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Correlated equilibria of games with many players

Hans Keiding[1], **Bezalel Peleg***[2]

[1] Institute of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark.
[2] Institute of Mathematics and Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Givat Ram, 91904 Jerusalem, Israel

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Abstract. Let $G_{m,n}$ be the class of strategic games with n players, where each player has $m \geq 2$ pure strategies. We are interested in the structure of the set of correlated equilibria of games in $G_{m,n}$ when $n \to \infty$. As the number of equilibrium constraints grows slower than the number of pure strategy profiles, it might be conjectured that the set of correlated equilibria becomes large. In this paper, we show that (1) the average relative measure of the set of correlated equilibria is smaller than 2^{-n}; and (2) for each $1 < c < m$, the solution set contains c^n correlated equilibria having disjoint supports with a probability going to 1 as n grows large. The proof of the second result hinges on the following inequality: Let $\mathbf{c}_1, \ldots, \mathbf{c}_l$ be independent and symmetric random vectors in $\mathbf{R}^k$, $l \geq k$. Then the probability that the convex hull of $\mathbf{c}_1, \ldots, \mathbf{c}_l$ intersects $\mathbf{R}^k_+$ is greater than or equal to $1 - 2^{-l}\left[\binom{l}{0} + \cdots + \binom{l}{k-1}\right]$.

Key words: correlated equilibrium, large games

1. Introduction

Correlated equilibria of a game were introduced by Aumann (1974) as a description of a situation where the randomization of the pure strategies is

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performed by an independent device rather than, as usual, by individual choices of mixed strategies. Thus, a correlated strategy is a probability distribution over the set of all pure strategy combinations in the game, and for a correlated strategy to be an equilibrium, it should satisfy the incentive compatibility constraints: players should not gain by deviating unilaterally from the choice prescribed by the randomization device. Correlated equilibria have subsequently been studied by several authors, e.g., Aumann (1987), Hart and Schmeidler (1989), Forges (1990a), (1990b), (1993), Nau and McCardle (1990), and Myerson (1997).

In this paper we consider the set of correlated equilibria for games with a large number n of players, where each player has a fixed number m of pure strategies. In such a game, there are m^n pure strategy profiles, and a correlated equilibrium is therefore a nonnegative solution to a system of inequalities, defined by the incentive compatibility constraints, in m^n variables. Since the incentive compatibility constraints state that no player should gain by changing any one prescribed pure strategy choice by any other, there are $nm(m-1)$ such constraints. Thus, the number of constraints becomes very small compared to the number of variables, and intuitively, the set of correlated equilibria becomes very large.

While there has been research in questions of the largeness of the set of Nash equilibria (cf., e.g., Gül, Pearce and Stachetti (1993), Quint and Shubik (1997), McLennan (1997), Keiding (1997)), rather little is known about the set of correlated equilibria. The purpose of the present paper is to investigate the extent to which the intuition sketched above, that the set of correlated equilibria becomes very large, can be reformulated into precise statements. There are some very simple examples of games where the set of correlated equilibria is very small indeed, consisting of a single point no matter how large n becomes, namely all games with an equilibrium in dominant strategies. Therefore, what can be obtained is at most results holding for some or perhaps almost all games. As a consequence, we shall work with random games, i.e. families of games where the payoffs are chosen in accordance with a specified probability distribution, and we shall be interested in properties which hold for all games in a set with probability tending to 1 as n becomes large.

However, even in this setup the "largeness" of the set of correlated equilibria has to be defined in a rather careful way. The rather straightforward definition in terms of relative measure in the set of all probability distributions on a finite set of cardinality m^n does not yield the desired result. Quite on the contrary, we show that the average relative measure of the set of correlated equilibria goes to 0 for n growing large. Thus, we must look for other ways of expressing largeness of a subset of the simplex in $\mathbf{R}^{m^n}$.

A way of expressing the fact that a set is large even if it does not have full dimension is that it contains distinct points which are far apart. Since we deal with subsets of a simplex, it seems reasonable to consider points as being distant from each other if they belong to subsimplices which have no connection with each other; in terms of probability distributions, if they have disjoint supports. From this point of view, a subset of the simplex is large if it contains a large number of points on mutually disjoint subsimplices; the maximal possible number is of course m^n, attained by the simplex itself.

In the main result of this paper, we show that for any real number $c < m$, we can make the probability of having c^n correlated strategies with disjoint supports as close to 1 as we want by choosing n large enough. Thus, even though its dimension is in most cases smaller than $m^n - 1$, the set of correlated equilibria will contain a large number of equilibria which are quite far from each other. One might therefore still argue that the set of correlated strategies is a very large set when there are many players.

The paper is organized as follows: In section 2, we give the necessary definitions and introduce the notion of a random game. In section 3, we show that the relative measure of the set of correlated equilibria is small on the average. In section 4, we consider the probability that a game has many (in a properly defined way) correlated equilibria with mutually disjoint supports. The second main result of the paper states that this probability tends to 1 as n grows large. To obtain this result, we need an assessment of the probability that a given vector can be written as a nonnegative linear combination of l vectors in $\mathbf{R}^k$, when the vectors are independently and symmetrically distributed. This is obtained using a combinatorial result on representing a vector as a nonnegative linear combination of given vectors. These results may have other applications and therefore be of independent interest, and we have placed them in an appendix.

