Payoff Equivalence in Sealed Bid Auctions and the Dual Theory of Choice Under Risk [1]

Oscar Volij [2]

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Abstract

This paper analyzes symmetric, single item auctions in the private values framework, with buyers whose preferences satisfy the axioms of Yaari's (1987) dual theory of choice under risk. It is shown that when their valuations are independently distributed, risk averse buyers are indifferent among all the auctions contained in a big family of mechanisms that includes the standard auctions. It is also shown that in the linear equilibria of the sealed bid double auction, as the degree of players' risk aversion grows arbitrarily large, the ex post inefficiency of the mechanism tends to vanish. *Journal of Economic Literature* classification numbers: D44; D81.

Keywords: Auctions; non-expected utility; risk aversion.

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[2] Department of Economics, Brown University, Providence, RI 02912, and Hebrew University, Center for Rationality and Interactive Decision Theory and Department of Economics, 91905 Jerusalem, Israel. E-mail: `oscar@volij.co.il`

Introduction

Since Vickrey (1961)'s seminal paper, the literature on auctions has grown very rapidly. Most of this literature, though, deals with the case of risk neutral buyers (see, for example, Riley and Samuelson (1981), Myerson (1981) and Milgrom and Weber (1982) to name only a few very influential papers). Notable exceptions are Matthews (1983), Maskin and Riley (1984), Moore (1984), and Matthews (1987), among others, who take risk aversion into account. If results about auctions with risk averse buyers are scarce, the analysis of auctions with non-expected utility maximizers is almost nonexistent. Salo and Weber (1995) and Lo (1998) are among the few papers that take auction theory outside the expected utility paradigm.

The difficulty in the analysis of risk aversion in auctions stems from the nonlinearity of preferences in money. This may sound tautological because under the expected utility hypothesis, risk aversion is equivalent to the concavity of the von Neumann-Morgenstern utility function. But if one is willing to drop the assumption of expected utility maximization, there is no reason to identify risk aversion with decreasing marginal utility of money. Yaari (1987) proposed a theory of choice under risk which allows for the co-existence of both risk aversion and linearity of preferences in payments. Although the linearity of preferences in money is not the most appealing assumption, it can make the analysis of auctions with risk averse buyers as simple as in the case of risk neutral buyers. Also, unlike other non-expected utility theories, Yaari's dual theory of choice under risk is not a generalization of the expected utility theory. This means that the auction theory developed from it is no weaker than the standard one and thus one can expect very different and sometimes contradictory predictions from them.

One of the most celebrated results in the theory of auctions is the revenue equivalence theorem. It states, roughly, that in the single item, private i.i.d. values framework, there is a large family of auction mechanisms (which contains the standard auctions) that yield the same expected revenue. The other side of the revenue equivalence theorem is the fact that all bidders are indifferent among the auctions in that family. While it is well known that revenue equivalence breaks down as soon as risk aversion is allowed, Matthews (1987) shows, that when buyers share the same constant absolute risk aversion von Neumann-Morgenstern utility function, they are indifferent between the first price and second price auctions. The main result of this paper is that in the private i.i.d. values framework, when buyers' risk preferences follow the dual theory axioms, there is a large family of auctions (which contains the standard ones), among which buyers are indifferent. That is, although there is no revenue equivalence when buyers's preferences exhibit risk aversion, it is still true that they are indifferent between participating in any auction in the above family.

The paper also shows the effect of changes in the degree of risk aversion on the revenue maximizing reserve price of the high bid auction. It turns out, as in the case of expected utility preferences, that an increase in buyers' degree of risk aversion results in a decrease in the seller's optimal reserve price.

Preferences that satisfy the expected utility axioms differ from those that satisfy the dual theory axioms. Nevertheless, we find that in the first price auction the equilibrium bidding function when buyers' von Neumann-Morgenstern utility function is given by $u(x) = x^{1/m}$, for $m > 1$, is identical to the equilibrium bidding function when buyers' Yaari "probability-evaluation" function is the inverse of u. This result does not generalize to other utility functions and can be regarded as purely coincidental.

Lastly, we calculate the linear equilibrium of the sealed-bid double auction analyzed in Chatterjee and Samuelson (1983), when buyer and seller behave according to the dual theory and have probability-evaluation functions given by simple polynomials. In this case, as in the case of risk neutral buyers, the equilibria are ex post inefficient, since it is not true that there is trade if and only if the value for the buyer is at least as large as the value for the seller. It turns out, however, that the inefficiency vanishes as the degree of the traders' risk aversion becomes arbitrarily large. Also, it is shown that an increase in the buyer's degree of risk aversion uniformly increases the equilibrium terms of trade. Similarly, an increase in the seller's degree of risk aversion uniformly decreases the equilibrium terms of trade.

The paper is organized as follows. Section 1 presents a short review of Yaari's theory of choice under risk. After presenting the equilibrium bidding strategies in a few standard auctions, Section 2 gives the main result of the paper: the utility equivalence of many auction mechanisms. It also contains some results concerning the effect of risk aversion on the first price auction. Section 3 looks into the effect of risk aversion on the linear equilibrium of the sealed-bid double auction. Section 4 concludes.

1 A Short Review of the Dual Theory of Choice Under Risk

Given a random variable r, defined on some probability space Ω and taking values in some real interval $[m, 1]$, let G_r be its *decumulative distribution function* (DDF), which is defined by

$$G_r(x) = \Pr\{r > x\}, \qquad m \le x \le 1.$$

It is known that G_r is nonincreasing, right-continuous and satisfies $G_r(1) = 0$. The random variable r, represents a lottery over monetary outcomes or, if the reader prefers, an asset that pays the monetary amount $r(\omega)$ at state $\omega \in \Omega$.

