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Value theory without symmetry*

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Abstract. We investigate quasi-values of finite games – solution concepts that satisfy the axioms of Shapley (1953) with the possible exception of symmetry.

Following Owen (1972), we define "random arrival", or *path*, values: players are assumed to "enter" the game randomly, according to independently distributed arrival times, between 0 and 1; the payoff of a player is his expected marginal contribution to the set of players that have arrived before him.

The main result of the paper characterizes quasi-values, symmetric with respect to some coalition structure with infinite elements (types), as *random path* values, with identically distributed random arrival times for all players of the same type.

General quasi-values are shown to be the random order values (as in Weber (1988) for a finite universe of players).

Pseudo-values (non-symmetric generalization of semivalues) are also characterized, under different assumptions of symmetry.

Key words: quasi-values, their representation as random path values

1. Introduction

The Shapley value is one of the central concepts in the cooperative game theory. Defined by Shapley (1953) as a solution concept on the set of finite superadditive games, a *value* satisfies the following requirements (axioms):

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(i) *efficiency* (the payoffs of all players add up to the worth of the grand coalition);

(ii) *additivity* (the value of the sum of two games is the sum of the values of the games);

(iii) the *zero player* axiom (the payoff of a player who contributes nothing to any coalition is zero);

(iv) the *symmetry* axiom (the payoffs are independent of the way the players are named, i.e., they are covariant under permutations of the universe of players).

All four axioms can be stated for a solution concept defined on the space of all finite games on a given universe of players. Another frequently assumed axiom is

(v) *positivity* (the payoff of a player is non-negative in monotonic games). It is a part of both Aumann and Shapley (1974) definition of value on games with a continuum of players, and Dubey et al (1981) definition of semivalue.

The first four axioms lead to a unique outcome, as shown in Shapley (1953), and the resulting solution is called the Shapley value. It have often been remarked, however, that the symmetry assumption is unrealistic in many applications (for a comprehensive summary see Kalai and Samet (1988)). It makes the characterization of solutions satisfying the above axioms with the possible exception of symmetry, an important topic. We call such solutions *quasi-values*.

The first possible relaxation of the symmetry axiom that comes to mind is the following one. Consider a *coalition structure* – a finite or countable partition of the universe of players, – whose elements we call *types*. A quasi-value is called symmetric with respect to the structure if it treats equally symmetric players of the same type, or, alternatively, is covariant under those permutations of the universe of players that preserve each type. One of the main targets of this paper is to describe all such quasi-values.

The Shapley value possesses the following intuitive interpretation, proposed by Owen (1968): Given a universe of players and a game with a finite support (carrier), we suppose that the players are arriving randomly to a meeting. Their arrival times are independent random variables, uniformly distributed on the unit interval $[0,1]$. The payoff of a player in the Shapley value is his expected marginal contribution to the set of players that have arrived before him.

The above interpretation of the Shapley value can be clearly used as an alternative definition. It is, moreover, very suggestive (Owen (1972)) – the aforementioned procedure will work if, instead of taking each random arrival time to be uniform, we consider a function (*path*) that attaches to each player an arbitrary distribution function (of his arrival time) on $[0,1]$. It can be instantly verified that it defines a quasi-value. We call it a *path value*.

In this paper we present a description of the extreme points of the (compact and convex) set of quasi-values symmetric with respect to a given coalition structure. Obviously, if the arrival times of the players of the same type are (independently) identically distributed, then the resulting path value is symmetric with respect to the coalitional structure. It turns out that if the types are infinite, then the extreme points are precisely these path values. Equivalently, any quasi-value symmetric with respect to such coalition structure is a *random path* value: assuming that a path is drawn at random according to some distribution, an expected path value is taken. The assumption of infiniteness of each type is crucial – the result breaks down even if only one type is finite.

To exemplify the result, note that the coalition structure value of Owen, certainly symmetric with respect to a given coalition structure, is indeed a random path value. Here the time is partitioned into disjoint intervals, one for each type. Arrival times of players of each type are uniformly distributed on the corresponding interval, which determines a path. It is then made random by taking a random, uniform matching between the intervals and the types.

Another well-known non-symmetric generalization of the Shapley value is the class of weighted values, defined explicitly as path values: we call a positive function w, defined on the universe of players, a weight function. The w-weighted value is a random arrival value with the random arrival time of a player is t^w-distributed. The w-weighted value is symmetric with respect to a coalition structure if w is constant on each type. In particular, w-weighted value is the Shapley value for a constant w. The weighted values are symmetric with respect to the group of permutations preserving the weight function w, and are of interest since the weights are the proportions in which the players share the windfall in unanimity games.

In the general non-symmetric case the quasi-values possess the following description. Given an order on the universe of players, an *order value* associated with this order gives a player his marginal contribution to the set of players preceding him. We show that the order values are the extreme points of the quasi-values, and any quasi-value is a mixture of order values, i.e., a *random order* value. Note that the Shapley value is obtained in this way by taking this distribution to be uniform. In the case of coalition structure values of Owen, the distribution is uniform on orders consistent with the coalition structure, i.e., the orders obtained by arranging first the types, and only then the players inside each type. Our result generalizes the one by Weber (1988), who proved it for a finite universe.

Our paper is organized as follows. In Section 2 we recall some basic definitions. In Section 3 the path values are formally defined, and examples are given. Characterizations of quasi-values are brought in Section 4. Characterizations of pseudo-values are given in Section 5.

2. Preliminaries

Let U be a set (the *universe*). The members of U are called *players*, the subsets of U – *coalitions*, and set functions which are 0-valued at $\emptyset$ – *games*. Denote by $Fin(U)$ the set of all finite coalitions. Given a game v, $T \subset U$ is a *support* of v if for each $S \subset U$, $v(S \cap T) = v(S)$. Define a restriction of v to $S \subset U$ by v_S. Define G_U to be the space of games with finite support in U. The minimal finite support of a game $v \in G_U$ is denoted by $Supp(v)$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$.

The group of permutations on a set V is denoted by S_V, and a subgroup of those permutations that fix all but finitely many elements is denoted by S'_V.

Let H be a subgroup of S_U and Q – an H-invariant subspace of G_U. Then every $\theta \in S_U$ induces a linear operator $\theta : Q \to Q$ by $\theta(v)(S) = v(\theta(S))$, where $\theta(S)$ is the image of S under θ, and for a given (linear) operator $\psi : Q \to Q$, θ induces $\theta\psi$ defined by $\theta\psi(v) = \psi(\theta v)$. We say that an operator $\psi : Q \to Q$ is *positive* if $\psi(v)$ is monotonic whenever v is so, and *efficient* if $\psi(v)(U) = v(U)$ for each $v \in Q$. The operator ψ satisfies *zero player* axiom if $Supp(\psi(v)) \subset$

$Supp(v)$. An operator ψ is called H-symmetric if $\theta\psi\theta^{-1} = \psi$ for every $\theta \in H$. Denote by FA_U the subspace of all additive set functions in G_U.

