Economics Letters 60 (1998) 157–162

economics letters

Almost all equilibria in dominant strategies are coalition-proof

Bezalel Peleg*

Institute of Mathematics and Center for Rationality and Interactive Decision Theory, the Hebrew University of Jerusalem, Jerusalem, Israel

Received 12 November 1997; accepted 16 April 1998

Abstract

We prove that almost all equilibria in dominant strategies of finite strategic games are coalition-proof. Also, we investigate existence of coalition-proof equilibria of separable games and strategyproof mechanisms. In particular, we give an example of a strategyproof mechanism, which is defined for strict orderings, and which is coalition-proof but not coalitionally strategyproof. © 1998 Elsevier Science S.A. All rights reserved.

Keywords: Dominant strategies; Coalition-proof equilibria; Strategyproof mechanisms

JEL classification: C72

1. Introduction

Coalition-proof Nash equilibria were introduced by Bernheim et al. (1987). Many reformulations and applications of the original definition have appeared since 1987. However, no general existence results were found until recently. Moreno and Wooders (1996) have proved that a Pareto-best member of the set of actions that survive the iterated elimination of strictly dominated strategies is coalition-proof. Using this result we observe that almost all equilibria in dominant strategies of finite games are coalition-proof.

The following example might illustrate the general situation. Let $G(\varepsilon)$, $\varepsilon \geq 0$, be the following family of 2×2 games

$$G(\varepsilon) = \begin{array}{ccc} & L & R \\ T & 2, 2 & 0, 2+\varepsilon \\ B & 2+\varepsilon, 0 & 1, 1 \end{array}$$

(B, R) is the unique equilibrium in dominant strategies of $G(\varepsilon)$ for $\varepsilon \geq 0$. For $\varepsilon = 0$ (T, L) is the unique coalition-proof Nash equilibrium, whereas (B, R) is the unique coalition-proof Nash equilibrium for

*Corresponding Author. Address for correspondence: Copenhagen Business School, Dept. of Management Science and Statistics, Julius Thomsens Plads 10, 1925 Frederiksberg C, Denmark. Fax: +45 381 53500; e-mail: peleg@cbs.dk

PII: S0165-1765(98)00098-6

all $\varepsilon>0$. (For the sake of completeness we remark that an equilibrium in weakly dominant strategies also may be coalition-proof.)

There are finite games with coalition-proof equilibria but without equilibria in dominant strategies. (Indeed, every two-person finite game has a coalition-proof equilibrium.) On the other hand, in our Example 4.4 every strategy of every player is dominant and, nevertheless, the game has no coalition-proof equilibrium.

We now review briefly the contents of this note. Section 2 contains the relevant definitions. The short proof of the result is given in Section 3. In Section 4 we observe that games with separable payoff functions have dominant strategies. We conclude in Section 5 with an examination of the coalition-proofness of two strategy-proof mechanisms: Voting by committees and pivotal mechanisms. Voting by committees is coalition-proof whereas pivotal mechanisms may not be coalition-proof.

2. Definitions and notations

A game in strategic form is a system $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ where N is a finite set of players; A_i, $i\in N$, is the (non-empty) set of strategies of i; and $u_i\colon \times_{j\in N} A_j \to R$ is the payoff function of player $i\in N$. (Here R denotes the set of real numbers.) Let $S\subseteq N$, $S\neq\emptyset$. We denote $A_S=\times_{i\in S} A_i$ and $A=A_N$. If $x\in A$ then x_S denotes the restriction of x to S.

Let $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic game, let $S\subset N$, $S\neq\emptyset$, and let $x\in A$. The *reduced game* of G with respect to (w.r.t) S and x is the game $G^{S,x}=(S, (A_i)_{i\in S}, (u_i^x)_{i\in S})$ where $u_i^x(y_S)=u_i(y_S, x_{N\setminus S})$ for all $y_S\in A_S$ and $i\in S$.