The primitive of the dual theory is the set Γ of all nonincreasing, right-continuous functions $G : [m, 1] \to [0, 1]$ that satisfy $G(1) = 0$. This set is interpreted as the set of all DDFs associated with some random variable defined on some sufficiently rich probability space and taking values in $[m, 1]$. The area under a DDF in Γ resembles a production possibilities set and the problem of ranking DDFs is essentially the same as ranking production possibilities sets.

Let $\succeq$ be a complete preference relation on Γ. Yaari (1987) imposes the following axioms on

$\succeq$:

1. Continuity (with respect to L_1-convergence),
2. Monotonicity: if $G_r \geq G_s$ then $G_r \succeq G_s$,
3. Dual independence: r, s and t are pairwise comonotonic and $G_r \succeq G_s$, then $G_{\alpha r+(1-\alpha)t} \succeq G_{\alpha s+(1-\alpha)t}$.

Continuity is a technical requirement. Monotonicity requires that if G_r stochastically dominates G_s then $G_r \succeq G_s$. The dual independence axiom is where the dual theory departs from the traditional expected utility theory. It deals with portfolios of comonotonic random variables. Note that $\alpha r + (1-\alpha)t$ denotes the random variable that awards $\alpha r(\omega) + (1-\alpha)t(\omega)$ at state $\omega \in \Omega$. In particular, it is not the probability mixture of the random variables r and t. Two random variables, r and s, are comonotonic if for every pair of states, ω and ω', $(r(\omega) - r(\omega'))\,(s(\omega) - s(\omega')) \geq 0$. In words, r and s are comonotonic if, when going from state ω to ω' in Ω, both random variables move (weakly) in the same direction. Dual independence requires that whenever r, s and t are pairwise comonotonic and $G_r \succeq G_s$, then any portfolio containing a proportion α of r and $1-\alpha$ of t should be weakly preferred to a portfolio containing α of s and $1-\alpha$ of t.

Before we present Yaari's representation theorem we need the following notation. For any monetary outcome x and probability p, $[x;p]$ denotes the lottery that yields x with probability p and m with the complementary probability. With the aid of the above axioms Yaari (1987) shows the following:

Theorem 1 A complete preference relation $\succeq$ satisfies continuity, monotonicity and dual independence if, and only if, there exists a continuous and non decreasing real function, g, defined on the unit interval, such that for all G_r and G_s belonging to Γ,

$$G_r \succeq G_s \Leftrightarrow \int_m^1 g(G_r(t))\,dt \geq \int_m^1 g(G_s(t))\,dt.$$

Moreover, the function g, which is unique up to a positive affine transformation, can be selected in such a way that, for all $p \in [0,1]$, $g(p)$ solves the preference equation

$$[1;p] \sim [g(p);1]. \tag{1}$$

The function g is analogous to the von Neumann-Morgenstern utility function and, in a sense, we can say that g represents the agent's preferences. However, since it takes probabilities as an input, we can call it a "probability-evaluation" function instead of utility function.

Graphically, an agent whose preferences are described by the dual theory evaluates random variables according to the area under a suitable transformation (the function g) of their DDF. Analogously, an agent whose preferences satisfy the expected utility axioms, evaluates random variables according to the area "to the left" of a suitable transformation (the von Neumann-Morgenstern utility function) of the (generalized) inverse of their DDF.

For any random variable r, let

$$U(r) = m + \int_m^1 g(G_r(x))\, dx$$

where g is defined in equation 1. This representation ensures that the utility of \$$y$ with certainty is y, for all $y \in [m, 1]$. Theorem 1 says that agents will chose among random variables so as to maximize U. Further, Yaari (1987) also shows that for any random variable r, the agent is indifferent between r and getting $U(r)$ with certainty. In other words, $U(r)$ is the certainty equivalent of r. Another important property of he above representation is that the utility of the sum of two comonotonic random variables is equal to the sum of their respective utilities. Formally, if r and s are comonotonic random variables then $U(r+s) = U(r) + U(s)$.

One of the appealing features of the dual theory is that, unlike under the expected utility theory, the agent's attitude towards risk is not entangled with his attitude towards wealth. More specifically, under the dual theory, the marginal utility of wealth is constant and this feature is consistent with any attitude towards risk. Specifically, Yaari (1987) shows that a preference relation $\succeq$ that satisfies the dual theory's axioms exhibits risk aversion if and only if the function g that represents $\succeq$ is convex. The property of constant marginal utility of wealth is not a particularly appealing feature of individual preferences, but it is not especially unappealing when one wants to model firm behavior. One may want to think of a firm as being, on the one hand, risk averse, while on the other hand as evaluating each additional dollar independently of the level of wealth (or profits). This kind of behavior is precluded by the expected utility hypothesis. Also, while the linearity of preferences in money is and admittedly unrealistic assumption, within the dual theory it constitutes a relaxation of the straight-jacket imposed by the usual assumption of risk neutrality.

In this paper we are interested in the effect of changes in the degree of agents' risk aversion on equilibrium outcomes. For this purpose, it is necessary to understand what it means for one agent to be *more risk averse than* another. As Yaari (1986) says, since risk aversion is characterized by the convexity of the function g, it would be natural to define an agent as being more risk averse than the another if, and only if, the former's g is more convex than the latter's. Formally, we have the following:[1]

[1] See Yaari (1986) for various and equivalent interpretations of this definition.