Let Q be a subspace of G_U that includes FA_U. We call a positive linear operator $\psi : Q \to FA_U$, that satisfies the zero layer axiom, a *quasi-value*, and a positive linear projection $\psi : Q \to FA_U$ a *pseudo-value*. Note that any quasi-value is always a pseudo-value, and also that any pseudo-value satisfies the zero player axiom (this is implicit in the proof of the first part of the Lemma in Dubey et al (1981)).

The set of H-symmetric pseudo-values on G_U is denoted by $PV(U)[H]$, and that of quasi-values by $V(U)[H]$. The set of pseudo-values (respectively, quasi-values) are denoted by $PV(U)$ (respectively, $V(U)$).

Remark 1: For any H, both $V(U)[H]$ and $PV(U)[H]$ are convex, and they are compact in the minimal topology in which for every $v \in G_U$, $a \in U$, the function $\varphi(v)(a)$ is continuous (Ruckle (1982) and Monderer (1988)).

A finite or countable partition of U, Π, is called a *coalition structure*. The elements of Π are called *types*. The subgroup of S_U that consists of permutations θ preserving the structure, i.e., $\theta(\pi) = \pi$ for each $\pi \in \Pi$, is identifiable with $\prod_{\pi \in \Pi} S_\pi$. A $\prod_{\pi \in \Pi} S_\pi$-symmetric pseudo-value is called Π-*symmetric*. Observe that

$$PV(U)\left[\prod_{\pi \in \Pi} S_\pi\right] = PV(U)\left[\prod_{\pi \in \Pi} S_\pi \cap S'_U\right].$$

To simplify the notations, we shorten $PV(U)[\prod_{\pi \in \Pi} S_\pi]$ to $PV(U)[\Pi]$, and $V(U)[\prod_{\pi \in \Pi} S_\pi]$ to $V(U)[\Pi]$.

Remark 2: If each $\pi \in \Pi$ is infinite, then any positive, efficient and Π-symmetric operator φ on G_U satisfies the zero player axiom, i.e., φ is a Π-symmetric quasi-value.

Given a compact Hausdorf space X, we denote by $M(X)$ the space of probability measures on Bair-measurable subsets of X. We endow $M(X)$ with the weak topology of measures, in which this space is compact. If X is metrizable, then so is $M(X)$, and Borel and Bair σ-fields on X coincide. By $NA(X)$ we denote the subset of non-atomic probabilities. If a group H acts on X, then it induces an action on $M(X)$, by $(\theta P)(A) = P(\theta A)$ for any $P \in M(X)$, $\theta \in H$, and a Bair set A. If $\theta P = P$ for each $\theta \in H$ then P is called H-invariant.

Following Owen (1972), we define a *multilinear extension* to $[0,1]^U$ of a game $v \in G_U$, $\bar{v} : [0,1]^U \to R$, by

$$\bar{v}(x) = \sum_{T \subset Supp(v)} \prod_{a \in T} x(a) \prod_{a \in Supp(v) - T} (1 - x(a)) v(T).$$

In other words, we think x as a vector of probabilities according to which the players, independently of each other, join a random coalition, and $\bar{v}(x)$ represents the expected worth of this random coalition.

The function $\bar{v}$ is determined by its restriction on $Supp(v)$, $\bar{v}|_{[0,1]^{Supp(v)}}$. Therefore, by (Owen 1972), $\bar{v}$ is a unique multilinear function on $[0,1]^U$ that coincides with v on the indicator functions (when the indicator functions are identified with appropriate sets). It also enables us to define the differentiability of $\bar{v}$: for every $a \in U$ the derivative of $\bar{v}$ with respect to a-th coordinate, $(\partial/\partial x_a)\bar{v}$, is the corresponding derivative of $\bar{v}|_{[0,1]^S}$, for some $S \in Fin(U)$ that contains $Supp(v)$. The derivatives are well-defined, and satisfy

$$\frac{\partial}{\partial x_b}\bar{v}(x) = \sum_{T \subset Supp(v)} \prod_{a \in T-\{b\}} x(a) \prod_{a \in Supp(v)-T} (1 - x(a))[v(T) - v(T - \{b\})]$$

for every $a \in U$ and $x \in [0,1]^U$. In other words, $(\partial/\partial x_b)\bar{v}(x)$ is an expected marginal contribution of the player b to the random coalition, which each player $a \neq b$ joins with the probability $x(a)$.

3. Definitions and examples

3.1. Path values

In this subsection we introduce our concept of path values. We fix a universe of players, U.

Definition 3.1. *A function $\gamma : [0,1] \to [0,1]^U$ is called a path if for each $a \in U$, $\gamma(t)(a)$ is a distribution function on $[0,1]$, and for each $a \neq a'$ the discontinuity sets of $\gamma(.)(a)$, $\gamma(.)(a')$ are disjoint.*

Definition 3.2. *Given a path γ on U, define for every $v \in G_U$ and $a \in U$*

$$\varphi_\gamma(v)(a) = E[v(\{b \in T_a \mid X_b \leq X_a\}) - v(\{b \in T_a \mid X_b < X_a\})], \tag{3.1}$$

where $T_a = Supp(v) \cup \{a\}$, $\{X_b\}_{b \in T_a}$ are independent random variables such that each X_b is $\gamma(.)(b)$ distributed, and E is the expectation operator. Note that $\varphi_\gamma(v)(a) = 0$ for $a \notin Supp(v)$. This definition induces a linear operator $\varphi_\gamma : G_U \to FA_U$. We call it a path value.

The definition of a path value φ_γ is based on the principle of random arrival: $\{X_a\}_a$ are random arrival times of the players, and player's a share is his expected marginal contribution to the set of players that have arrived earlier. It is clearly a quasi-value. We will check only the efficiency: for $v \in G_U$,

$$\begin{aligned}
&\varphi_\gamma(v)(U)\\
&\quad = \varphi_\gamma(v)(Supp(v))\\
&\quad = \sum_{a \in Supp(v)} E[v(\{b \in Supp(v) \mid X_b \leq X_a\}) - v(\{b \in Supp(v) \mid X_b < X_a\})]\\
&\quad = E\left(\sum_{a \in Supp(v)} [v(\{b \in Supp(v) \mid X_b \leq X_a\}) - v(\{b \in Supp(v) \mid X_b < X_a\})]\right)
\end{aligned} \tag{3.2}$$

$$= E(v(Supp(v)) - v(\emptyset)) = v(I). \tag{3.3}$$

When equalizing (3.2) and (3.3), we use the fact that $X_b \neq X_a$ with probability 1 for all $b \in Supp(v) - \{a\}$. This fact holds due to the assumption that distributions of the arrival times have no common discontinuities (Definition 3.1). Without this assumption the efficiency is lost (take the extreme case of deterministic arrival times).

