollary 2.1. The value of (3.3) is simply

$$b_1 P(A_1) + E(Z - b_1)^+ P(\bar{A}_1) = \tfrac{1}{2}(b_1 + E(Z - b_1)^+) \geq \tfrac{1}{2} EZ,$$

where we use $P(A_1) = \tfrac{1}{2}$.

Now suppose the assertion is true for any Z and $k - 1$. Let $c > 0$ be an arbitrary threshold, let $0 \leq \alpha \leq 1$ and let

$$A = \{(Z > c) \cup (Z = c \text{ and } U \leq \alpha)\}, \tag{3.4}$$

where we assume that $P(\bar{A}) > 0$. Consider a new sample space, which is the previous sample space conditional on $\bar{A}$, and let $\hat{Z}$ be a random variable distributed like Z conditionally on $\bar{A}$. Denote the probability measure on this sample space by $\hat{P}$, that is, for any event B define $\hat{P}(B) = P(B \cap \bar{A})/P(\bar{A})$. The expectation $\hat{E}$ is defined similarly.

Now consider the $k - 1$ exact j/k percentiles of $\hat{Z}$, and denote the corresponding thresholds and α-values by $\hat{b}_{k-1}, \ldots, \hat{b}_1$ and $\hat{\alpha}(k-1), \ldots, \hat{\alpha}(1)$. Let

$$B_j = \{(\hat{Z} > \hat{b}_j) \cup (\hat{Z} = \hat{b}_j, \hat{U}_j \leq \hat{\alpha}(j))\}, \qquad j = 1, \ldots, k-1$$

and $B_0 = \varnothing$, where the definition of $\hat{U}_j$ is self-explanatory, and the $\hat{U}_j$s are independent of U in (3.4). Then the induction hypothesis about $\hat{Z}$ implies

$$\sum_{j=1}^{k-1} \hat{b}_j \hat{P}(B_j \cap \bar{B}_{j-1}) + \sum_{j=1}^{k-1} \hat{E}[(\hat{Z} - \hat{b}_j)^+ \mid \bar{B}_{j-1}] \hat{P}(\bar{B}_j) \geq \frac{k-1}{k} \hat{E}(\hat{Z})$$

$$= \frac{k-1}{k} E(Z \mid \bar{A}). \tag{3.5}$$

Let $b_{j+1} = \hat{b}_j$, $U_{j+1} = \hat{U}_j$ and $\alpha(j+1) = \hat{\alpha}(j)$ for $j = 1, \ldots, k-1$, and define

$$A_j = \{(Z > b_j) \cup (Z = b_j, U_j \leq \alpha(j))\}, \qquad j = 2, \ldots, k.$$

In terms of A_j and b_j the left-hand side of (3.5) can be rewritten as

$$b_2 \frac{P(A_2 \cap \bar{A})}{P(\bar{A})} + \sum_{j=3}^{k} b_j \frac{P(A_j \cap \bar{A}_{j-1} \cap \bar{A})}{P(\bar{A})} + E[(Z - b_2)^+ \mid \bar{A}] \frac{P(\bar{A}_2 \cap \bar{A})}{P(\bar{A})}$$

$$+ \sum_{j=3}^{k} E[(Z - b_j)^+ \mid \bar{A}_{j-1} \cap \bar{A}] \frac{P(\bar{A}_j \cap \bar{A})}{P(\bar{A})}. \tag{3.6}$$

Multiplying (3.6) by $P(\bar{A})$ we can rewrite (3.5) as

$$b_2 P(A_2 \cap \bar{A}) + \sum_{j=3}^{k} b_j P(A_j \cap \bar{A}_{j-1} \cap \bar{A})$$

$$+ E[(Z - b_2)^+ \mid \bar{A}] P(\bar{A}_2 \cap \bar{A})$$

$$+ \sum_{j=3}^{k} E[(Z - b_j)^+ \mid \bar{A}_{j-1} \cap \bar{A}] P(\bar{A}_j \cap \bar{A}) \geq \frac{k-1}{k} E(Z \mid \bar{A}) P(\bar{A}). \tag{3.7}$$