We shall use the following notational conventions in the paper: For A a finite set, $\# A$ denotes the cardinality of A. If $A = \{a_1, \ldots, a_r\} \subset \mathbf{R}^d$ is a finite set of vectors in Euclidean space $\mathbf{R}^d$, then $\operatorname{span}(a_1, \ldots, a_r)$ is the subspace of $\mathbf{R}^d$ spanned by $a_1, \ldots, a_r$, that is the smallest linear subspace of $\mathbf{R}^d$ containing $a_1, \ldots, a_r$. Furthermore, $\operatorname{conv}(a_1, \ldots, a_r)$ denotes the convex hull of $a_1, \ldots, a_r$ (the smallest convex subset of $\mathbf{R}^d$ containing $a_1, \ldots, a_r$), and $\operatorname{cone}(a_1, \ldots, a_r)$ is the convex cone spanned by $a_1, \ldots, a_r$,

$$\operatorname{cone}(a_1, \ldots, a_r) = \left\{ \sum_{i=1}^{r} \lambda_i a_i | \lambda_i \geq 0, i = 1, \ldots, r \right\}.$$

Finally, for $t \in \mathbf{R}$, $\lfloor t \rfloor$ denotes the integral part of t.

2. Definitions

We consider games with a fixed finite number $m \geq 2$ of strategies for each player and a finite, but possibly large, number n of players. We assume that all players have the same number of (pure) strategies; this assumption may be relaxed without changing the methods of proofs of the results to follow, as long as the number of pure strategies of any player remains uniformly bounded.

An n-person game form is an n-fold cartesian product

$$\mathscr{A} = A^n$$

of a finite set $A = \{a^1, \ldots, a^m\}$. Elements of $\mathscr{A}$ are called strategy profiles and are denoted by $a = (a_1, \ldots, a_n)$. We use the notational convention $a_{-i} = (a_j)_{j \neq i}$ and write $a = (a_i, a_{-i})$.

A correlated strategy on $\mathscr{A}$ is an element p of the set $\triangle(\mathscr{A})$ of all probability distributions over $\mathscr{A}$, that is a vector

$$p = (p(a))_{a \in \mathscr{A}} \in \mathbf{R}_+^{\mathscr{A}}$$

such that

$$\sum_{a \in \mathscr{A}} p(a) = 1.$$

We use the shorthand notation $\triangle$ instead of $\triangle(\mathscr{A})$ in the rest of the paper, since no confusion can arise.

For any player $i \in \{1, \ldots, n\}$, a utility assignment is a vector $u_i \in \mathbf{R}^{\mathscr{A}}$, whereby $u_i(a)$ is interpreted as the utility payoff to player i if the strategy profile $a \in \mathscr{A}$ is chosen. The family of utility assignments is thus $\mathbf{R}^{\mathscr{A}}$; for ease of notation we denote this space by $\mathscr{U}$. Now a game over $\mathscr{A}$ may be defined as an n-tuple $u = (u_1, \ldots, u_n) \in \mathscr{U}^n$ of utility assignments.

Let p be a correlated strategy on $\mathscr{A}$ and u a game over $\mathscr{A}$; then p is a correlated equilibrium in the game u if for each $i \in \{1, \ldots, n\}$ and each $a_i, a_i' \in A$,

$$\sum_{a_{-i}} p(a_i, a_{-i})[u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})] \geq 0. \tag{1}$$

The set of correlated equilibria in u, which is a polyhedral convex subset of $\triangle$, is denoted $\mathscr{C}(u)$.

A (mixed strategy) Nash equilibrium in $u \in \mathscr{U}^n$ is a correlated equilibrium p which is the product of n probability distributions on A. The set $\mathscr{C}(u)$ thus contains the convex hull of all the Nash equilibria of u, but it may contain other elements of $\triangle$ as well. These additional elements may give a better expected payoff to the players, but, as the following example shows, there are games where some correlated equilibrium is worse for all players than any Nash equilibrium. Thus, a large set of correlated equilibria might include undesirable outcomes.

Example 1. Consider the following three-person game u, where each player has two pure strategies (player 1 choosing row, player 2 column and player 3 matrix):

	1	2
1	(1, 1, 1)	(−1, −1, 1)
2	(−1, −1, 1)	(−1, −1, −1)

1

	1	2
1	(1, 1, 0)	(1, 0, 0)
2	(0, 1, 0)	(0, 0, 0)

2

Here we may define $A = \{1, 2\}$; thus, $a_1 = j$ means that player 1 chooses the jth row, etc. The game has a unique Nash equilibrium with each player choosing strategy 1 (with probability 1), and the payoff of each player is 1. However, as it may easily be checked, the correlated strategy which gives probability 1/2 to each of the strategy profiles $(1, 1, 1)$ and $(2, 2, 1)$ is a correlated equilibrium, and it gives each player the average payoff of 0.

Let $u \in \mathscr{U}^n$ be a game; we introduce a matrix description of (1) which will be useful in the sequel. The matrix will have $|\mathscr{A}|$ columns, one for each strategy profile, and $nm(m-1)$ rows, indexed by a player i and two pure strategies of player i occurring in one of the equilibrium constraints (1) above.