Definition 1 Let $\succeq_1$ and $\succeq_2$ be two preference relations that satisfy the dual theory's axioms and that are represented by the functions g_1 and g_2, respectively. We say that $\succeq_1$ is more risk averse than $\succeq_2$ if, and only if, there exists a convex function h, defined on the unit interval, such that $g_1 = h \circ g_2$.

The linearity of preferences in monetary outcomes and the characterization of risk aversion as the convexity of the function g that represents them make the dual theory very appealing for the analysis of auctions. The linearity of the utility functional in wealth makes the analysis not too cumbersome and in some cases as simple as the case of risk neutral buyers. The characterization of the risk attitude by means of the convexity of a univariate nondecreasing function makes it relatively easy to analyze the effect of risk aversion on the outcome of auctions. These observations determine our task in the following sections.

2 Auctions with the Dual Theory

In order to motivate the main result of this paper, we start by calculating the equilibrium strategies of three standard auctions and the corresponding buyers' utilities.

There are n potential bidders, each of whose valuations for the object is drawn independently from a strictly increasing and twice continuously differentiable distribution function $F : [0,1] \to [0,1]$.

Bidders' preferences satisfy the dual theory's axioms and are represented by the dual function $g : [0,1] \to [0,1]$ which is normalized so that $g(0) = 0$ and $g(1) = 1$.

Second price auction

According to the second price auction with reserve price b_0, all participating bidders simultaneously bid a price $b \in [b_0, \infty)$ and the object is awarded to the bidder who bids the highest price. (In case of a tie, the object is randomly awarded to one of the bidders who made the highest bid, according to some fixed random rule.) The winner pays a price equal to the highest bid among the losers' bids or b_0, whichever is highest.

As bidders whose valuation of the object is less than the reserve price have no incentive to participate in the auction, consider bidder 1 with a valuation of $v \geq b_0$. Any bid b defines the random variable of bidder 1's earnings, which is denoted by $\tilde{e}_b$. It is well known that the random variable $\tilde{e}_v$ stochastically dominates $\tilde{e}_b$ for all $b \neq v$ and therefore, since the agents' preferences are monotonic with respect to stochastic dominance, we conclude that a symmetric equilibrium in this game dictates that all buyers with valuation $v \geq b_0$ should bid their true valuation.

Letting $\tilde{v}_i$ denote the random variable of buyer i's true valuation, in equilibrium the earnings of bidder 1, whose valuation for the object is $v \geq b_0$, are distributed as follows:

$$\begin{aligned}\Pr(\tilde{e}_v > e) &= \Pr([v - \max(b_0, \tilde{v}_2, \ldots, \tilde{v}_n)]_+ > e) \\ &= \begin{cases} 1 & \text{if } e < 0 \\ F^{n-1}(v-e) & \text{if } 0 \leq e < v - b_0 \\ 0 & \text{if } e \geq v - b_0. \end{cases}\end{aligned}$$

Since all the buyers are ex ante identical and the rules of the game are anonymous, the interim utility of a bidder with valuation v of participating in a second price auction is

$$\int_0^{v-b_0} g(F^{n-1}(v-e))de$$

which, by the change of variables $x = v - e$, is equal to

$$\int_{b_0}^{v} g(F^{n-1}(x))dx.$$

First price auction

According to this auction mechanism, bidders who wish to participate submit bids above the reservation price b_0, and the bidder with the highest bid gets the object for which he pays his own bid.

Again, we are interested in finding a symmetric equilibrium, where the common equilibrium bidding strategy, β, is invertible. Buyers who value the object below the reservation price have no incentive to participate. Therefore consider a bidder with valuation $v \geq b_0$, and assume that he bids $b \leq v$ (bidding more than the true valuation can never be optimal), while the other bidders play according to the equilibrium strategy β. We first need to calculate the decumulative distribution of his earnings, $\tilde{e}_b$, which is easily seen to be

$$\Pr(\tilde{e} > e) = \begin{cases} 1 & \text{if } e < 0 \\ F^{n-1}(\beta^{-1}(b)) & \text{if } 0 \leq e < v - b \\ 0 & \text{if } e \geq v - b. \end{cases}$$

Therefore the agent's utility when the other bidders behave according to β is

$$\begin{aligned}U(v,b) &= \textstyle\int_0^1 g(\Pr(\tilde{e} > e))\, de \\ &= (v-b)g(F^{n-1}(\beta^{-1}(b))).\end{aligned} \tag{2}$$

Using a standard argument we can now deduce that the symmetric equilibrium strategy of a bidder with valuation $v \geq b_0$ is

$$\beta(v) = v - \frac{\int_{b_0}^{v} g(F^{n-1}(x))\, dx}{g(F^{n-1}(v))}. \tag{3}$$

Substituting into equation (2) we get that the equilibrium interim utility of a bidder is

$$U(v, \beta(v)) = \int_{b_0}^{v} g(F^{n-1}(x))\, dx,$$

which is the same utility he gets from participating in the second price auction.

All pay auction

According to this auction mechanism, each bidder pays his own bid but only the bidder who bids the highest price gets the object.