Let $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic game. $x\in A$ is a Nash equilibrium (*NE*) of G if for every $i\in N$ $u_i(x)\geq u_i(y_i, x_{N\setminus\{i\}})$ for all $y_i\in A_i$. $d_i\in A_i$ is a *dominant* strategy of player i if $u_i(d_i, x_{N\setminus\{i\}})\geq u_i(e_i, x_{N\setminus\{i\}})$ for all $e_i\in A_i$ and all $x_{N\setminus\{i\}}\in A_{N\setminus\{i\}}$. We denote

$D(G)=\{d\in A \mid d_i$ is a dominant strategy for every $i\in N\}$

$d_i\in A_i$ is a *strictly* dominant strategy of player i if for every $e_i\in A_i$, $e_i\neq d_i$, $u_i(d_i, x_{N\setminus\{i\}})>u_i(e_i, x_{N\setminus\{i\}})$ for all $x_{N\setminus\{i\}}\in A_{N\setminus\{i\}}$. We denote

$\mathring{D}(G)=\{d\in A \mid d_i$ is strictly dominant for every $i\in N\}$.

Clearly, if $\mathring{D}(G)$ is non-empty, then it is a singleton.

Let $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic game, let $x\in A$, and let $S\subset N$, $S\neq\emptyset$. An *internally consistent improvement* (ICI) of S upon x is defined by induction on $|S|$, the number of members of S. If $|S|=1$, that is $S=\{i\}$ for some $i\in N$, then $y_i\in A_i$ is an ICI of i upon x if $u_i(y_i, x_{N\setminus\{i\}})>u_i(x)$. If $|S|>1$ then $y_S\in A_S$ is an ICI of S upon x if $(i)u_i(y_S, x_{N\setminus S})>u_i(x)$ for all $i\in S$, and (ii) no $T\subset S$, $T\neq\emptyset$, S, has an ICI upon $(y_S, x_{N\setminus S})$. x is a *coalition-proof Nash equilibrium* (CPNE) if no $T\subseteq N$, $T\neq\emptyset$, has an ICI upon x. Clearly, every CPNE is an *NE*. The reader is referred to Bernheim et al. (1987) for a discussion of the foregoing definition and motivation.

The following definition is closely related to Kaplan's definition of semi-strong equilibrium (see Kaplan (1992)).

Definition 2.1: Let $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic game and let $x\in A$. x is a *strong* CPNE (SCPNE) if

(i) x is an *NE* of G; and

(ii) for every $S \subset N$, $S \neq \emptyset$, and every *NE* y_S of $G^{S,x}$, there exists $i \in S$ such that $u_i(x) \geq u_i(y_S, x_{N \setminus S})$.
Clearly, if x is an SCPNE of G, then x is a CPNE of G.

Lemma 2.2. *Let* $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ *be a strategic game. If* $\mathring{D}(G) \neq \emptyset$ *and* $x = \mathring{D}(G)$, *then x is an SCPNE.*

Proof: Lemma 2.2 follows from Moreno and Wooders (1996), [p. 92]. For the sake of completeness we give the short proof. Assume, on the contrary, that there exist $S \subseteq N$, $S \neq \emptyset$, and an NE y_S of $G^{S,x}$ such that $u_i(y_S, x_{N\setminus S}) > u_i(x)$ for all $i \in S$. Choose $i \in S$ such that $y_i \neq x_i$. Clearly, $u_i(y_{S\setminus\{i\}}, x_i, x_{N\setminus S}) > u_i(y_S, x_{N\setminus S})$ because x_i is strictly dominant. Thus, y_S is not an *NE* of $G^{S,x}$ and the desired contradiction has been obtained. Q.E.D.

3. Almost all equilibria in dominant strategies are strongly coalition-proof

We now fix a set of players N and finite strategy sets A_i, $i \in N$. Denote by $\Gamma = \Gamma(N, (A_i)_{i\in N})$ the set of all strategic games $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$. We may identify Γ with the Euclidean space $R^{A\times N}$. Denote further $\delta(\Gamma) = \{G \in \Gamma | D(G) \neq \emptyset\}$ and $\sigma(\Gamma) = \{G \in \delta(\Gamma) | D(G) = SCPNE(G)\}$ where $SCPNE(G)$ is the set of *SCPNE*'s of G.