Thus, for $a = (a_i, a_{-i}) \in \mathscr{A}$, $i \in \{1, \dots, n\}$ a player and $(j, k) \in \{1, \dots, m\}$ any pair of indices of distinct strategies for i, so that $a^j, a^k \in A$ are pure strategy choices by player i, and (a^j, a_{-i}) the element of $\mathscr{A}$ where players different from i choose a_{-i} and i chooses a^j, we let

$$c_{i;j,k}[u](a) = \begin{cases} u_i(a^j, a_{-i}) - u_i(a^k, a_{-i}) & \text{if } a_i = a^j \text{ (so that } a^j \text{ is chosen} \\ & \text{in } a \text{ and compared by } i \text{ to } a^k\text{),} \\ 0 & \text{otherwise,} \end{cases}$$

and we let $c_{i;j,k}[u] \in \mathbf{R}^{\mathscr{A}}$ be the vector with coordinates $c_{i;j,k}[u](a)$, $a \in \mathscr{A}$. Then $p \in \triangle$ belongs to $\mathscr{C}(u)$ if and only if

$$p \cdot c_{i;j,k}[u] \geq 0$$

for all $i \in \{1, \dots, n\}$ and all $j, k \in \{1, \dots, m\}$ with $j \neq k$. In matrix notation, this condition may be written as

$$\mathbf{C}[u]p \geq 0,$$

where

$$\mathbf{C}[u] = \begin{pmatrix} c_{1;1,2}[u] \\ c_{1;1,3}[u] \\ \cdots \\ c_{n;m,m-1}[u] \end{pmatrix}$$

has the vectors $c_{i;j,k}[u]$ as rows. This again means that some vector in $\mathbf{R}_+^{nm(m-1)}$ can be written as a nonnegative linear combination of the columns of $\mathbf{C}(u)$,

$$\sum_{a \in \mathscr{A}} p(a)c[u](a) \geq 0,$$

where $c[u](a) \in \mathbf{R}^{nm(m-1)}$ is the vector with coordinates $c_{i;j,k}[u](a)$, for $i \in \{1, \dots, n\}$, $j, k \in \{1, \dots, m\}$, $j \neq k$.

Example 2. As in Example 1, we consider a three-person game u, where each player has two pure strategies:

	1	2
1	$(2, 2, 2)$	$(-1, 3, 0)$
2	$(3, -1, 0)$	$(0, 0, -1)$

1

	1	2
1	$(0, 0, -1)$	$(3, -1, 0)$
2	$(-1, 3, 0)$	$(2, 2, 2)$

2

The set $\mathscr{A}$ of strategy profiles has cardinality 8, and there are $3 \cdot 2 = 6$ constraints of the type (1) (since there are 3 players and for each player, two comparisons of pure strategies), so the matrix $\mathbf{C}[u]$ has 6 rows and 8 columns;

a simple computation gives

	(1,1,1)	(1,1,2)	(1,2,1)	(1,2,2)	(2,1,1)	(2,1,2)	(2,2,1)	(2,2,2)
(1;1,2)	−1	1	−1	1	0	0	0	0
(1;2,1)	0	0	0	0	1	−1	1	−1
(2;1,2)	−1	1	0	0	−1	1	0	0
(2;2,1)	0	0	1	−1	0	0	1	−1
(3;1,2)	3	0	0	0	0	0	−3	0
(3;2,1)	0	−3	0	0	0	0	0	3

Here, the notation (i, j, k) in the columns denotes the strategy profile where 1 chooses i, 2 chooses j, and 3 chooses k, whereas the notation $(i; j, k)$ in rows means that player i compares the choice of j with that of k.

Definition 1. *A random game over $\mathscr{A}$ is a probability distribution F over $\mathscr{U}^n$ (where $\mathscr{U}^n$ is identified with the Euclidean space $\mathbf{R}^{nm^n}$ with the σ-algebra of Borel sets). In this paper, we consider only random games such that the distribution F is the product of distributions $F_1, \dots, F_n$ on $\mathscr{U}$.*

Using the notion of random games, we may investigate properties of the set of correlated equilibria in order to see whether such properties hold often, that is for a set of games of probability close to 1. In particular, we may study the limiting behavior of such probabilities when the number of players gets large. The results of the next two sections give some examples of this.

3. The set of correlated equilibria is small on the average

An intuitively obvious way of studying largeness of the set of correlated equilibria is by considering the relative measure of this set in the set of all probability distributions on $\mathscr{A}$. In the present section, we show that this relative measure is typically quite small.

Consider the random variable $u \mapsto \mathscr{C}(u)$ with values in the family of convex subsets of $\triangle$. We denote by $\mu(\mathscr{C}(u))$ the relative volume of $\mathscr{C}(u)$, i.e.

$$\mu(\mathscr{C}(u)) = \frac{m(\mathscr{C}(u))}{m(\triangle)}$$

where m is Lebesgue measure in $\mathbf{R}^{m^n-1}$.