Again, we are interested in finding a symmetric equilibrium, β. Consider a bidder with valuation $v > b_0$, and assume that he bids $b \leq v$ while the other bidders play according to the equilibrium strategy β. The decumulative distribution of his earnings, $\tilde{e}_b$ is given by

$$\Pr(\tilde{e}_b > e) = \begin{cases} 1 & \text{if } e < -b \\ F^{n-1}(\beta^{-1}(b)) & \text{if } -b \leq e < v - b \\ 0 & \text{if } e \geq v - b. \end{cases}$$

Therefore the utility for an agent with valuation v of a bid b when the other bidders behave according to β is

$$\begin{aligned} U(v, b) &= -b + \textstyle\int_{-b}^{v-b} g(F^{n-1}(\beta^{-1}(b)))\, dx \\ &= -b + v\, g(F^{n-1}(\beta^{-1}(b)). \end{aligned} \tag{4}$$

Using a standard argument we can deduce that the symmetric equilibrium strategies are given by

$$\beta(v) = vg(F^{n-1}(v)) - \int_{b_0}^{v} g(F^{n-1}(x))dx.$$

Substituting into equation (4) we get that the equilibrium interim utility of a bidder is

$$U(v, \beta(v)) = \int_{b_0}^{v} g(F^{n-1}(x))\, dx,$$

which is the same utility he gets from participating in a first or second price auction.

Utility equivalence

The utility equivalence of the above three auctions, suggests that there might be a large family of auctions among which ex ante identical buyers are indifferent. This is what we want to investigate in this section.

There are n bidders, each of whose valuations of the single object is independently drawn from a strictly increasing and twice continuously differentiable distribution function $F_i : [0,1] \to [0,1]$. We denote by $F_{-i} : [0,1]^{n-1} \to [0,1]$ the joint distribution of all the bidders' valuations except for bidder i.

Every equilibrium of an auction game, can be associated to an incentive compatible direct revelation mechanism (see Myerson (1981)). In our case, letting $V = [0,1]^n$ be the set of possible valuation profiles, a *direct revelation mechanism* is defined as an n-tuple $\langle (I_i, t_i)_{i=1}^n \rangle$ of pairs of functions, one for each agent where $I_i : V \to \{0,1\}$ is such that $\sum_{i=1}^n I_i(v) \leq 1$ for all $v \in V$, and $t_i : V \to [0,1]$. The function I_i is an indicator function that takes the value 1 if and only if agent 1 wins the object. The function t_i returns player i's payment to the seller as a function of the valuations profile. The restriction that $\sum_{i=1}^n I_i(v) \leq 1$ just says that no more than one agent can get the object. If the mechanism awards the object to a bidder with the highest valuation, we say that auction is ex-post efficient. Note that the probability that a bidder i of type v_i wins the auction is given by $Q_i(v_i) = \int_{V_{-i}} I_i(v_i, v_{-i}) dF_{-i}(v_{-i})$, where $V_{-i} = [0,1]^{n-1}$ is the set of valuation profiles of the agents other than i. In particular, if the mechanism is efficient and if the agents' valuations are identically distributed according to F, then $Q_i(v_i) = F^{n-1}(v_i)$. Together with the primitives of the model, a direct revelation mechanism defines a Bayesian game. The direct mechanism is said to be *incentive compatible* if truthful revelation (the identity function) is an equilibrium of the corresponding Bayesian game. We can now state our payoff-equivalence result:

Proposition 1 Assume that a risk averse buyer behaves according to the dual theory of choice under risk with "probability-evaluation" function g. Any incentive compatible auction mechanism that gives 0 utility to the bidders with valuation 0 yields a utility level of $\int_0^{v_i} g(Q_i(t))\, dt$ to a bidder with valuation v_i. Consequently, bidder i is indifferent among all auction mechanisms that induce the same winning probability function Q_i and that yield a 0 utility to his lowest valuation type.

Proof : Consider a risk averse buyer whose risk preferences are represented by the dual function g. Let $x_i : V \to \mathbb{R}$ be the random variable of bidder i's payoff determined by the mechanism $\langle (I_i, t_i)_{i \in N} \rangle$:

$$x_i(v_i, v_{-i}) = v_i I(v_i, v_{-i}) - t_i(v_i, v_{-i}).$$

Letting $G_{x_i(v_i)}$ be the DDF of the above random variable conditional on v_i being the type of agent i, type v_i's utility of $x_i(v_i, v_{-i})$ is:

$$U_i(v_i) = m + \int_m^1 g(G_{x_i(v_i)}) dx_i$$

where m is a lower bound of x_i.

Since the mechanism is incentive compatible, type v_i of agent i should prefer the above random variable to the random payoff that he would obtain if he reports any other type v_i':

$$y(v_i, v_i') = v_i I(v_i', v_{-i}) - t_i(v_i', v_{-i}).$$

Adding and subtracting $v_i' I_i(v_i', v_{-i})$ in the above expression we get

$$y_i(v_i, v_i') = x_i(v_i', v_{-i}) + (v_i - v_i') I_i(v_i', v_{-i}).$$

That is, by reporting v_i', type v_i gets a sum of two random variables: the random payoff type v_i' would get and the difference in valuations $v_i - v_i'$ that he obtains when he wins the object. Let r and s be two comonotonic random variables such that r and $x_i(v_i', v_{-i})$ have the same DDF and so do s and $(v_i - v_i')I_i(v_i', v_{-i})$. By comonotonicity, type v_i's utility of the sum $r + s$ is the sum of the utilities. By the way r and s were chosen, and since, by independence, both types v_i and v_i' evaluate random variables (defined on the other bidders' types) in the same way, we have that this sum of utilities is