Remark 3: A complementary definition of the path value is available, which stresses more the geometrical role of a path. Note that for every $v \in G_U$ and $a \in U$,

$$\varphi_\gamma(v)(a) = \int_0^1 E[v(\{b \in T_a \mid X_b \leq t\}) - v(\{b \in T_a \mid X_b < t\}) \mid X_a = t]\, d\gamma(t)(a)$$

$$= \int_0^1 \frac{\partial}{\partial x_a} \bar{v}(\gamma(t))\, d\gamma(t)(a).$$

This is precisely a definition suggested in Owen (1972), for differentiable $\gamma(t)(a)$.

Remark 4: Given a path γ, each $\theta \in S_U$ induces a path $\theta\gamma$, by $(\theta\gamma)(t)(a) = \gamma(t)(\theta(a))$ for any $a \in U$, $t \in [0,1]$. Then $\theta\varphi_\gamma\theta^{-1} = \varphi_{\theta\gamma}$.

Proposition 3.3. *If $s : [0,1] \to [0,1]$ is a nondecreasing and surjective function, then for each path γ*

$$\varphi_\gamma = \varphi_{s\gamma},$$

where the path $s\gamma$ is defined by $(s\gamma)(t) = \gamma(s(t))$.

Proof: If X_a is a random variable with distribution function $(s\gamma)(.)(a)$, then the random variable $s(X_a)$ is $\gamma(.)(a)$-distributed. The result now follows from Definition 3.2, since s preserves order. ∎

All previously known quasi-values on the space of finite games are particular cases of path values, as is shown Example 3.4, and later, in Example 3.6.

Example 3.4. Consider a positive function $w \in [0,\infty)^U$, the weight function, and an ordered partition $(U_1, \ldots, U_k)$ of U, and define a path $\gamma^{w,(U_1,\ldots,U_k)}$ by

$$\gamma^{w,(U_1,\ldots,U_k)}(t)(a) = (kt - i + 1)^{w(a)} I_{[(i-1)/k,\, i/k]}(t) + \sum_{j=i+1}^{k} I_{((j-1)/k,\, j/k]}(t)$$

for $a \in U_i$ and $1 \leq i \leq k$. In other words, player's a arrival time belongs to the interval $[(i-1)/k, i/k]$, with a distribution function $(kt - i + 1)^{w(a)}$ there, provided $a \in U_i$. The quasi-value $\varphi_{\gamma^{w,(U_1,\ldots,U_k)}}$ is a $w, (U_1, \ldots, U_k)$-weighted value, defined in Kalai and Samet (1988). If the partition is trivial, i.e., $k = 1$, then it is the w-weighted value. For $w \equiv 1$ it is the symmetric, Shapley, value.

There is a canonical way to construct new quasi-values from the path values, by allowing a path to be chosen at random.

Definition 3.5. *Given a probability space (Γ, Σ, μ) on the set of paths on U, such that for each $v \in G_U$, $a \in U$ the function $\varphi_\gamma(v)(a)$ is μ-integrable, define the operator $E_{d\mu(\gamma)}(\varphi_\gamma)$ by*

$$[E_{d\mu(\gamma)}(\varphi_\gamma)](v)(a) = \int_\Gamma \varphi_\gamma(v)(a)\, d\mu(\gamma).$$

One can easily verify that $E_{d\mu(\gamma)}(\varphi_\gamma)$ is a quasi-value on G_U. We call any quasi-value of this form a *random* path value.

Given a coalition structure Π on U and a vector $F = (F_\pi)_{\pi \in \Pi}$ of continuous distribution function on $[0,1]$, define a path γ_F^Π by $\gamma_F^\Pi(t) = \sum_{\pi \in \Pi} F_\pi(t) \cdot \pi$ (π, a coalition, also stands for the indicator function I_π). In other words, the players from the type π have the same distribution of their arrival times – F_π. Such path is called a Π-symmetric path, or Π-path. For the sake of convenience we will not distinguish between such F and $F \in (NA([0,1]))^\Pi$ – the vector of corresponding probability measures.

By Remark 4 the quasi-value $\varphi_{\gamma_F^\Pi}$ is Π-symmetric. The quasi-value $E_{d\mu(F)}(\varphi_{\gamma_F^\Pi})$ for any measure μ on $(NA([0,1]))^\Pi$ is also Π-symmetric.

Example 3.6. Let $\Pi = \{U_1, \ldots, U_k\}$ be a coalition structure on U. Taking $w = 1$ in Example 3.4, $\varphi_{\gamma^{w,(U_1,\ldots,U_k)}} = \varphi_{\gamma_F^\Pi}$, where F_i is a uniform distribution on $[(i-1)/k, i/k]$. Define a random path value φ_Π as

$$\varphi_\Pi = \frac{1}{k!} \sum_{\theta \in S_{\{1,\ldots,k\}}} \varphi_{\gamma^\Pi_{F_{\theta(1)},\ldots,F_{\theta(k)}}}.$$

The Π-symmetric φ_Π is precisely the Π-value of Owen.

Definition 3.7. *Given a coalition structure Π on U and a probability vector $q \in M(\Pi)$ (i.e., probability vector on Π), which is strictly positive, i.e., $q_\pi > 0$ for each $\pi \in \Pi$, the vector $F = (F_\pi)_{\pi \in \Pi} \in (M([0,1]))^\Pi$ is called q-normalized if*

$$\sum_{\pi \in \Pi} q_\pi \cdot F_\pi(t) = t$$

for each $t \in [0,1]$.

The set of q-normalized vectors is denoted by $(NA([0,1]))_q^\Pi$; it is a subset of $(NA([0,1]))^\Pi$ and a compact subspace of $(M([0,1]))^\Pi$.

Proposition 3.8. *Given a coalition structure Π on U, strictly positive $q \in M(\Pi)$ and $F \in (NA([0,1]))^\Pi$, there is a unique q-normalized $F_q \in (NA([0,1]))_q^\Pi$, such that $\varphi_{\gamma^\Pi_{F_q}} = \varphi_{\gamma^\Pi_F}$.*

Proof: Define for every $t \in [0,1]$ $s(t) = \sum_{\pi \in \Pi} q_\pi \cdot F_\pi(t)$. For each $\pi \in \Pi$ define the function $(F_q)_\pi$ by $(F_q)_\pi(t) = F_\pi(\max[s^{-1}(t)])$. One can easily verify that $(F_q)_\pi$ are well-defined and continuous distribution functions, and the corresponding vector F_q is q-normalized. Since $s\gamma^{\Pi}_{F_q} = \gamma^{\Pi}_F$, Proposition 3.3 implies that $\varphi_{\gamma^{\Pi}_{F_q}} = \varphi_{\gamma^{\Pi}_F}$. The uniqueness of such path will be proved in Corollary 5.7. ■

3.2. The random order values

Given a universe of players U and $T \subset U$, denote by Ω_T the set of all (irreflexive and transitive) orders on T. This set is a compact Hausdorf space in the minimal topology in which sets of the form $\{\omega \in \Omega_I \mid b \prec_\omega a\}$ are open.