Theorem 3.1. $\delta(\Gamma)\setminus\sigma(\Gamma)$ *is contained in a closed subset of* $R^{A\times N}$ *of Lebesgue measure zero.*

Proof: For $x \in A$ we denote $\delta_x = \{G \in \Gamma | x \in D(G)\}$ and $\mathring{\delta}_x = \{G \in \Gamma | x = \mathring{D}(G)\}$. Clearly, $\delta(\Gamma) = \bigcup_{x\in A} \delta_x$. Also, by Lemma 2.2, if $G \in \mathring{\delta}_x$ then $x = SCPNE(G)$. Thus $\sigma(\Gamma) \supset \bigcup_{x\in A} \mathring{\delta}_x$. Therefore

$$\delta(\Gamma)\setminus\sigma(\Gamma) \subset \bigcup_{x\in A} \delta_x \setminus \bigcup_{x\in A} \mathring{\delta}_x \subset \bigcup_{x\in A} \delta_x \setminus \mathring{\delta}_x.$$

We now observe that each set $\delta_x \setminus \mathring{\delta}_x$, $x \in A$, is a closed set of Lebesgue measure zero. Q.E.D.

4. Separable games

We now introduce a class of games which have dominant strategies.

Definition 4.1: A strategic game $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ is *partially separable* if for each $i \in N$ there exist two functions u_i^i: $A_i \to R$ and u_i^{-i}: $A_{N\setminus\{i\}} \to R$ such that $u_i(x) = u_i^i(x_i) + u_i^{-i}(x_{N\setminus\{i\}})$ for all $x \in A$.

Lemma 4.2. *Let* $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ *be a partially separable game. If for every* $i \in N$ A_i *is a compact topological space, A has the product topology, and* u_i^i *is upper semi-continuous, then* $D(G) \neq \emptyset$.

The proof of Lemma 4.2 is straightforward.

Remark 4.3: *A game* $G=(N, (A_i)_{i\in N}, (u_i)_{i\in N})$ *is separable if for each* $i \in N$ *there exist functions* u_i^j:

$A_j \to R$, $j \in N$ such that $u_i(x) = \sum_{j \in N} u_i^j(x_j)$ for all $x \in A$. For a recent work on separable games see Balder (1997).

Clearly, by Lemmas 4.2 and 2.2, if G is a ''generic'' finite and partially separable game, then $SCPNE(G) \neq \emptyset$. (A partially separable finite game is ''generic'' if for each $i \in N$ u_i^i is injective.) However, a finite separable game may have no CPNE.

Example 4.4: Let $N = \{1, 2, 3, 4\}$, $A_i = \{a_i, b_i\}$, $i \in N$, and $u_i(x_1, x_2, x_3, x_4) = \sum_{j \neq i} u_i^j(x_j)$ for $x \in A$ and $i \in N$, where $u_i^j(b_j) = 0$ and $u_i^j(a_j) = (-1)^{j-i+1}$, i, $j \in N$, $i \neq j$. If $i \in N$ plays a_i with probability p_i, $0 \leq p_i \leq 1$, then $u_1(\underset{\sim}{p}) = p_2 - p_3 + p_4$, $u_2(\underset{\sim}{p}) = p_1 + p_3 - p_4$, $u_3(\underset{\sim}{p}) = -p_1 + p_2 + p_4$, and $u_4(\underset{\sim}{p}) = p_1 - p_2 + p_3$, where $\underset{\sim}{p} = (p_1, p_2, p_3, p_4)$. Every $\underset{\sim}{p}$ is in $D(G)$, where $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$. However, there is no CPNE. Assume, on the contrary, that $\underset{\sim}{q} = (q_1, q_2, q_3, q_4)$ is a CPNE (in mixed strategies). Then $\max(q_1, q_2) = 1$ (otherwise $\{1, 2\}$ has an ICI upon $\underset{\sim}{q}$). Similarly, $\max(q_1, q_4) = 1$, $\max(q_2, q_3) = 1$, and $\max(q_3, q_4) = 1$. If $q_1 < 1$ then $q_2 = q_4 = 1$ and $\{2, 4\}$ has an ICI. Thus $q_1 = 1$. Similarly, $q_3 = 1$. Hence $\{1, 3\}$ has an ICI, which is the desired contradiction.