$$U_i(v_i') + (v_i - v_i') g(Q_i(v_i')).$$

Since $x_i(v_i', v_{-i})$ and $(v_i - v_i')I_i(v_i', v_{-i})$, however, are not comonotonic, the random variable $y_i(v_i, v_i')$ second-order stochastically dominates $r + s$ (see Müller (1997) or Goovaerts, Dahene, and De Schepper (2000)). Therefore, since agent i is risk averse, v_i's the utility of $y_i(v_i, v_i')$ is at least as high as $U_i(v_i') + (v_i - v_i')g(Q_i(v_i'))$. By incentive compatibility of the mechanism we have that type v_i prefers x_i to y_i, which in turn implies that

$$U_i(v_i) \geq U_i(v_i') + (v_i - v_i') g(Q(v_i')).$$

Since this is true for every pair of possible types, we have that U_i is differentiable and

$$U_i'(v_i) = g(Q_i(v_i)) \qquad \forall v_i \in [0, 1].$$

As a result, since the lowest type gets 0 utility we get

$$U_i(v_i) = \int_0^{v_i} g(Q_i(t)\, dt.$$

$\square$

Corollary 1 Assume that bidders' valuations are independently and identically distributed according to F. In any ex-post efficient mechanism that yields 0 utility to the lowest type of bidder i, the equilibrium utility of a bidder with valuation v_i is $\int_0^{v_i} g[F^{n-1}(t)]\, dt$. Consequently, bidders are indifferent among all the ex-post efficient auction mechanisms that give 0 utility to the lowest valuation bidders.

Matthews (1987) shows that an expected utility maximizer bidder with CARA von Neumann Morgenstern utility function is indifferent between participating in a first price auction or in a second price auction with IID ex ante identical buyers. Since a buyer who satisfies the dual theory axioms exhibits constant average risk aversion, the above result suggests that constant absolute risk aversion is what lies behind the utility equivalence result. This conjecture, however, is left for further research.

Risk aversion in the first price auction

The equilibrium bidding function that appears in (3) allows us to predict the effect of an in increase buyers' risk aversion on their bids, and therefore on the seller's revenue. Remember that a dual utility maximizer becomes more risk averse when his dual function undergoes a convex transformation.[2]

Proposition 2 Suppose that the buyers' valuations are identically and independently distributed and that all buyers share a common dual utility function. In the first price auction, as bidders become more risk averse, they make uniformly higher bids.

Proof : An immediate corollary of the following lemma.

Lemma 1 Let $h : [0,1] \to [0,1]$ and $H : [0,1] \to [0,1]$ be two continuous and increasing functions with $h(0) = 0$ and $H(0) = 0$. Assume further that H is strictly increasing and that h is convex. Then, for all $v \in (0,1]$,

$$\frac{\int_{b_0}^{v} h(H(t))\, dt}{h(H(v))} \leq \frac{\int_{b_0}^{v} H(t)\, dt}{H(v)}.$$

Proof : Take $v \in (0,1]$. Since g is convex, for all λ and p in $[0,1]$ we have $g(\lambda p) \leq \lambda\, g(p)$. Letting $\lambda = H(t)/H(v)$ and $p = H(v)$, this means that

$$g(H(t)) \leq \frac{H(t)}{H(v)} g(H(v)), \qquad \forall t \leq v.$$

Consequently, taking integrals on both sides,

$$\int_{b_0}^{v} g(H(t))\, dt \leq \frac{g(H(v)) \int_{b_0}^{v} H(t)\, dt}{H(v)}.$$

Rearranging, we get the desired result. □

[2]Proposition 2 can be found in Salo and Weber (1995), though with a different dressing. Since the proof is simple, we give it here for the reader's convenience.

Since under the dual theory, becoming more risk averse means applying a convex transformation to the dual utility function, the above lemma implies that more risk averse buyers shade their bids less. □

Riley and Samuelson (1981) show that in the case of ex ante identical risk averse expected utility maximizing buyers, the reserve price that maximizes the seller's revenue decreases with the buyers' degree of risk aversion. The following proposition shows that the same holds in the case of risk averse buyers who behave according to the dual theory.

Proposition 3 Assume that the buyers' valuations are drawn independently and identically from a strictly increasing and twice continuously differentiable distribution $F : [0,1] \to [0,1]$, and that all buyers share a common probability-evaluation function, g. Assume that the seller is risk neutral. Then, in the first price auction, an interior optimal seller reserve price is defined implicitly by

$$b_* = \frac{\frac{g[F^{n-1}(b_*)]}{F^{n-1}(b_*)} \int_{b_*}^{1} \frac{F^{n-1}(v)}{g[F^{n-1}(v)]}\, dF(v)}{F'(b_*)}.$$

Further, this optimal reserve price is a non-increasing function of the buyers' risk aversion.

Proof : The expected revenue of a first price auction with reserve price b, and when the bidders' preferences are represented by the function g, is given by

$$R_g(b) = \int_b^1 (v - \frac{\int_b^v g[F^{n-1}(x)\, dx]}{g[F^{n-1}(v)]})\, dF^n(v).$$

A risk neutral seller chooses a reserve price so as to maximize the expected revenue. Taking derivatives with respect to b and equalizing to yields

$$-nb_* F^{n-1}(b_*)F'(b_*) + n \int_{b_*}^{1} g[F^{n-1}(b_*)]\, \frac{F^{n-1}(v)}{g[F^{n-1}(v)]}\, dF(v) = 0. \tag{5}$$

Consequently, an interior solution to the seller's problem satisfies

$$b_* = \frac{\frac{g[F^{n-1}(b_*)]}{F^{n-1}(b_*)} \int_{b_*}^{1} \frac{F^{n-1}(v)}{g[F^{n-1}(v)]}\, dF(v)}{F'(b_*)}, \tag{6}$$

which proves the first part of the claim.