Definition 3.9. *Given a probability measure $P \in M(\Omega_U)$, denote for every $S \in Fin(U)$, $v \in G_S$, $a \in S$*

$$\phi_P(v)(a) = \int_{\Omega_U} [v(\{b | b \preceq_\omega a\}) - v(\{b | b \prec_\omega a\})]\, dP(\omega).$$

Observe that ϕ_P defines a quasi-value on G_U. We call ϕ_P a random order value.

Given $\omega \in \Omega_U$, denote by δ_ω the Dirac measure supported on $\{\omega\}$. The quasi-value ϕ_{δ_ω} is called an *order value*.

Remark 5: Given $v \in G_U$, $a \in U$, the function $\phi_{\delta_\omega}(v)(a)$ is continuous on Ω_U (in ω), and $\phi_P(v)(a)$ is continuous on $M(\Omega_U)$ (in P).

The random path values, and the path values in particular, are random order values. For instance, given a path γ on U, $\varphi_\gamma = \phi_{P(\gamma)}$, where

$$P(\gamma)(a_1 \prec_\omega a_2 \prec_\omega \cdots \prec_\omega a_n) = P(X_1 < X_2 < \cdots < X_n),$$

for independent $\{X_i\}_{i=1}^n$, such that X_i is $\gamma(.)(a_i)$-distributed.

Definition 3.10. *Define an action of a permutation $\theta \in S_U$ on the order $\omega \in \Omega_U$ by*

$$a \prec_{\theta(\omega)} b \Leftrightarrow \theta(a) \prec_\omega \theta(b)$$

for each $a, b \in U$.

Proposition 3.11. (i) *For each $P \in M(\Omega_U)$, $\theta \in S_U$ and $v \in G_U$,*

$$\theta^{-1}\phi_P\theta(v) = \phi_{\theta P};$$

(ii) *If H is a subgroup of S_U and $P \in M(\Omega_U)$ is H-invariant, then the quasi-value ϕ_P is H-symmetric.*

(iii) *If H is locally finite (i.e., its finitely generated subgroups are all finite) and the quasi-value ϕ_P, $P \in M(\Omega_U)$, is H-symmetric, then there is an H-invariant $\bar{P} \in M(\Omega_U)$ such that $\phi_P = \phi_{\bar{P}}$.*

Proof: (i) and (ii) are self-evident;

(iii) By (i) and the H-symmetry of ϕ_P, for any finite subgroup H' of H an H'-invariant measure $P_{H'} \in M(\Omega_U)$, defined by

$$P_{H'} = \frac{1}{|H'|} \sum_{\theta \in H'} \theta P,$$

satisfies $\phi_P = \phi_{P_{H'}}$. Take a subnet $(P_{H'_\alpha})_\alpha$ of $(P_{H'})$ that converges weakly to $\bar{P} \in M(\Omega_U)$, which is necessarily H-invariant. Then, by Remark 5, $\phi_P(v)(a) = \lim_\alpha \phi_{P_{H'_\alpha}}(v)(a) = \phi_{\bar{P}}(v)(a)$ for any $v \in G_U$, $a \in U$. ■

4. Quasi-values – a characterization

4.1. General quasi-values

The following result provides a characterization of all quasi-values on G_U.

Theorem 4.1. *Given a quasi-value φ on G_U, there is $P \in M(\Omega_U)$ such that $\varphi = \phi_P$.*

Proof: By Theorem 13 of Weber (1988) there exists probability measure p_S on Ω_S for each $S \in Fin(U)$ such that $\varphi|_{G_S} = \phi_{p_S}$. Extend p_S arbitrarily to a probability measure on Ω_U. The net $\{P_S\}_{S \in Fin(U)}$ has a subnet $\{P_{S_\alpha}\}_{\alpha \in A}$, that converges weakly to some $P \in M(\Omega_U)$. Thus $\varphi(v)(a) = \lim_\alpha \phi_{P_{S_\alpha}}(v)(a) = \phi_P(v)(a)$ for any $v \in G_U$, $a \in U$, by Remark 5. ■

By Proposition 3.11, we have the following corollary.

Corollary 4.2. *If a quasi-value φ on G_U is H-symmetric for some locally finite subgroup H of S_U, then the measure P in Theorem 4.1 can be chosen H-invariant.*

Corollary 4.3. *The order values $\{\phi_{\delta_\omega}\}_{\omega \in \Omega_U}$ are precisely the extreme points of the set $V(U)$.*

Proof: No extreme point can be obtained by averaging (mixing) of points in a set excluding it. By Theorem 4.1 any quasi-value can be represented as a random order value, and so the set of extreme points of $V(U)$ is partial to $\{\phi_{\delta_\omega}\}_{\omega \in \Omega_U}$.

We prove next that if $\phi_{\delta_\omega} = \phi_P$ for some $\omega \in \Omega_U$ and $P \in M(\Omega_U)$, then $P = \delta_\omega$. Indeed, given $a, b \in U$ such that $b \prec_\omega a$, consider a game v given by

$$v(S) = \begin{cases} 1, & a, b \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\phi_{\delta_\omega}(v) = \phi_P(v),$$

$$\phi_P(v)(a) = \Pr_P(\{\omega' \in \Omega_U \mid b \prec_{\omega'} a\}),$$

and

$$\phi_{\delta_\omega}(v)(a) = 1,$$

it follows that

$$\Pr_P(\{\omega' \in \Omega_U \,|\, b \prec_{\omega'} a\}) = 1.$$

It holds for all such a, b, and so $P = \delta_\omega$.

Thus, by definition, every order value ϕ_{δ_ω} is an extreme point of $V(U)$, and so $\{\phi_{\delta_\omega}\}_{\omega \in \Omega_U}$ is included in the set of extreme points of $V(U)$. Since we have already established the other inclusion, equality must hold. ■

4.2. Partially symmetric quasi-values

We now present a characterization of the quasi-values symmetric with respect to a coalition structure with infinite types.

Theorem 4.4. *Let Π be a coalition structure on U in which all types are infinite, and φ – a Π-symmetric quasi-value on G_U. For any strictly positive $q \in M(\Pi)$ there is a probability measure $\mu \in M[(NA([0,1]))_q^{\Pi}]$ such that $\varphi = E_{d\mu(F)}(\varphi_{\gamma_F^{\Pi}})$.*

Proof: Assume that $\Pi = \{U_j\}_{j=0}^n$, where $n \in N \cup \{\infty\}$. A type function t is defined by $t(a) = j$ if $a \in U_j$. Fix a countable subset of U, V, which has an infinite intersection with each type. For each j pick an increasing family $\{V_{j,m}\}_{m=1}^{\infty}$ such that $\bigcup_{m \geq 1} V_{j,m} = V \cap U_j (= V^j)$ and $|V_{j,m}| = [m \cdot q_j]$ (i.e., the largest integer less or equal to $m \cdot q_j$). Denote $V_m = \bigcup_{j=0}^n V_{j,m}$. By Theorem 4.1 and Corollary 4.2 there exists $\prod_{j=0}^n S_{V_{j,m}}$-invariant probability measure p_m on Ω_{V_m} such that $\varphi|_{G_{V_m}} = \phi_{p_m}$.