5. Strategyproof mechanisms

Strategyproof Mechanisms yield strategic games with equilibria in dominant strategies. For a survey of such mechanisms see Chapter 8 of Moulin (1988). We shall consider in this section two examples: Voting by committees and pivotal mechanisms. In Voting by committees the sincere equilibrium is coalition-proof, whereas truthtelling in pivotal mechanisms may not be coalition-proof.

*5.1. Voting by committees (*Barberá et al. (1991)*)*

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of voters, and let $K = \{1, \ldots, k\}$, $k \geq 2$ be a set of candidates. Each voter $i \in N$ has a preference relation P_i on 2^K, the set of all subsets of K. We assume that P_i is complete, transitive, and asymmetric. We denote by π the set of all preference relations on 2^K which satisfy the foregoing three properties. A *voting scheme* f is a function from a set $B \subset \pi^N$ to 2^K. A *committee* is a pair $C = (N, W)$, where N is the set of voters and W is a set of nonempty subsets of N, which satisfies the following conditions: (i) $N \in W$; and (ii) $[M \in W$ and $M \subset M^* \subset N] \Rightarrow M^* \in W$. If $P \in \pi$ then we denote by $B(P)$ the best element in 2^K according to P. Let $B \subset \pi^N$. A voting scheme f: $B \to 2^K$ is *voting by committees*, if for each $x \in K$ there exists a committee $C_x = (N, W_x)$ such that: for all $P^N \in B$, $x \in f(P^N)$ if and only if $\{i | x \in B(P_i)\} \in W_x$. A preference relation $P \in \pi$ is *additively representable* if there exists a function u: $K \cup \{\emptyset\} \to R$, with $u(\emptyset) = 0$ such that for all D, $E \subset K$,

$$DPE \text{ if and only if } \sum_{z \in D} u(z) > \sum_{z \in E} u(z)$$

The set of all additively representable members of π is denoted by π_A.

Let f: $\pi_A^N \to 2^K$ be a (surjective) voting scheme. Then f is strategyproof if and only if f is voting by committees (see Theorem 2 of Barberá et al. (1991)). If f is strategyproof then it is also coalition-proof, that is, sincere voting is always a CPNE (and not only an NE). The proof is similar to the proof of Theorem 4.1 of Peleg and Sudhölter (1997); it is too long to be included here. We now shall present an example of a strategyproof voting scheme f: $\pi_A^N \to 2^K$ which is not strongly coalition-proof.

Let $N=\{1, 2, 3\}$ and let $K=\{a, b, c\}$. We shall consider the following voting scheme f: $\pi_A^N \to 2^K$. Let $W_a = W_b = W_c = \{S \subset N | \ |S| \geq 2\}$. f will be voting by committees with the foregoing three committees. Thus, if $P^N \in \pi_A^N$ then $x \in f(P^N)$ if and only if $|\{i | x \in B(P_i)\}| \geq 2$. Consider the profile u^N given by the following table

	a	b	c	$\emptyset$
u_1	4	−1	−2	0
u_2	−2	4	−1	0
u_3	−1	−2	4	0

Here u_i, $i \in N$, is an additive representation of a member P_i of π_A. Clearly, $f(P^N) = \emptyset$. Now consider the profile $\hat{u}^N$ where $\hat{u}_1 = \hat{u}_2 = \hat{u}_3$ and $\hat{u}_1(a)=4$, $\hat{u}_1(b)=2$, $\hat{u}_1(c)=1$, and $\hat{u}_1(\emptyset)=0$. Let $\hat{P}^N$ be the corresponding profile in π_A^N. Then $f(\hat{P}^N)=K$, $KP_i\emptyset$ for $i=1, 2, 3$, and $\hat{P}^N$ is an NE of f (with respect to the sincere profile P^N). Thus, f is not strongly coalition-proof. (This example is similar to the example in Section 5 of Peleg and Sudhölter (1997).)