Now let $h : [0,1] \to [0,1]$ be a convex function such that $h(0) = 0$ and $h(1) = 1$ and let $f \equiv h \circ g$. The function f represents preferences that are more risk averse than the preferences represented by g. Letting R_f be the revenue function when the bidders' preferences are represented by f, we will show first that the difference $R_g - R_f$ is a non-decreasing function of the reserve price. Since

h is convex, for all λ and p in $[0,1]$ we have $h(\lambda p) \leq \lambda\, h(p)$. Taking $\lambda = g[F^{n-1}(b_*)]/g[F^{n-1}(v)]$ and $p = g[F^{n-1}(v)]$ we have

$$f(F^{n-1}(b_*)) \leq \frac{g[F^{n-1}(b_*)]}{g[F^{n-1}(v)]}\, f(F^{n-1}(v)).$$

Rearranging and multiplying both sides by $F^{n-1}(v)$ we get

$$f(F^{n-1}(b_*))\, \frac{F^{n-1}(v)}{f(F^{n-1}(v))} \leq g[F^{n-1}(b_*)]\, \frac{F^{n-1}(v)}{g[F^{n-1}(v)]}. \tag{7}$$

Using equation (5) we get

$$\frac{d(R_g - R_f)}{db} = n \int_b^1 (g[F^{n-1}(b_*)]\, \frac{F^{n-1}(v)}{g[F^{n-1}(v)]} - f[F^{n-1}(b_*)]\, \frac{F^{n-1}(v)}{f[F^{n-1}(v)]})\, dF(v)$$

which by (7) is non-negative. Therefore the difference $R_g - R_f$ is non-decreasing. Now let b_g and b_f be optimal reserve prices when the preferences are represented by g and f, respectively. We must have, $R_g(b_g) \geq R_g(b_f)$ and $R_f(b_f) \geq R_f(b_g)$, which implies that

$$R_g(b_g) - R_f(b_g) \geq R_g(b_f) - R_f(b_f).$$

Since $R_g - R_f$ is non-decreasing, we conclude that $b_g \geq b_f$. □

The Case of Expected Utility Maximizers: A Comparison

An agent who satisfies the axioms of the expected utility theory is characterized by his von Neumann-Morgenstern utility function, u, that can be chosen so as to solve the following preference equation: $[1, u(x)] \sim [x, 1]$ for all x in the domain (see Yaari (1987) or Fishburn (1982)). Similarly, the dual "probability-evaluation" function that represents the preferences of an agent who behaves according to the dual theory can be chosen so as to solve $[1, p] \sim [g(p), 1]$ for all $p \in [0,1]$ (see Theorem 1). Consequently, given fixed preferences over lotteries, the functions u and g that solve the above preference equations are the inverses of each other. It would be interesting to compare the equilibrium bidding functions of buyers who behave according to the von Neumann-Morgenstern utility function u with those of buyers who behave according to the dual functions g. Although no general result is available, in the case of constant relative risk aversion the following claim is easily proved.

Proposition 4 Consider an individual with von Neumann-Morgenstern (CRRA) utility function $u(x) = x^{1/m}$ participating in a n-bidder first price auction. His equilibrium behavior is identical to the equilibrium behavior of a bidder whose dual function is given by $g(p) = p^m$. In other words, in both cases the best response correspondences coincide.

Proof : Note that an individual whose valuation is v_i submits a bid of b, then he gets the lottery $[v_i - b, p(b)]$, where $p(b)$ is the probability that the individual wins the object given his bid and the other bidders' strategies. For every type, bidding above his valuation is a strictly dominated strategy. Therefore it is enough to compare the two different preferences over the family of lotteries of the form $[x, p]$, where $x \geq 0$. Now, in this case an individual whose risk preferences can be represented by the von Neumann-Morgenstern utility function u prefers lottery $[x, p]$ to lottery $[y, q]$ if and only if $x^{1/m} p > y^{1/q} q$, which in turn holds if and only if $xp^m > yq^m$. But this last inequality holds if and only if an individual whose risk preferences can be represented by the dual function g prefers lottery $[x, p]$ to lottery $[y, q]$. □

Therefore, the behavior in a first price auction of an expected utility maximizer with a von Neumann-Morgenstern utility function $u(x) = x^{1/m}$ is indistinguishable from the behavior of a bidder whose preferences can be described by the dual theory with "probability-evaluation" function $g(p) = p^m$. Clearly, this does not mean that the utility equivalence holds for this kind of expected utility maximizer. We know that this kind of bidder is not indifferent between first price and second price auctions. The result, however, does not generalize for other von Neumann-Morgenstern utility functions.

3 Risk aversion in a Sealed-Bid Double Auction

In this section we look at the sealed-bid double auction introduced by Chatterjee and Samuelson (1983), when both bargainers behave according to the dual theory of choice under risk. As will be seen, our analysis follows closely that of Leininger, Linhart, and Radner (1989), which is again evidence that introducing risk aversion via the dual theory does not add any technical difficulty to the analysis. We focus on the "linear" strategies of this mechanism and check the effect of changes in the degree of risk aversion on the equilibrium strategies, on the probability of trade, and on the ex post inefficiency of the equilibrium outcome.