Let m be such that $|V_m| \geq 2$, and let A_m be the set of all $z \in [0,1]^{V_m}$ for which $\{z_a | a \in V_m\} = \{l/(|V_m| - 1)\}_{l=0}^{|V_m|-1}$. Given $\omega \in \Omega_{V_m}$ let z^ω be the unique point in A_m determined by the condition

$$(z^\omega)_a < (z^\omega)_b \Leftrightarrow a \prec_\omega b.$$

Consider $R_m \in M([0,1]^{V_m})$, supported on the set A_m, that satisfies

$$R_m(z^\omega) = p_m(\omega)$$

for every $\omega \in \Omega_{V_m}$. Since for each $\sigma \in S_{V_m}$ and $\omega \in \Omega_{V_m}$, $\sigma(z^\omega) = z^{\sigma(\omega)}$, the probability measure R_m is $\prod_{j=0}^n S_{V_{j,m}}$-invariant.

Lemma 4.5. *Fix an integer m, $\varepsilon > 0$ and $a, b \in V_m$. Then*

$$R_m(|z_a - z_b| < \varepsilon) \leq \frac{2\varepsilon |V_m| + 1}{|V_{t(a),m} - \{b\}|}.$$

Proof: Since R_m is supported on A_m,

$$\sum_{a': t(a') = t(a),\, a' \neq b} I_{\{|z_{a'} - z_b| < \varepsilon\}} \leq \sum_{a': a' \neq b} I_{\{|z_{a'} - z_b| < \varepsilon\}} \leq 2\varepsilon |V_m| + 1.$$

Conditionally on z_b, $\{z_d | t(d) \neq t(a)\}$, the random variables $\{z_{a'}\}_{t(a')=t(a), a' \neq b}$ are identically distributed, as follows from the $\prod_{j=0}^{n} S_{V_{j,m}}$-invariance of R_m. Therefore for each $a' \in V_{t(a),m}$

$$R_m(|z_{a'} - z_b| < \varepsilon | z_b, \{z_d | t(d) \neq t(a)\})$$

$$= R_m(|z_a - z_b| < \varepsilon | z_b, \{z_d | t(d) \neq t(a)\})$$

and thus it equals to

$$\frac{1}{|V_{t(a),m} - \{b\}|} E_m \left[\sum_{a' : t(a') = t(a), a' \neq b} I_{\{|z_{a'} - z_b| < \varepsilon\}} \middle| z_b, \{z_d | t(d) \neq t(a)\} \right]$$

$$\leq \frac{2\varepsilon |V_m| + 1}{|V_{t(a),m} - \{b\}|},$$

and the result follows by taking the expectation over z_b, $\{z_d | t(d) \neq t(a)\}$. ■

Extend this measure arbitrarily, preserving the $\prod_{j=0}^{n} S_{V_{j,m}}$-invariance, to $[0,1]^V$. By the compactness of $M([0,1]^V)$, the sequence $\{R_m\}_{m \geq 1}$ has a subsequence $\{R_{m_i}\}_{i \geq 1}$ which converges weakly to some $R \in M([0,1]^V)$. Let $Z = (Z_a)_{a \in V}$ be a R-distributed random variable.

Given $l \geq 1$, define for every $\omega \in \Omega_{V_l}$ the set

$$A_\omega^l = \{z \in [0,1]^V \mid z_a < z_b \ \forall a, b \in V_l \text{ such that } a \prec_\omega b\}.$$

We show that

$$R(A_\omega^l) = \lim_i R_{m_i}(A_\omega^l) \tag{4.1}$$

Clearly

$$\partial A_\omega^l = \bigcup_{\{a,b \in V_l \mid a \neq b\}} A^{a,b} \subset \bigcup_{\{a,b \in V_l \mid a \neq b\}} A_\varepsilon^{a,b},$$

where $A^{a,b} = \{z \| z_a = z_b\}$, $A_\varepsilon^{a,b} = \{z \| z_a - z_b| < \varepsilon\}$. For $m \geq l$ Lemma 4.5 yields

$$\limsup_m R_m(A_\varepsilon^{a,b}) \leq 2\varepsilon q_{t(a)}^{-1},$$

and so

$$R(A^{a,b}) \leq 2\varepsilon q_{t(a)}^{-1}$$

for every $\varepsilon > 0$. This implies that $R(A^{a,b}) = 0$, and so $R(\partial A_\omega^l) = 0$. Therefore, by Theorem 2.1(d) of Billingsley (1971), (4.1) holds.

Next observe that for every $\theta \in \prod_{j=0}^{n} S_{V^j} \cap S'_V$ and a Borel subset A of $[0,1]^V$,

$$R(A) = \theta R(A).$$

Indeed, for a given $\theta \in \prod_{j=0}^{n} S_{V^j} \cap S'_V$, $R_m(A) = \theta R_m(A)$ for all m large enough, hence it is also the case for the limiting measure, R.

The de-Finetti partial exchangeability principle (Theorem 6.1 of the Appendix) says that for $\prod_{j=0}^{n} S_{V^j} \cap S'_V$-invariant probability measure R on $[0,1]^V$ there is a probability measure μ on the product of n copies of $M([0,1])$ (in other words, $\mu \in M[(M([0,1]))^{\Pi}]$), such that for every Borel set A,

$$R(A) = \int P_F(A)\, d\mu(F) \tag{4.2}$$

where $P_F(A)$ denotes the probability of A determined under the following hypotheses:

1) the random variables $\{Z_a\}_{a \in V}$ are independent;

2) for each $1 \leq j$, $\{Z_a\}_{t(a)=j}$ are i.i.d. with common distribution function F_j.

For each l, $v \in G_{V_l}$, $b \in V_l$, the use of (4.1), (4.2) now yields

$$\begin{aligned}
\varphi(v)(b) &= \lim_{i\to\infty} E_{R_{m_i}}\left[\sum_{\Omega_{V_l}} [v(\{a|a \preceq_\omega b\}) - v(\{a|a \prec_\omega b\})] I_{A^l_\omega}\right] \\
&= E_R\left[\sum_{\Omega_{V_l}} [v(\{a|a \preceq_\omega b\}) - v(\{a|a \prec_\omega b\})] I_{A^l_\omega}\right] \\
&= \int E_{P_F}[v(\{a|Z_a \leq Z_b\}) - v(\{a|Z_a < Z_b\})]\, d\mu(F) \\
&= \int \varphi_{\gamma^{\Pi}_F}(v)(b)\, d\mu(F),
\end{aligned}$$

by Definition 3.2.