Now consider Z and the k thresholds $b_1 = c$ and $b_2, \ldots, b_k$ with corresponding α and $\alpha(2), \ldots, \alpha(k)$. Note that $b_1 \geq b_2 \geq \cdots \geq b_k$. If $b_1 > b_2$, then $\bar{A}_j \cap \bar{A} = \bar{A}_j$ for $j = 2, \ldots, k$ and, with $\bar{A}_1 = \bar{A}$, (3.7) can be written as

$$\sum_{j=2}^{k} b_j \mathrm{P}(A_j \cap \bar{A}_{j-1}) + \sum_{j=2}^{k} \mathrm{E}[(Z - b_j)^+ \mid \bar{A}_{j-1}] \mathrm{P}(\bar{A}_j) \geq \frac{k-1}{k} \mathrm{E}(Z \mid \bar{A}) \mathrm{P}(\bar{A}). \qquad (3.8)$$

If $c = b_1 = b_2 = \cdots = b_s$ for some $s \geq 2$, then, for $j = 2, \ldots, s$,

$$\bar{A}_j \cap \bar{A} = \{(Z < c) \cup (Z = c, U_j > \alpha(j), \ U > \alpha)\}.$$

Define $\alpha^*(j)$ by $\alpha^*(1) = \alpha$ and $1 - \alpha^*(j) = (1 - \alpha(j))(1 - \alpha)$, $j = 2, \ldots, s$. Notice that $\alpha^*(1) \leq \alpha^*(2) \leq \cdots \leq \alpha^*(s)$. Let

$$A_j^* = \begin{cases} \{(Z > c) \cup (Z = c, U \leq \alpha^*(j))\}, & j = 1, \ldots, s, \\ A_j & j = s+1, \ldots, k. \end{cases} \qquad (3.9)$$

Then $A_1^* = A$ and (3.7) is equivalent to

$$\sum_{j=2}^{k} b_j \mathrm{P}(A_j^* \cap \bar{A}_{j-1}^*) + \sum_{j=2}^{k} \mathrm{E}[(Z - b_j)^+ \mid \bar{A}_{j-1}^*] \mathrm{P}(\bar{A}_j^*) \geq \frac{k-1}{k} \mathrm{E}(Z \mid \bar{A}) \mathrm{P}(\bar{A}), \qquad (3.10)$$

which coincides with (3.8) if we let $A_j = A_j^*$ there. Now consider (3.3), with A_j there replaced by A_j^*, for the thresholds and α-values as defined in (3.9). The first terms ($j = 1$) of both sums in (3.3) yield

$$c \mathrm{P}(A) + \mathrm{E}(Z - c)^+ \mathrm{P}(\bar{A}) \qquad (3.11)$$

and the remaining terms are just the left-hand side of (3.10).

Let $p = \mathrm{P}(\bar{A})$. Then (3.11) can be written as

$$c(1 - p) + \mathrm{E}(Z \mid A)(1 - p)p - c(1 - p)p = \mathrm{E}(Z \mid A) \mathrm{P}(A) p + c(1 - p)^2. \qquad (3.12)$$

Note that $c \geq \mathrm{E}(Z \mid \bar{A})$. Thus the sum of the right-hand sides of (3.10) and (3.12) is greater than or equal to

$$\left(\frac{k-1}{k} + \frac{(1-p)^2}{p} \right) \mathrm{E}(Z \mid \bar{A}) \mathrm{P}(\bar{A}) + p \mathrm{E}(Z \mid A) \mathrm{P}(A). \qquad (3.13)$$

If in (3.13) we solve for p the equation $((k-1)/k + (1-p)^2/p) = p$, and denote the solution by p^*, then the value of (3.13) will be exactly $p^* \mathrm{E} Z$. Now the solution is $p^* = k/(k+1)$, which shows that with this choice of thresholds the value of (3.3) is greater than or equal to $(k/(k+1)) \mathrm{E} Z$. Also (c, α) is the exact $k/(k+1)$ percentile of Z. To see that the exact j/k percentiles for $\hat{Z}$, $j = 1, \ldots, k-1$ are actually the exact $j/(k+1)$ percentiles for Z, note that $(j/k) \mathrm{P}(\bar{A}) = (j/k)(k/(k+1)) = j/(k+1)$.