A potential buyer and a potential seller are bargaining over the price of a single object. Their respective valuations of the object are independently, identically and uniformly distributed over $[0, 1]$. The *sealed-bid mechanism* is defined as follows: The buyer and the seller simultaneously submit their bids, b and s, respectively. If $b \geq s$, trade takes place at the price $(b + s)/2$. If $b < s$ there is no trade.

We are interested in finding equilibrium strategies when both players' preferences satisfy the axioms of the dual theory of choice under risk, the buyer's dual function being f and the seller's g. Further, we are interested in the influence of changes in the degree of risk aversion on the equilibrium strategies and on the equilibrium terms of trade.

Consider a buyer with valuation v and let $\tilde{e}_b$ be the random variable of his earnings when he bids b and the seller's bid is distributed according to the cumulative distribution function S. Then we have

$$\begin{aligned}
\Pr\{\tilde{e}_b > x\} &= \Pr\{v - \tfrac{b+s}{2} > x \text{ and } s < b\} \\
&= \Pr\{s < 2(v-x) - b \text{ and } s < b\} \\
&= \Pr\{s < \min\{2(v-x) - b; b\}\} \\
&= S(\min\{2(v-x) - b; b\}).
\end{aligned}$$

Consequently, since the buyer's earnings are bounded from above by $v - b/2$, his utility from bidding b is

$$\begin{aligned}
U(v,b) &= \textstyle\int_0^{v-b/2} f[S(\min\{2(v-x) - b; b\})]\,dx \\
&= \textstyle\int_0^{v-b} f[S(b)]\,dx + \int_{v-b}^{v-b/2} f[S(2(v-x) - b)]\,dx.
\end{aligned}$$

Similarly, the utility of a seller with valuation c of a bid s when the buyer's bid is distributed according to the cumulative distribution function B is given by

$$\begin{aligned}
V(c,s) &= \textstyle\int_0^{\frac{1+s}{2}-c} g[(1 - B(\max\{2(c+x) - s; s\}))]\,dx \\
&= \textstyle\int_0^{s-c} g[1 - B(s)]\,dx + \int_{s-c}^{\frac{1+s}{2}-c} g[1 - B(2(c+x) - s)]\,dx.
\end{aligned} \tag{8}$$

The buyer's optimal bid must satisfy the first order condition:

$$\frac{\partial U}{\partial b} = -(f \circ S(b)) + \int_0^{v-b} (f \circ S)'(b)\,dx - \int_{v-b}^{v-b/2} (f \circ S)'(2(v-x)-b)\,dx - 1/2(f \circ S)(0) + f \circ S(b) = 0.$$

Making the change of variable $y = 2(v-x) - b$, the above equation can be written

$$-(f \circ S(b)) + (v-b)(f \circ S)'(b) - 1/2\int_0^b (f \circ S)'(y)\,dy - 1/2(f \circ S)(0) + (f \circ S)(b) = 0$$

or

$$-(f \circ S(b)) + (v-b)(f \circ S)'(b) - \frac{(f \circ S)(b) - (f \circ S)(0)}{2} - 1/2(f \circ S)(0) + (f \circ S)(b) = 0$$

which implies

$$v = \frac{1}{2}\frac{(f \circ S)(b)}{(f \circ S)'(b)} + b. \tag{9}$$

The second order condition requires that

$$\frac{-3\,(f\circ S)'(b)}{2}+\frac{(f\circ S)(b)\,(f\circ S)''(b)}{2\,(f\circ S)'(b)}<0 \tag{10}$$

which is satisfied if and only if the right hand side of (9) is strictly increasing. Therefore, if (10) holds, equation (9) defines the inverse, β^{-1}, of the buyer's equilibrium strategy β which is strictly increasing.

Similarly, taking derivatives of $V(c,s)$ with respect to s in equation (8) we get

$$c=\frac{1}{2}\frac{g[1-B(s)]}{(g\circ(1-B))'(s)}+s \tag{11}$$

which determines the inverse σ^{-1} of the seller's strictly increasing equilibrium strategy σ. Since β is strictly monotone and the buyer's valuation is uniform, we have

$$\begin{aligned} B[\beta(x)] &= \Pr[\beta(v)\le\beta(x)] \\ &= \Pr(v\le x) \\ &= x. \end{aligned}$$

Therefore

$$B=\beta^{-1}.$$

Similarly,

$$S=\sigma^{-1}.$$

Consequently, the first order conditions can be written as

$$B(x)=\frac{1}{2}\frac{(f\circ S)(x)}{(f\circ S)'(x)}+x \tag{12}$$

and

$$S(x)=\frac{1}{2}\frac{[g\circ(1-B)](x)}{[g\circ(1-B)]'(x)}+x. \tag{13}$$

In order to get an explicit analytic solution to the system of differential equations (12)–(13), we must make some simplifying assumptions about the dual functions f and g. Specifically, consider the case where the buyer's and seller's dual functions are given by

$$f(p)=k_B p^{\alpha}$$

and

$$g(p)=k_S p^{\delta},$$

respectively. In this case, the first order conditions (12) and (13) become

$$B(x) = x + \frac{1}{2}\frac{S(x)}{\alpha S'(x)}$$

and

$$S(x) = x - \frac{1}{2}\frac{1 - B(x)}{\delta B'(x)}.$$

It can be checked that the following cumulative distribution functions solve the above system of differential equations:

$$B(b) = \begin{cases} 0 & \text{if } b < \frac{a\,(1-h)}{1-a} \\ a + (b-a)\dfrac{1-a}{h-a} & \text{if } \frac{a\,(1-h)}{1-a} \leq b \leq h \\ 1 & \text{if } b \geq h \end{cases}$$

$$S(s) = \begin{cases} 0 & \text{if } 0 \leq s < a \\ \dfrac{(s-a)h}{h-a} & \text{if } a \leq s \leq 1 - a\frac{1-h}{h} \\ 1 & \text{if } s \geq 1 - a\frac{1-h}{h} \end{cases}$$

where

$$a = \frac{\alpha}{\alpha + \delta + 2\,\alpha\,\delta} \tag{14}$$

and

$$h = \frac{\alpha\;(1 + 2\,\delta)}{\alpha + \delta + 2\,\alpha\,\delta}. \tag{15}$$

This solution corresponds to the following linear equilibrium strategies:

$$\begin{aligned} \beta(v) &= h - \frac{(h-a)\;(1-v)}{1-a} \\ \sigma(c) &= a + \frac{h-a}{h}c. \end{aligned} \tag{16}$$

It is not difficult to check that this is the only linear equilibrium of the sealed-bid mechanism. According to these equilibrium strategies, the buyer will never bid more than h and the seller will never bid less than a. Therefore, the range of equilibrium prices is the interval $[a, h]$. Also, since

$$\beta(1) = h = \sigma(h) < \sigma(1)$$

and

$$\beta(0) < \beta(a) = a = \sigma(0),$$

the linearity of the strategies imply that $\beta(v) < \sigma(c)$, for all $v \leq c$. In other words, there is trade only if $v > c$. On the other hand, the equilibrium is ex-post inefficient because there might be no trade even if $v > c$.

Proposition 5 Assume that the buyer's preferences satisfy the dual theory axioms and can be represented by the dual function $f(p) = p^{\alpha}$. Similarly, assume that the seller's preferences can be represented by the dual function $g(p) = p^{\delta}$. In the only linear equilibrium, the equilibrium range of prices moves to the right as the buyer becomes more risk averse, and moves to the left as the seller becomes more risk averse. Further, the probability of trade is an increasing function of the trades' degree of risk aversion. When both players have the same risk preferences, namely $\alpha = \delta = m$, the equilibrium price interval converges to $[0, 1]$ as the degree of risk aversion, m, becomes arbitrarily large. In this case the probability of trade tends to $1/2$ as m goes to ∞.

Proof : It can be checked that both a and h are strictly increasing functions of α and strictly decreasing functions of δ. Consequently, the range of equilibrium prices moves to the left as the buyer becomes more risk averse, and moves to the right as the seller becomes more risk averse.

Also, when both players have the same risk preferences, namely $\alpha = \delta = m$, the range of equilibrium prices is

$$[\frac{1}{2+2\,m}, \frac{1+2\,m}{2+2\,m}].$$

It can be seen that as m tends to ∞, the price interval tends to $[0, 1]$, which means that as risk aversion becomes arbitrarily large, the ex post inefficiency of the mechanism tends to vanish.

This feature can also be seen from the probability of trade. Let v be the valuation of the potential buyer. Given the equilibrium strategies in (16) the probability of trade, conditional on the buyer's valuation, is

$$\begin{aligned} \Pr\{\sigma(c) \leq \beta(v)\} &= \Pr\{c \leq \tfrac{h\,(v-a)}{1-a}\} \\ &= \begin{cases} 0 & v < a \\ \frac{h\,(v-a)}{1-a} & v \geq a. \end{cases} \end{aligned}$$

Since there is trade only if $v \geq a$, the overall probability of trade is

$$\begin{aligned}\Pr\{\sigma(c) \leq \beta(v)\} &= \int_a^1 \frac{h\,(v-a)}{1-a}\,dv \\ &= \frac{(1-a)\,h}{2} \\ &= \frac{(\alpha+2\,\alpha\,\delta)\,(\delta+2\,\alpha\,\delta)}{2\,(\alpha+\delta+2\,\alpha\,\delta)^2}.\end{aligned}$$

When both traders are risk averse, namely $\alpha, \delta > 1$, the above expression is increasing in α and δ. This means that an increase in a player's risk aversion, leads to an increase in the probability of trade. When the seller and the buyer have the same risk preferences, namely when $\alpha = \delta = m$, the probability of trade is

$$\frac{(1+2\,m)^2}{8\,(1+m)^2}$$

and tends to $1/2$ as m tends to ∞. In other words, when the common degree of risk aversion tends to infinity, the probability of trade tends to $1/2$. This is because in the limit there is trade if and only if the value of the object to the buyer is at least as large as the value of the object to the seller. □

4 Concluding Remarks

It has been shown that the dual theory of choice under risk is quite apt for the analysis of single item auctions. In particular, the introduction of risk aversion does not seem to complicate the analysis of auctions beyond the standard difficulty of auctions with risk neutral buyers. Many open issues remain that the theory needs to deal with before we conclude that it is a most appropriate tool for the analysis of auctions. Among the topics for future research we can mention the analysis of common value auctions, auctions with an endogenously determined number of bidders, and especially, the design of optimal auctions.

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