As $R(A^{a,b}) = 0$ for each $a \neq b$, $P_F(A^{a,b}) = 0$ μ-a.e. Therefore $F \in (NA([0,1]))^{\Pi}$ μ-a.e. By Proposition 3.8 we can assume that μ is supported on $(NA([0,1]))^{\Pi}_q$.

The proof is now completed by the Π-symmetry of φ. ■

Remark 6: The measure μ in Theorem 4.4 is not necessarily unique. We exhibit the following counter-example for $|\Pi| = 2$. Let F be a continuous distribution function such that there exists $0 < t_0 < 1$ with $F(t_0) = t_0$, and $F|_{[0,t_0]}(t) < t$, $F|_{[t_0,1]} > t$. Define $F_1(t) = \min(F(t), t)$ and $F_2(t) = \max(F(t), t)$. Obviously F_1, F_2 are continuous distribution functions. Let $\Pi = (U_1, U_2)$. The paths associated with the pairs (t,t), (t,F), (t,F_1), (t,F_2) are all different (even under any q-normalization), yet

$$\tfrac{1}{2}(\varphi_{\gamma^{\Pi}_{t,t}} + \varphi_{\gamma^{\Pi}_{t,F}}) = \tfrac{1}{2}(\varphi_{\gamma^{\Pi}_{t,F_1}} + \varphi_{\gamma^{\Pi}_{t,F_2}}). \quad ■$$

The path value, however, is determined uniquely. The following proposition will be proved as Corollary 5.8 in Section 5.

Proposition 4.6. *If* $F_0 \in (NA([0,1]))_q^{\Pi}$ *and* $\mu \in M[(NA([0,1]))_q^{\Pi}]$ *such that* $\varphi_{\gamma_{F_0}^{\Pi}} = E_{d\mu(F)}(\varphi_{\gamma_F^{\Pi}})$, *then* μ *is Dirac measure supported on* F_0.

Theorem 4.4 and Proposition 4.6 lead to the following corollary (we argue as in the proof of Corollary 4.3).

Corollary 4.7. *The set* $\{\varphi_{\gamma_F^{\Pi}} | F$ *is q-normalized*$\}$ *is the set of the extreme points of* $V(U)[\Pi]$.

Allowing a coalition structure to be an uncountable infinite partition of the universe, the set of extreme points is much larger. Nevertheless, a weaker result is available, in which the randomization is taken with respect to a finitely additive measure.

Theorem 4.8. *Let* Π *be a coalition structure on* U *in which each type is infinite. Let* φ *be a* Π*-symmetric quasi-value on* G_U. *Then there is a finitely additive probability measure* μ *on* $(NA([0,1]))^{\Pi}$, *such that* $\varphi = E_{d\mu(F)}(\varphi_{\gamma_F^{\Pi}})$.

Proof: By Theorem 4.4, for each $S \in Fin(\Pi)$ there is $\mu^S \in M[(NA([0,1]))^S]$ such that $\varphi|_{G_{\cup S}} = E_{d\mu^S}(\varphi_{\gamma^S_{(F_\pi)_{\pi \in S}}})$. The set of finitely additive probability measures on $(NA([0,1]))^{\Pi}$ is compact (in the weak* topology). Extending the measures $\{\mu^S\}_{S \in Fin(\Pi)}$ arbitrarily on the entire $(NA([0,1]))^{\Pi}$, take a convergent subnet of $\{\mu^{S_\beta}\}_{\beta \in B}$ and then its limit, μ, yields $\varphi = E_{d\mu(F)}(\varphi_{\gamma_F^{\Pi}})$. ■

The following remark shows that the assumption of the infiniteness of each type cannot be disposed of.

Remark 7: Consider the universe $U = \{1,2,3\}$, and the structure (U_1, U_2), for $U_1 = \{1,2\}$, $U_2 = \{3\}$. Consider the distribution P on Ω_U which chooses each one of the orders $(1,3,2)$ and $(2,3,1)$ with probability $\frac{1}{2}$. The quasi-value ϕ_P is (U_1, U_2)-symmetric, although it is not a random (U_1, U_2)-path value. Indeed, the distribution P is determined uniquely (it is a direct observation, or use, alternatively, Remark 8). But any random (U_1, U_2)-symmetric path value induces probability measure on Ω_U (via Theorem 4.1) in which the set of orders in which U_1 appears (w.l.o.g.) before U_2 has a positive probability.

In a similar manner, given a coalitional structure Π on U with at least one finite type one can construct a quasi-value symmetric with respect to this structure, which is not a random Π-path value. ■

It shows that the structure of $V(U)[\Pi]$, when the types of Π are of arbitrary cardinality, is more complicated. We state the result characterizing $V(U)[\Pi]$ in the case of only one finite type. Similar assertions can be made for any number of finite types. The proof requires a generalization of Theorem 6.1, and in general follows the spirit of the proof of Theorem 4.4, so it will be omitted.

Theorem 4.9. *Let Π be a coalition structure on U in which only one type, π_0, is finite. Identify every $y \in [0,1]$ with the $\{0,1\}$-valued distribution function with discontinuity at y. The path values of the form $\sum_{\theta \in S_{\pi_0}} \theta \varphi_{\gamma_{F,\theta(Y)}^{\Pi-\{\pi_0\},\pi_0}} \theta^{-1}$, where $\Pi - \{\pi_0\}, \pi_0$ denotes the coalition structure obtained from Π, in which each player in π_0 is a separate type, $F \in (M([0,1]))^{\Pi-\{\pi_0\}}$* and *$Y \in [0,1]^{\pi_0}$, are the extreme points of the set $V(U)[\Pi]$. Every $\varphi \in V(U)[\Pi]$ is obtained by taking a random $F, \theta(Y)$.*

5. Pseudo-values

In this section we characterize the pseudo-values. We follow rather closely the layout and methods developed in Dubey et al (1981) and Monderer (1988) for the case of symmetric values, so many of the proofs in this section are merely sketched out, and full details are given only in cases of conceptually different proofs.

Given a set V, we view the set of all subcoalitions of V, 2^V, as the product space $\{0,1\}^V$ with a discrete topology on each component.

Proposition 5.1. *Let U be a finite universe of players. Each $p = (p^a)_{a \in U} \in \Pi_{a \in U} M(2^{U \setminus \{a\}})$ defines a pseudo-value φ_p on G_U by the formula*

$$\varphi_p(v)(b) = E_{p^b}\left[\frac{\partial}{\partial x_b}\bar{v}\left(\sum_{a \in U-\{b\}} s^a a\right)\right],$$

for each $b \in U$, where $(s^a)_{a \in U}$ is p^b-distributed random variable. The map $\Upsilon : \Pi_{a \in U} M(2^{U \setminus \{a\}}) \to PV(U)$, defined by $\Upsilon(p) = \varphi_p$, is an affine isomorphism.