Lemma 1. *Let $F = (F_1, \dots, F_n)$ be a random game over $\mathscr{A}$. Then the map $u \mapsto \mu(\mathscr{C}(u))$ is integrable.*

Proof: The set

$$\mathbf{U} = \{(u, p) \in \mathscr{U}^n \times \triangle \mid p \in \mathscr{C}(u)\}$$

is measurable, so that its characteristic function $\chi_{\mathbf{U}}$ defined by

$$\chi_{\mathbf{U}}(u, p) = \begin{cases} 1 & \text{if } (u, p) \in \mathbf{U}, \\ 0 & \text{otherwise,} \end{cases}$$

is measurable and integrable with respect to the product measure defined by F on $\mathscr{U}^n$ and μ on $\triangle$. By Fubini's theorem, the function $\mu(\mathscr{C}(\cdot))$ defined by

$$\mu(\mathscr{C}(u)) = \int_{\triangle} \chi_{\mathbf{U}}(u,p)\,d\mu$$

is integrable. □

Having introduced the relative measure of the set of correlated equilibria, we have now a way of expressing whether this set is "large" or "small", and taking averages over all games we can even obtain statements about the largeness of the set of correlated equilibria for most games. It turns out, as shown in Theorem 1 below, that measured in this way, the set of correlated equilibria of a game with many players is small: The average relative measure goes to zero as the number of players increases.

Theorem 1. *Let $F = (F_1,\ldots,F_n)$ be a random game over $\mathscr{A}$, such that the F_i are independent, absolutely continuous, and symmetric in the sense that*

$$\int_{\mathscr{U}} \chi_B\,dF_i = \int_{\mathscr{U}} \chi_{-B}\,dF_i \quad \textit{for each } i,$$

for any measurable subset B of $\mathscr{U}$, where $-B = \{-x|x \in B\}$. Then

$$E_u[\mu(C(u))] \le \frac{1}{2^n}.$$

Proof: For each player i, define the set

$$M_+^i = \left\{(u_i,p) \in \mathscr{U} \times \triangle \;\middle|\; \sum_{a_{-i}} p(a^1,a_{-i})[u_i(a^1,a_{-i}) - u_i(a^2,a_{-i})] > 0\right\}$$

of all pairs (u_i,p) consisting of a payoff assignment to player i and a correlated strategy, such that player i gets a better average payoff from choosing the pure strategy a^1 rather than a^2, and similarly, define

$$M_-^i = \left\{(u_i,p) \in \mathscr{U} \times \triangle \;\middle|\; \sum_{a_{-i}} p(a^1,a_{-i})[u_i(a^1,a_{-i}) - u_i(a^2,a_{-i})] < 0\right\},$$

$$M_0^i = \left\{(u_i,p) \in \mathscr{U} \times \triangle \;\middle|\; \sum_{a_{-i}} p(a^1,a_{-i})[u_i(a^1,a_{-i}) - u_i(a^2,a_{-i})] = 0\right\}.$$

Let $M^i(u_i) = \{p \in \triangle \mid (u_i,p) \in M_+^i \cup M_0^i\}$, then $\mathscr{C}(u) \subset \bigcap_{i=1}^n M^i(u_i)$. Moreover, $(u_i,p) \in M_+^i$ implies that $(-u_i,p) \in M_-^i$ for all $u_i \in \mathscr{U}$. For $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \in \{-,+,0\}^n$, let

$$M_\varepsilon = \{(u,p) \in \mathscr{U}^n \times \triangle \mid (u_i,p) \in M_{\varepsilon_i}^i, i = 1,\ldots,n\}.$$

We then have that the sets M_ε and $M_{\varepsilon'}$ are disjoint for $\varepsilon \neq \varepsilon'$, so that for each $u \in \mathscr{U}^n$,

$$1 = \int_\triangle \chi_\triangle \, d\mu = \int_\triangle \left[\sum_{\varepsilon \in \{+,-,0\}^n} \chi_{M_\varepsilon} \right] d\mu$$

and consequently, using Fubini's theorem, we get that

$$1 = \int_{\mathscr{U}^n} \int_\triangle \chi_\triangle \, d\mu \, dF = \int_\triangle \left[\sum_{\varepsilon \in \{+,-,0\}^n} \int_{\mathscr{U}^n} \chi_{M\varepsilon} \, dF \right] d\mu. \tag{2}$$

If $\varepsilon_i = 0$ for some $i \in \{1, \dots, n\}$, then for each $p \in \triangle$ we have

$$\begin{aligned} \int_{\mathscr{U}^n} \chi_{M_\varepsilon}(u,p) \, dF &= \int_{\mathscr{U}^{-i}} \left[\int \chi_{M_\varepsilon}(u,p) \, dF_i \right] dF_{-i} \\ &\leq \int_{\mathscr{U}^{-i}} \left[\int \chi_{M_0^i}(u_i,p) \, dF_i \right] dF_{-i}; \end{aligned}$$

now either $p(a^1, a_{-i}) = 0$ for all a_{-i}, and the set of such p has relative volume 0, or $p(a^1, a_{-i}) \neq 0$ for some a_{-i}, in which case the set of u_i such that $(u_i, p) \in M_0^i$ is a set of Lebesgue measure zero in $\mathbf{R}^{m^n}$. Thus we have by absolute continuity of F_i that $\int \chi_{M_0^i}(u_i,p) \, dF_i = 0$, so we may restrict the sum in (2) to all $\varepsilon \in \{+,-\}^n$. Now, by symmetry,

$$\int_\triangle \int_{\mathscr{U}^n} \chi_{M_\varepsilon} \, dF \, d\mu = \int_\triangle \int_{\mathscr{U}^n} \chi_{M_{\varepsilon'}} \, dF \, d\mu$$

for all $\varepsilon, \varepsilon' \in \{+,-\}^n$, so by (2) we have

$$\int_\triangle \int_{\mathscr{U}^n} \chi_{M_{(+,\dots,+)}} \, dF \, d\mu = \frac{1}{2^n}.$$