Remark 3.1. Let $k = 2$. When using the $\frac{1}{3}$ and $\frac{2}{3}$ percentile points, the value $\frac{2}{3} \mathrm{E} X_n^*$ on the right-hand side of (3.2) cannot be improved upon, as seen by the following simple example, where $n = 3$. Let

$$X_1 \equiv a, \qquad X_2 = \begin{cases} b & \text{w.p. } \frac{1}{2}, \\ 0 & \text{w.p. } \frac{1}{2}, \end{cases} \qquad X_3 = \begin{cases} c & \text{w.p. } \frac{1}{3}, \\ 0 & \text{w.p. } \frac{2}{3}, \end{cases}$$

where $a < b < c$. Then X_3^* takes values a, b, c each with probability $\frac{1}{3}$. The rule with the two thresholds b and c has value $(b+c)/3$, thus

$$\frac{\mathrm{E}[X_{t(b)} \vee X_{t(c)}]}{\mathrm{E}X^*} = \frac{b+c}{a+b+c}$$

which tends to $\frac{2}{3}$ as $a, b \to c$. Note also that the inequality (2.5) becomes an equality for any $a < b < c$ in this example. Similar examples can be constructed for any $k > 1$.

Remark 3.2. For $k = 2$ the following example shows that $\mathrm{E}X_n^* < \gamma \max_{t_1,t_2} \mathrm{E}(X_{t_1} \vee X_{t_2})$ cannot hold for $\gamma < \frac{5}{4}$, whether one considers threshold rules or all stopping rules. Let $n = 3$ and for $0 < \beta < \frac{1}{2}$ let

$$X_1 \equiv \beta, \qquad X_2 = \begin{cases} 2\beta & \text{w.p. } \frac{1}{2}, \\ 0 & \text{w.p. } \frac{1}{2}, \end{cases} \qquad X_3 = \begin{cases} 1 & \text{w.p. } \beta, \\ 0 & \text{w.p. } 1 - \beta. \end{cases}$$

Here $\mathrm{E}X_3^* = \beta(5 - 3\beta)/2$. For this example there are only three pairs of thresholds to be considered: (i) $(\beta, 1)$; (ii) $(\beta, 2\beta)$; (iii) $(2\beta, 1)$. The corresponding values for the statistician are (i) $\beta(2-\beta)$, (ii) $\beta(2-\beta/2)$ and (iii) $\beta(2-\beta)$. No gain can be achieved through randomization, and the pair $(\beta, 2\beta)$ is optimal. The corresponding rule is optimal also among all two-choice rules. The ratio between the value to the prophet and to the statistician is therefore $(5 - 3\beta)/(4 - \beta)$. Letting β tend to 0 yields the value $\frac{5}{4}$.

The next result is an immediate consequence of Theorem 3.1.

Corollary 3.1. *For any $k = 1, 2, \ldots$ and any $(X_1, \ldots, X_n) \in \mathcal{C}$*

$$\mathrm{E}X_n^* \le \frac{k+1}{k} \sup_{t_1,\ldots,t_k \in \tilde{T}_n} \mathrm{E}[X_{t_1} \vee \cdots \vee X_{t_k}].$$

For any infinite sequence $(X_1, X_2, \ldots)$ of non-negative independent random variables with $\mathrm{E}(\sup X_i) < \infty$

$$\mathrm{E}(\sup X_i) \le \frac{k+1}{k} \lim_{n \to \infty} \sup_{t_1,\ldots,t_k \in \tilde{T}_n} \mathrm{E}[X_{t_1} \vee \cdots \vee X_{t_k}],$$

where $\tilde{T}_n$ denotes the set of all threshold rules (including randomization).

Remark 3.3. In a similar manner to the proof in Rinott and Samuel-Cahn (1987), it can be shown that Theorem 3.1 and Corollary 3.1 hold also if, instead of non-negative independent X_is, we assume only that the X_is are non-negative and negatively lower orthant dependent in sequence, which includes, for example, negatively associated random variables.

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