Proof: This is a non-symmetric version of Proposition 1.1 of Dubey et al (1981), which can be proved by essentially the same means. Alternatively, it is an immediate result of the standard separation theorem. ■

Corollary 5.2. *Proposition 5.1 remains valid for any universe U.*

Proof: We call a sequence $(P_T)_{T \in Fin(U)} \in \prod_{T \in Fin(U)} [\Pi_{a \in T} M(2^{T \setminus \{a\}})]$ consistent if for each $a \in T_1 \subset T_2$ $P^a_{T_2 \setminus \{a\}}$ induces a measure $P^a_{T_1 \setminus \{a\}}$ on $2^{T_1 \setminus \{a\}}$. Denote the set of consistent sequences by C_M. By the Kolmogorov consistency principle, the affine map $c_M : \Pi_{a \in U} M(2^{U \setminus \{a\}}) \to C_M$, given by $c_M(P) = (P^a|_{2^{T \setminus \{a\}}})_{T \in Fin(U)}$, is an isomorphism. A sequence $(\varphi_T)_{T \in Fin(U)} \in \prod_{T \in Fin(U)} PV(T)$ is called consistent if for each $T_1 \subset T_2$ φ_{T_1} is a restriction of φ_{T_2} to G_{T_1}. Denote the set of such by C_V. The affine map $c_V : PV(U) \to C_M$, given by $c_V(\varphi) = (\varphi|_{G_T})_{T \in Fin(U)}$ is an isomorphism. By Proposition 5.1, the affine map $\tilde{\Upsilon} : C_M \to C_V$, given by $\tilde{\Upsilon}((P_T)_{T \in Fin(U)}) = (\varphi_{P_T})_{T \in Fin(U)}$, is also an isomorphism. Then $\Upsilon = c_V^{-1} \circ \tilde{\Upsilon} \circ c_M$ is an affine isomorphism as well. ■

Remark 8: For $P \in M(\Omega_U)$, the measure $\Upsilon^{-1}(\phi_P)_a$ is given on sets $A_T = I_T \times \{0,1\}^{U-T}$ by

$$p^b(A_T) = P(\{\omega \in \Omega_U \mid a \prec_\omega b \ \forall a \in T\}).$$

Thus P, P' induce the same distribution on the initial segments $I(a,\omega) = \{a' \in U \mid a' \prec_\omega a\}$ if and only if $\phi_P = \phi_{P'}$. Since the distribution on the initial segments determines uniquely the distribution on orders in the case of $|U| \leq 3$, the distribution P is determined uniquely for values of three players games. ■

Proposition 5.3. *If $\varphi_p \in PV(U)[H]$ for a subgroup H of S_U, then p satisfies*

$$p^a = \theta p^{\theta(a)}$$

for each $a \in U$ and $\theta \in H$.

Proof: The proof follows from the simple observation that $\theta\varphi_p\theta^{-1} = \varphi_{(\theta p^{\theta(a)})_{a\in U}}$, and Corollary 5.2. ■

Our next result provides a characterization of pseudo-values on G_U, symmetric with respect to some coalition structure. The following proposition is easily verifiable:

Proposition 5.4. *Given a coalition structure Π on U, every $p = (p_\pi)_{\pi\in\Pi} \in (M([0,1]^\Pi))^\Pi$ defines a Π-symmetric pseudo-value φ_p on G_U, by the formula*

$$\varphi_p(v)(a) = E_{p_{t(a)}}\left[\frac{\partial}{\partial x_a}\bar{v}\left(\sum_{\pi\in\Pi} s_\pi \pi\right)\right],$$

for each $a \in U$, where $t(a) = \pi$ if $a \in \pi$, and the vector $(s_\pi)_{\pi\in\Pi}$ is $p_{t(a)}$-distributed.

Theorem 5.5. *Let Π be a coalition structure on U in which each type is infinite. Then the map $\Upsilon : (M([0,1]^\Pi))^\Pi \to PV(U)[\Pi]$, defined by $\Upsilon(p) = \varphi_p$, is an affine isomorphism.*

Proof: For each $p \in (M([0,1]^\Pi))^\Pi$ there is a unique $\bar{p} \in \prod_{a\in U} M(2^{U\setminus\{a\}})$ such that $\varphi_p = \varphi_{\bar{p}}$. If $a \in U$, then $\bar{p}^a$ is $[\prod_{\pi\neq t(a)} S_\pi] \times S_{t(a)-\{a\}}$-invariant, and using Theorem 6.1 in Appendix, we prove that Υ is onto as in Dubey et al (1981).

As for the 1:1 part, it suffices to show that p is determined by $\bar{p}$. If $a \in U$, for each $m \geq 1$ and $\pi \in \Pi$ pick subsets $\pi_m^a \subset \pi - \{a\}$ with $|\pi_m^a| = m$. If $d \in [0,1]^\Pi$ such that $d_\pi = 0$ for all but finitely many $\pi \in \Pi$ and $p_{t(a)}(\partial[\prod_{\pi\in\Pi}(0,d_\pi)]) = 0$, then for $\bar{p}^a$-distributed random coalition S

$$\bar{p}^a\{0 \leq |S \cap \pi_m^a| \leq d_\pi m \;\forall \pi \in \Pi\}$$

$$= E_{p_{t(a)}}\left[\prod_{\pi\in\Pi} \Pr\left\{0 \leq \sum_{b\in\pi_m^a} X_b(s_\pi) \leq d_\pi m\right\}\right],$$

where $\{X_b\}_{b\in\bigcup\pi_m^a}$ are $\{0,1\}$-valued random variables, that, conditionally on $(s_\pi)_{\pi\in\Pi}$, are independently distributed with the mean $s_{t(b)}$, and $(s_\pi)_{\pi\in\Pi}$ is $p_{t(a)}$ distributed. By the law of large numbers and the bounded convergence theorem, the limit as $m \to \infty$ of the above expression, determined solely by $\bar{p}$, is $p_{t(a)}(\Pi_{\pi \in \Pi}(0, d_\pi))$. As $p_{t(a)}$ has only a countably many atoms, it is determined uniquely. ■

Corollary 5.6. *Under the assumptions of Theorem 5.5, the set of extreme points of $PV(U)[\Pi]$ is $\{\varphi_p | \forall \pi \in \Pi p_\pi$ is a Dirac measure$\}$.*

Remark 9: As in Remark 7, one can see that the assumption of infiniteness of each type is crucial for the validity of Theorem 5.5 and the above corollary. One can obtain similar results for coalition structures with finite types, by considering each player in the finite type a separate type, and then symmetrizing (in the way it was presented in Theorem 4.9).