The conclusion of the theorem now follows, since

$$\begin{aligned} E_u[\mu(\mathscr{C}(u)] &= \int_{\mathscr{U}^n} \int_\triangle \chi_{\{(u,p) | p \in \mathscr{C}(u)\}} \, d\mu \, dF \\ &\leq \int_\triangle \int_{\mathscr{U}^n} \chi_{M_{(+,\dots,+)}} \, dF \, d\mu = \frac{1}{2^n}. \end{aligned}$$ □

The result of Theorem 1 shows that the average relative measure of the set of correlated equilibria is quite small, even approaching 0 as the number of players grows large. However, even if the set of correlated equilibria must be considered as small in the sense of average relative measure, it may still contain a large number of correlated strategies which are far away from each other and even in some sense evenly spread out in $\triangle$. This is the topic of the next section.

4. The probability of several correlated equilibria with disjoint supports

In order to find correlated equilibria which are not close to each other, we shall exhibit correlated equilibria with disjoint supports. In terms of the representation of correlated equilibria of the game u as solutions to the system of inequalities

$$\mathbf{C}[u]p \geq 0$$

as described in Section 2, we shall look for solutions using mutually disjoint sets of columns of $\mathbf{C}[u]$. This means that we shall be interested in sets of (column) vectors $\{c_1, \ldots, c_l\}$ in $\mathbf{R}^k$ such that the cone spanned by these vectors, $\text{cone}(c_1, \ldots, c_l)$, intersects the nonnegative orthant in $\mathbf{R}^k$.

In the appendix, we have collected a number of purely technical results on the probability that the cone spanned by l vectors in a k-dimensional space intersects the nonnegative orthant. In the present section, we show how these results may be applied to prove that games with a large number of players have many correlated equilibria which are far apart from each other.

In our application, the dimension k is determined by the number of rows in the matrix $\mathbf{C}(u)$ for u a realization of a random game, cf. Section 2. In order to apply the generel results stated in the Appendix, we need to exhibit columns which are stochastically independent. The next lemmas will show that there is a sufficient supply of subsets of the m^n columns such that this condition is satisfied.

Lemma 2. *Let $\lambda(n)$ be a natural number between* 1 *and m^n such that*

$$\lambda(n)(n(m-1)+1) \leq m^n.$$

Then there exist $\lambda(n)$ mutually disjoint subsets S_j, $j = 1, \ldots, \lambda(n)$, of $\mathscr{A}$ such that (1) $\# S_j$, the cardinality of the set S_j, satisfies

$$\# S_j = \left\lfloor \frac{m^n}{\lambda(n)[n(m-1)+1]} \right\rfloor$$

for each j, and (2)

$$a \in \bigcup_{j=1}^{\lambda(n)} S_j \quad and \quad \bar{a}_{-i} = a_{-i} \quad for\ some\ \bar{a} \in \mathscr{A} \quad and \quad i \in \{1, \ldots, n\}, \quad a \neq \bar{a}$$

$$\Rightarrow \bar{a} \notin \bigcup_{j=1}^{\lambda(n)} S_j.$$

Proof: Choose $\lambda(n)$ satisfying the conditions of the lemma, and let $\succ$ be an arbitrary total ordering of $\mathscr{A}$. Define a subset $D = \{a^{(1)}, \ldots, a^{(R)}\} \subset \mathscr{A}$

inductively as follows: We let $a^{(1)}$ be the $\succ$-minimal element of $\mathscr{A}$ and let

$$G_1 = \{a \neq a^{(1)} \mid \exists i \in \{1, \ldots, n\} : a_{-i} = a_{-i}^{(1)}\};$$

then G_1 contains $n(m-1)$ strategies.

In the kth step, assume that we have defined $a^{(1)}, \ldots, a^{(k-1)} \in \mathscr{A}$ and $G_1, \ldots, G_{k-1}$; choose $a^{(k)}$ as the $\succ$-minimal element of

$$\mathscr{A} \setminus \left[\bigcup_{j=1}^{k-1} G_j \cup \{a^{(1)}, \ldots, a^{(k-1)}\} \right]$$

and let

$$G_k = \{a \neq a^{(k)} \mid \exists i \in \{1, \ldots, n\} : a_{-i} = a_{-i}^{(k)}\};$$

This is possible as long as $k(n(m-1)+1) \leq m^n$; letting R be the maximal such k and partitioning D into $\lambda(n)$ subsets of equal cardinality (and a residual set), we get the result of the lemma. □

Lemma 3. *Let $F = (F_1, \ldots, F_n)$ be a random game over $\mathscr{A}$ such that $u_i(a)$ for $i = 1, \ldots, n$, $a \in \mathscr{A}$ are independent random variables with distributions $F_i(a)$ which are symmetric. If $\lambda(n)$ and $S_1, \ldots, S_{\lambda(n)}$ are as in Lemma 2, then the vectors $c[u](a)$, where*

$$c_{i;\,j,k}[u](a) = \begin{cases} u_i(a^j, a_{-i}) - u_i(a^k, a_{-i}) & \text{if } a_i = a^j, \\ 0 & \text{otherwise,} \end{cases}$$

for $a \in \bigcup_{j=1}^{\lambda(n)} S_j$, are independent and symmetrically distributed random vectors in $\mathbf{R}^{nm(m-1)}$.