We now describe the relationship between the foregoing example and the general framework of Dasgupta et al. (1979). The voting scheme f is strategyproof. Hence, it satisfies IPM (see Theorem 4.3.1 of Dasgupta et al. (1979)). Also, f is not coalitionally strategyproof, and, furthermore, π_A consists of strict orderings. Hence, π_A is not rich. Indeed, assume, on the contrary, that π_A is rich. Then, by Corollary 3.2.3 of Dasgupta et al. (1979) f would satisfy IWM. Thus, by Theorem 4.5.1 of Dasgupta et al. (1979), f would also be coalitionally strategyproof, which is the desired contradiction. We conclude from the foregoing discussion that a strategyproof (direct) mechanism on a domain (consisting of strict orderings) which is not rich may be coalition-proof without being coalitionally strategyproof.

*5.2. Pivotal mechanisms (*Clarke (1971) *and* Groves (1973)*)*

We shall only consider a two-person example. Thus, let $N=\{1, 2\}$ be the set of agents and let $A=\{a_1, \ldots, a_6\}$ be the set of public decisions. The pivotal mechanism is given by a triple $(\alpha; t_1, t_2)$ where α: $R^{A \times N} \to A$ and t_i: $R^{A \times N} \to \mathrm{R}$, $i=1, 2$. For $v=(v_1, v_2) \in R^{A \times N}$

$$\alpha(v) \in \arg\max_{a \in A}[v_1(a) + v_2(a)] \text{ and}$$

$$t_i(v) = v_j(\alpha(v)) - \max_{a \in A} v_j(a),\ j = 3 - i,\ i = 1, 2.$$

The payoff of i is $H^i(v) = u_i(\alpha(v)) + t_i(v)$, $i=1, 2$, where $u=(u_1, u_2)$ is the profile of true preferences.

In the specific example that we consider $u=(u_1, u_2)$ is given by the following table

	a_1	a_2	a_3	a_4	a_5	a_6
u_1	17	11	0	10	7	0
u_2	0	11	17	0	7	10

Table 1

Clearly, $\alpha(u)=a_2$, $t_1(u)=t_2(u)=11-17=-6$, and $H^1(u)=H^2(u)=11-6=5$. Now consider the profile $v=(v_1, v_2)$ given by Table 2

	a_1	a_2	a_3	a_4	a_5	a_6
v_1	0	0	0	12	11	0
v_2	0	0	0	0	11	12

Table 2

$$\alpha(v)=a_5,\ t_1(v)=t_2(v)=11-12=-1, \text{ and } H^1(v)=H^2(v)=7-1=6.$$

Thus v Pareto dominates u. We claim that v is an NE (and therefore u is not a CPNE). We have only to prove that v_1 is a best response to v_2. Clearly, u_1 is a best response to v_2 and, furthermore, $H^1(u_1, v_2)=6$. Hence, v_1 also is a best response to v_2.

We conclude by observing that the foregoing example is robust, that is, there is an open neighborhood of u in $R^{A\times N}$ such that each profile (of true preferences) in that neighborhood is not a CPNE (of the corresponding game).

References

Balder, E.J., 1997. Remarks on Nash equilibria for games with additively coupled payoffs. Economic Theory 9, 161–167.

Barberá, S., Sonnenschein, H., Zhou, L., 1991. Voting by committees. Econometrica 59, 595–609.

Bernheim, B.D., Peleg, B., Whinston, M.D., 1987. Coalition-proof Nash equilibria I. Concepts. Journal of Economic Theory 42, 1–12.

Clarke, E.H., 1971. Multipart pricing of public goods. Public choice 11, 17–33.

Dasgupta, P., Hammond, P., Maskin, E., 1979. The implementation of social choice rules: some general results on incentive compatibility. The Review of Economic Studies 46, 185–216.

Groves, T., 1973. Incentives in teams. Econometrica 41, 617–663.

Kaplan, G., 1992. Sophisticated outcomes and coalitional stability, M. Sc. Thesis, Department of Statistics, Tel-Aviv University (in English).

Moreno, D., Wooders, J., 1996. Coalition-proof equilibrium. Games and Economic Behavior 17, 80–112.

Moulin, H., 1988. Axioms of Cooperative Decision Making, (Econometric Society monograph no. 15), Cambridge University Press, Cambridge.

Peleg, B., Sudhölter, P., 1997. Single-peakedness and coalition-proofness, Institute of Mathematics, The Hebrew University of Jerusalem, mimeo.