Remark 10: Let Π be a coalition structure on U in which each type is infinite, and $F \in (NA([0,1]))^{\Pi}$. For each $0 \leq \alpha \leq \beta \leq 1$ denote

$$W_{\alpha,\beta}(F) = \{(F_\pi(t))_{\pi \in \Pi} | t \in [\alpha, \beta]\}.$$

Consider the path value $\varphi_{\gamma_F^\Pi}$. Then $\Upsilon^{-1}(\varphi_{\gamma_F^\Pi})_\pi$ is the measure supported on the set $W_{0,1}$ and for each $0 \leq \alpha \leq \beta \leq 1$ it satisfies

$$\Upsilon^{-1}(\varphi_{\gamma_F^\Pi})_\pi(W_{\alpha,\beta}(F)) = F_\pi(\beta) - F_\pi(\alpha),$$

as follows from Remark 3.

Corollary 5.7. *If Π is a coalition structure on U in which each type is infinite, and F, $F_0 \in (NA([0,1]))^{\Pi}$ are such that $\varphi_{\gamma_F^\Pi} = \varphi_{\gamma_{F_0}^\Pi}$, then*

$$W_{0,1}(F) = W_{0,1}(F_0).$$

As a result, F, F_0 have the same q-normalization (defined in Proposition 3.8).

Corollary 5.8. *If Π is a coalition structure on U in which each type is infinite, $F_0 \in (NA([0,1]))_q^{\Pi}$ and $\mu \in M[(NA([0,1]))_q^{\Pi}]$ are such that $\varphi_{\gamma_{F_0}^\Pi} = E_{d\mu(F)}(\varphi_{\gamma_F^\Pi})$, then μ is the Dirac measure on $\{F_0\}$.*

Proof: Suppose that μ is not supported on $\{F_0\}$. Therefore $W_{0,1}(F)$ differs from $W_{0,1}(F_0)$ with some nonzero probability according to μ (otherwise $W_{0,1}(F) = W_{0,1}(F_0)$ μ-a.e., and this determines q-normalized F uniquely, as equal to F_0). By the definition of Υ, the measures $\Upsilon^{-1}(E_{d\mu(F)}(\varphi_{\gamma_F^\Pi}))_\pi$ are not supported on $W_{0,1}(F_0)$, which is precisely $\Upsilon^{-1}(\varphi_{\gamma_{F_0}^\Pi})_\pi$ (by Remark 10). It implies that $\Upsilon^{-1}(E_{d\mu(F)}(\varphi_{\gamma_F^\Pi}))_\pi \neq \Upsilon^{-1}(\varphi_{\gamma_{F_0}^\Pi})_\pi$. This, however, cannot be the case, since $E_{d\mu(F)}(\varphi_{\gamma_F^\Pi}) = \varphi_{\gamma_{F_0}^\Pi}$ by our assumption. Therefore μ is supported on $\{F_0\}$. ■

Monderer (1988) showed that a semivalue on a subspace of G_U, containing FA_U, can be extended to the whole space G_U. This provides, in particular, a characterization of semivalues on those subspaces. Similar results hold in the context of the pseudo-values symmetric with respect to some coalition structure.

The most general extension result is Theorem 1.8 in Monderer (1988):

Theorem 5.9. *Every pseudo-value on a subspace of G_U including FA_U can be extended to a pseudo-value on G_U.*

The proof of the following result is analogous to that of Theorem 2.1 of Monderer (1988).

Theorem 5.10. *Given a coalition structure Π on U, every Π-symmetric pseudo-value φ on a subspace of G_U including FA_U can be extended to Π-symmetric pseudo-value on G_U. If each type in Π is infinite, then $\varphi = \varphi_p$ for some $p \in (M([0,1]^{\Pi}))^{\Pi}$.*

6. Appendix

The partial exchangeability principle of de-Finetti is well-known. We give a precise statement and a proof of the principle, as we have found no convenient reference to it. The proof follows the one given in Loeve (1960) for the case of full exchangeability.

Denote by N the set of integers.

Theorem 6.1. *Let $\{Z_{i,j}\}_{i,j\geq 1}$ be an infinite random matrix. Suppose that the distribution P of this matrix is row-exchangeable, i.e., for $(\theta_j)_{j\geq 1} \in (S'_N)^N$, such that $\theta_j = id$ for all but finitely many j, $\{Z_{i,j}\}_{i,j\geq 1}$ and $\{Z_{\theta_j(i),j}\}_{i,j\geq 1}$ have the same distribution. Then there is a probability measure μ on $(M[0,1])^N$, such that for each Borel set $A \subset [0,1]^{N\times N}$*

$$P(A) = \int_{(M[0,1])^N} P_F(A)\, d\mu(F),$$

where P_F denotes the probability of A being determined by the hypotheses that

(i) *$\{Z_{i,j}\}_{i\geq 1}$ are i.i.d. with common distribution function $F_j \in M[0,1]$, for each $1 \leq j$;*

(ii) *the processes $\{Z_{i,j}\}_{i\geq 1}$, $1 \leq j$, are independent between themselves.*

Proof: For any $m, j \geq 1$ and $t \in [0,1]$, define

$$\xi_{m,j}(t) = \frac{1}{m}\sum_{i=1}^{m} I_{\{Z_{i,j}\leq t\}}.$$

Observe now that for any j, m, n, and $t \in [0,1]$,

$$E(\xi_{m,j}(t) - \xi_{n,j}(t))^2 \leq \frac{|m-n|}{mn},$$

thus there is a subsequence $\{\xi_{m_k,j}(t)\}_{k=1}^{\infty}$ that converges P-a.e. to the limit $\xi_j(t)$ for each rational t and $j \geq 1$, hence for any t. One can also assume that ξ_j is a distribution function P-a.e.

It suffices to prove the assertion for cylindrical sets $A_{n_1,n_2} = \Pi_{l,j\geq 1}[0, s_{l,j}]$, where $s_{l,j} = 1$ for $l > n_1$ or $j > n_2$. We show that

$$P(A_{n_1,n_2}) = E(\Pi_{l\leq n_1;\, j\leq n_2}\xi_j(s_{l,j})). \tag{6.1}$$

Indeed, for each j, m

$$\Pi_{l\leq n_1;\, j\leq n_2}\xi_{m,j}(s_{l,j})$$
$$= \frac{1}{m^{n_1,n_2}} \sum_{i\in\{1,\dots,m\}^{n_1\times n_2}} I_{\{\forall l\leq n_1;\, j\leq n_2 \ Z_{i(l,j),j}\leq s_{l,j}\}},$$

and so, by the exchangeability assumption,

$$E[\Pi_{l\leq n_1;\, j\leq n_2}\xi_{m,j}(s_{l,j})]$$
$$= E[I_{\{\forall l\leq n_1;\, j\leq n_2 \ Z_{l,j}\leq s_{l,j}\}}] + \varepsilon(m) = P(A_{n_1,n_2}) + \varepsilon(m),$$

with $\lim_{m\to\infty}\varepsilon(m) = 0$. The equality (6.1) follows by the bounded convergence theorem. ■

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