Proof: Independence: Let $c[u](a)$ and $c[u](a')$ be different vectors satisfying the assumptions of the lemma. Then there is no $i \in \{1, \ldots, n\}$ such that $a_{-i} = a'_{-i}$, and consequently the coordinates in $c[u](a)$ consist of sums of other random variables than those of $c[u](a')$, so that the two random vectors are independent. Symmetry is a consequence of the next lemma. □

Lemma 4. *Assume that $\mathbf{c}_1$ and $\mathbf{c}_2$ are independent stochastic variables with distributions F_1 and F_2. If F_1 and F_2 are symmetric, then $\mathbf{c}_1 - \mathbf{c}_2$ has a symmetric distribution.*

Proof: Let $\varphi_1(t)$ and $\varphi_2(t)$ be the characteristic functions of $\mathbf{c}_1$ and $-\mathbf{c}_2$, that is $\varphi_1(t) = Ee^{it\mathbf{c}_1}$ and $\varphi_2(t) = Ee^{it\mathbf{c}_2}$. Then φ_1 and φ_2 are real-valued because $\mathbf{c}_1$ and $\mathbf{c}_2$ are symmetrically distributed. If $\varphi_3(t)$ is the characteristic function of $\mathbf{c}_1 - \mathbf{c}_2$, then $\varphi_3(t) = \varphi_1(t)\varphi_2(t)$ since $\mathbf{c}_1$ and $\mathbf{c}_2$ are independent. Thus, $\varphi_3(t)$ is real-valued, and therefore $\mathbf{c}_1 - \mathbf{c}_2$ is symmetrically distributed. □

Now we may state and prove the main result of this section:

Theorem 2. *Let* $1 \le \lambda(n) \le c^n$, *where* $c < m$, *be a function of* n, *such that* $\lambda(n)$ *is a natural number for each* n. *Assume that* F^n *is a random game over* A^n *such that the distributions* $F_i^n(a)$ *are independent and symmetric. Then the probability* π_n *of obtaining at least* $\lambda(n)$ *correlated equilibria with disjoint supports in a game* $u \in \mathscr{U}^n$ *satisfies*

$$\lim_{n\to\infty} \pi_n = 1.$$

Proof: In order to prove the theorem, we use the following general result stated as Corollary 4 in the Appendix: If $\mathbf{c}_1, \ldots, \mathbf{c}_l$ are independent random vectors in $\mathbf{R}^k$, $l \ge k$, with distributions $F_1, \ldots, F_l$ which are symmetric, then

$$\pi = \Pr(\{\mathrm{conv}(\mathbf{c}_1, \ldots, \mathbf{c}_l) \cap \mathbf{R}_+^k \neq \varnothing\}) \ge 1 - \frac{\bar{f}(k,l)}{2^l} \tag{3}$$

where $\bar{f}(k,l) = \binom{l}{0} + \cdots + \binom{l}{k-1}$. Now, since $\lambda(n) \le c^n$ for some $c < m$, we have that $\lambda(n)(n(m-1)+1) \le m^n$ for large enough n, so that Lemma 2 applies. Consequently, by Lemmas 2 and 3, the set $\mathscr{A}$ contains $\lambda(n)$ mutually disjoint sets S_j of cardinality

$$\# S_j = \left\lfloor \frac{m^n}{\lambda(n)(n(m-1)+1)} \right\rfloor,$$

such that the set of column vectors $c[u](a)$, for $a \in \bigcup_{j=1}^{\lambda(n)} S_j$, are independent and symmetrically distributed random vectors in $\mathbf{R}^{nm(m-1)}$.

For each j, we may apply (3) to the set S_j; using the fact that $\bar{f}(k,l) \le 2l^{k-1}$ (for completeness, a proof is given in the Appendix); since a correlated equilibrium with support in S_j corresponds to a a convex combination of $(c[u](a))_{a \in S_j}$ belonging to $\mathbf{R}_+^{nm(m-1)}$, we have that the probability of such an event is at least

$$\left[1 - \frac{2\left(\frac{m^n}{\lambda(n)(m-1)n}\right)^{n(m-1)m-1}}{2^{\lfloor m^n/(\lambda(n)(n(m-1)+1))\rfloor}} \right]$$

for n large enough; since there are $\lambda(n)$ such sets S_j, we get by independence that

$$\pi_n \ge \left[1 - \frac{2\left(\frac{m^n}{\lambda(n)(m-1)n}\right)^{n(m-1)m-1}}{2^{\lfloor m^n/(\lambda(n)(n(m-1)+1))\rfloor}} \right]^{\lambda(n)}$$

for large n.

Now,

$$\ln(1-x) \geq \frac{-x}{1-x};$$

hence

$$\ln \pi_n \geq -\frac{\lambda(n)x_n}{1-x_n},$$

where

$$x_n = \frac{2\left[\dfrac{m^n}{\lambda(n)(m-1)n}\right]^{n(m-1)m-1}}{2^{\lfloor m^n/(\lambda(n)((m-1)n+1))\rfloor}}.$$

Clearly, $0 \leq x_n \leq 1$. We claim that $\lim_{n\to\infty} \lambda(n)x_n = 0$, which in its turn implies that x_n is bounded away from 1, so that $\lim_{n\to\infty} \ln \pi_n = 0$. Now

$$\ln(\lambda(n)x_n) \leq n \ln c + \ln x_n \leq n \ln c + n^2(m-1)m \ln m$$
$$- \left\lfloor \frac{m^n}{\lambda(n)(n(m-1)+1)} \right\rfloor \ln 2 \underset{n\to\infty}{\longrightarrow} -\infty,$$

which proves our claim and thereby the theorem. □

The result of Theorem 2 tells us that in most cases (in the sense that the probability of this event tends to 1 as the number of players grows large), the convex set $\mathscr{C}(u)$ of correlated equilibria is quite dispersed in $\triangle$, since it meets a large number (namely $\lambda(n)$) of the faces of $\triangle$ of relatively small dimension (namely $\# S_j$). Since the intersection of $\mathscr{C}(u)$ with any of these mutually disjoint faces of $\triangle$ must contain an extreme point of $\mathscr{C}(u)$, we have the following corollary pertaining to the number of extreme points of the set of correlated equilibria of a game.

Corollary 1. *Let $1 \leq \lambda(n) \leq c^n$, where $c < m$, be a function of n, such that $\lambda(n)$ is a natural number for each n. Assume that F^n is a random game over A^n such that the distributions $F_i^n(a)$ are independent and symmetric. Then the probability π_n of obtaining at least $\lambda(n)$ affinely independent extreme points of $\mathscr{C}(u)$ for $u \in \mathscr{U}^n$ satisfies*

$$\lim_{n\to\infty} \pi_n = 1.$$ □

Appendix. Some results on the probability that the cone spanned by *l* random vectors intersects the nonnegative orthant

Let k and l be natural numbers, $l \geq k > 0$, and let $c_1, \dots, c_l \in V$, where V is a k-dimensional (real) vector space. Then $c_1, \dots, c_l$ is maximally independent (m.i.) if every subset of k vectors is linearly independent. Let $\Sigma_{k,l}$ be the family of m.i. l-tuples $(c_1, \ldots, c_l)$ of vectors in V. We define

$$f(k,l) = \max_{\substack{(c_1,\ldots,c_l)\in\Sigma_{k,l} \\ x\in V\setminus\{0\}}} \#\{(\varepsilon_1,\ldots,\varepsilon_l)\in\{-1,1\}^l \mid x\notin \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_l c_l)\}.$$

Lemma 5. *Let $2 \le k \le l$. Then $f(k,l+1) \le f(k,l) + f(k-1,l)$.*

Proof: Let $(c_1,\ldots,c_{l+1}) \in \Sigma_{k,l+1}$, and let $x \in V\setminus\{0\}$. We show that

$$\#\{(\varepsilon_1,\ldots,\varepsilon_{l+1})|x\notin \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_{l+1}c_{l+1})\} \le f(k,l) + f(k-1,l). \quad (4)$$

If $x\notin \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_{l+1}c_{l+1})$, then also $x\notin \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_l c_l)$. Clearly, $(c_1,\ldots,c_l)\in\Sigma_{k,l}$. Hence, there are at most $f(k,l)$ such l-tuples. Now, if $x\in \text{span}(c_{l+1})$, then

$$\#\{(\varepsilon_1,\ldots,\varepsilon_{l+1})|x\notin \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_{l+1}c_{l+1})\} \le f(k,l)$$

and (4) is satisfied since $f(k-1,l)\ge 0$. Otherwise, there may be l-tuples $(\varepsilon_1 c_1,\ldots,\varepsilon_l c_l)$ such that

$$x\notin \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_l c_l, c_{l+1}) \cup \text{cone}(\varepsilon_1 c_1,\ldots,\varepsilon_l c_l, -c_{l+1}). \quad (5)$$

Consider the quotient space $V' = V/\text{span}(c_{l+1})$, and let $q: V\to V'$ be the quotient mapping. Then $q(x)\neq 0$ and $q(c_1),\ldots,q(c_l)$ is m.i. (in V'). Clearly, $q(x)\notin \text{cone}(\varepsilon_1 q(c_1),\ldots,\varepsilon_l q(c_l))$. Hence the number of l-tuples which satisfy (5) is at most $f(k-1,l)$. Thus, $f(k,l+1)\le f(k,l)+f(k-1,l)$. □

Theorem 3. *Let $\bar f(k,l) = \binom{l}{0} + \cdots + \binom{l}{k-1}$. Then $f(k,l)\le \bar f(k,l)$.*

Proof: We show by induction on l that $f(k,l)\le\bar f(k,l)$ for $1\le k\le l$. We have that

(i) $f(1,l) = 1 = \binom{l}{0}$ for $l = 1,2,\ldots$, and

(ii) $f(k,k) = 2^k - 1 = \binom{k}{0} + \cdots + \binom{k}{k-1} = \bar f(k,k)$, $k = 1,2,\ldots$.

Now suppose that the theorem is true for all $l'\le l$. If $l\ge k$, then by the induction hypothesis

$$f(k,l+1) \le f(k,l) + f(k-1,l) \le \sum_{i=0}^{k-1}\binom{l}{i} + \sum_{i=0}^{k-2}\binom{l}{i}$$

$$= \binom{l}{0} + \left[\binom{l}{1} + \binom{l}{0}\right] + \cdots + \left[\binom{l}{k-1} + \binom{l}{k-2}\right]$$

$$= \sum_{i=0}^{k-1}\binom{l+1}{i}.$$

Since $f(l+1,l+1) = \bar f(l+1,l+1)$ by (ii), we have that $f(k,l+1)\le \bar f(k,l+1)$ for all $k\le l+1$. □

Corollary 2. $f(k,l) \leq 2l^{k-1}$.

Proof: Since $\binom{l}{i+1} \leq l\binom{l}{i}$, we get that

$$f(k,l) \leq \sum_{i=0}^{k-1}\binom{l}{i} \leq \sum_{i=0}^{k-1} l^i = \frac{l^k-1}{l-1} \leq 2l^{k-1}$$

for $l > 1$. □

Corollary 3. *Let* $c_1, \ldots, c_l \in \mathbf{R}^k$, $l \geq k$. *Then*

$$\#\{(\varepsilon_1, \ldots, \varepsilon_l) \in \{-1,1\}^l \,|\, \mathrm{conv}(\varepsilon_1 c_1, \ldots, \varepsilon_l c_l) \cap \mathbf{R}^k_+ \neq \emptyset\} \geq 2^l - \bar{f}(k,l).$$

Proof: Let $d_1(t), \ldots, d_l(t)$, $t = 1, 2, \ldots$ satisfy:
(1) $d_j(t) \in \mathbf{R}^k$ for all j and t, and $\lim_{t\to\infty} d_j(t) = 0$, $j = 1, \ldots, l$.
(2) $(c_1 + d_1(t), \ldots, c_l + d_l(t))$ is m.i. for $t = 1, 2, \ldots$.
By Theorem 3,

$$\#\{(\varepsilon_1, \ldots, \varepsilon_l) \in \{-1,1\}^l \,|\, (1, \ldots, 1) \in \mathrm{cone}(\varepsilon_1(c_1+d_1(t)), \ldots, \varepsilon_l(c_l+d_l(t)))\}$$
$$\geq 2^l - \bar{f}(k,l). \tag{6}$$

Clearly, (6) implies

$$\#\{(\varepsilon_1, \ldots, \varepsilon_l) \in \{-1,1\}^l \,|\, \mathrm{conv}(\varepsilon_1(c_1+d_1(t)), \ldots, \varepsilon_l(c_l+d_l(t)) \cap \mathbf{R}^k_+ \neq \emptyset\}$$
$$\geq 2^l - \bar{f}(k,l).$$

Because $d_j(t) \to 0$ for $t \to \infty$, $j = 1, 2, \ldots, l$, the corollary follows. □

Corollary 4. *If* $\mathbf{c}_1, \ldots, \mathbf{c}_l$ *are independent random vectors in* $\mathbf{R}^k$, $l \geq k$, *with distributions* $F_1, \ldots, F_l$ *which are symmetric, then*

$$\pi = \Pr(\{\mathrm{conv}(\mathbf{c}_1, \ldots, \mathbf{c}_l) \cap \mathbf{R}^k_+ \neq \emptyset\}) \geq 1 - \frac{\bar{f}(k,l)}{2^l}.$$

Proof: For $(\varepsilon_1, \ldots, \varepsilon_l) \in \{-1,1\}^l$ let

$$\chi_{\varepsilon_1,\ldots,\varepsilon_l}(c_1, \ldots, c_l) = \begin{cases} 1 & \text{if } \mathrm{conv}(\varepsilon_1 c_1, \ldots, \varepsilon_l c_l) \cap \mathbf{R}^k_+ \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

By corollary 3,

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_l)\in\{-1,1\}^l} \chi_{\varepsilon_1,\ldots,\varepsilon_l}(c_1, \ldots, c_l) \geq 2^l - \bar{f}(k,l) \tag{7}$$

for all $c_1, \ldots, c_l$. By the symmetry of $F_1, \ldots, F_l$,

$$\int_{\mathbf{R}^{kl}} \chi_{\varepsilon_1,\ldots,\varepsilon_l}(c_1,\ldots,c_l)\, dF_1(c_1)\cdots dF_l(c_l) = \pi \tag{8}$$

for all $(\varepsilon_1,\ldots,\varepsilon_l) \in \{-1,1\}^l$. From (7) and (8), $2^l\pi \geq 2^l - \bar{f}(k,l)$, that is

$$\pi \geq 1 - \frac{\bar{f}(k,l)}{2^l}. \qquad \square$$

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