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The generation of formulas held in common knowledge

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Abstract. Common knowledge of a finite set of formulas implies a special relationship between syntactic and semantic common knowledge. If S, a set of formulas held in common knowledge, is implied by the common knowledge of some finite subset of S, and A is a non-redundant semantic model where exactly S is held in common knowledge, then the following are equivalent: (a) S is maximal among the sets of formulas that can be held in common knowledge, (b) A is finite, and (c) the set S determines A uniquely; otherwise there are uncountably many such A. Even if the knowledge of the agents are defined by their knowledge of formulas, 1) there is a continuum of distinct semantic models where only the tautologies are held in common knowledge and, 2) not assuming that S is finitely generated (a) does not imply (c), (c) does not imply (a), and (a) and (c) together do not imply (b).

Key words: Interactive epistemology, Common knowledge, Cantor sets

1. Introduction

In Simon (1999) we considered the space of maximally consistent sets of formulas using the multi-agent epistemic logic S5 and we showed that there can be a large descrepancy between syntactic and semantic common knowledge.

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In this paper, we re-examine this descrepancy in the light of how the syntactic common knowledge, namely the set of formulas held in common knowledge, is generated. Informally, a set of formulas held in common knowledge is finitely generated if the common knowledge of some finite subset of formulas implies logically the common knowledge of all the formulas in the set.

Let Ω be the space of maximally consistent sets of formulas that are generated by finitely many agents and primitive propositions. For every agent j let $\mathcal{Q}^j$ be its knowledge partition of Ω, where the knowledge of an agent is defined to be its knowledge of formulas, (see Aumann, 1999). Let $\mathcal{Q} := \bigwedge_j \mathcal{Q}^j$ be the meet partition of the $\mathcal{Q}^j$. A member of $\mathcal{Q}$ we call a "cell". To every cell $C \in \mathcal{Q}$ corresponds a set of formulas $F(C)$, the set of formulas held in common knowledge at any point in C. We say that C is "centered" if and only if C is the only cell that corresponds to $F(C)$.

We prove **Theorem 1**: if $F(C)$ is finitely generated then the following are equivalent:

(1) $F(C)$ is maximal among all the sets of formulas that can be held in common knowledge,
(2) C is finite, and
(3) C is centered.

In Fagin (1994) and Heifetz and Samet (1999) canonical semantic models for the multi-agent logic S5 corresponding to ordinal numbers were constructed. Although for a fixed ordinal α their models U_α were different, both constructions shared an important property. For any semantic model M for the S5 logic there are canonical maps to all the canonical models such that for some ordinal α the image of M in U_α is equivalent to M as a structure representing the knowledge of the agents. Furthermore U_α maps injectively to U_β for $\alpha < \beta$. We define a semantic model M to be *non-redundant* if there exists some ordinal α such that M maps injectively to the model U_α. (It is easy to see that the property of non-redundancy is not dependent on the type of canonical models chosen.) For both constructions, Ω is the canonical model associated with ω, the first infinite ordinal. Since a non-redundant semantic model is finite if and only if its image in Ω is finite, we can extend Theorem 1 to the formulation contained in the abstract of this paper.

Theorem 1 allows us to determine for many infinite semantic models that the formulas held in common knowledge cannot be finitely generated. Due to topological characterizations of the centered property in Simon (1999) the following simple and well known example of a denumerable semantic model can be described completely by the formulas held in common knowledge, but by Theorem 1 not by finitely many such formulas. Let there be two agents, 1 and 2, let the set S be $\{1, 2, \ldots\}$, and let the two partions of S corresponding to the agents be $\mathcal{P}^1 = \{\{1\}, \{2, 3\}, \{4, 5\} \ldots\}$ and $\mathcal{P}^2 = \{\{1, 2\}, \{3, 4\}, \ldots\}$. Let x, a primitive proposition, be true only at $\{1\}$. (At no point in S is there common knowledge that x is not true, yet there are arbitrarily high levels of mutual knowledge that x is not true.)

In the last section of this paper, we investigate what happens if the set of formulas held in common knowledge is not finitely generated. We think that this part of the paper is the most important. We present three examples that demonstrate the subtlety of the centered property. Example 1 is that of an un-centered cell corresponding to a maximal set of formulas that can be held in common knowledge. Second, as one goes up a tower of sets of formulas that can be held in common knowledge, one can go from a corresponding centered cell to uncountably many un-centered cells (Example 2). Lastly, Example 3 is that of a centered cell (topologically equivalent to a Cantor set) corresponding to a maximal set of formulas held in common knowledge such that at no point in the cell is the knowledge of any player "finitely generated," following the definition of Samet, (1990). All three of these examples employ three agents. With regard to Theorem 1, Example 1 demonstrates (1) does not imply (3), Example 3 demonstrates (1) and (3) do not imply (2), and Example 2 demonstrates a claim much stronger than (3) does not imply (1).

Example 1 is of special interest. Before demonstrating Example 1, we prove **Theorem 2:** the cardinality of cells in Ω for which only the tautologies are held in common knowledge is that of the continuum. In Simon (1999), it was proven that the number of uncentered cells sharing the same set of formulas in common knowledge is always uncountable, including those sharing the tautologies, but without assuming the Continuum Hypothesis a continuum cardinality was not proven. Theorem 2 is proven with a hierachical construction of Ω that allows one at every stage to construct or de-construct the common knowledge of some formula. Because of the maximality of the formulas held in common knowledge in Example 1, it belongs to a class of cells that demonstrate how the techniques of proving Theorem 2 do not suffice to conclude the stronger cardinality result for all uncentered cells.

All three examples use critically Theorem 3, a way to relate topologically certain partitions of Ω defined with two agents to semantic models with three agents. It is possible that there is a special structure to Ω valid only for two agents; it is unknown if there are counter-examples to Theorem 1 when there are only two agents and finite generation is not assumed.

The rest of the paper is organized as follows. Section 2 provides background information. Section 3 contains the proof of Theorem 1, and Section 4 contains the proof of Theorem 2. The three examples are presented in Section 5, along with the proof of Theorem 3.

2. Background

2.1. Formulas

Construct the set $\mathscr{L}(X,J)$ of formulas using the finite sets X and J in the following way:

1) If $x \in X$ then $x \in \mathscr{L}(X,J)$,
2) If $g \in \mathscr{L}(X,J)$ then $(\neg g) \in \mathscr{L}(X,J)$,
3) If $g,h \in \mathscr{L}(X,J)$ then $(g \wedge h) \in \mathscr{L}(X,J)$,
4) If $g \in \mathscr{L}(X,J)$ then $k_j g \in \mathscr{L}(X,J)$ for every $j \in J$,
5) Only formulas constructed through application of the above four rules are members of $\mathscr{L}(X,J)$.

We write simply $\mathscr{L}$ if there is no ambiguity. We define $g \vee h$ to be $\neg(\neg g \wedge \neg h)$ and $g \rightarrow h$ to be $\neg g \vee h$. For every subset $L \subseteq J$ $E_L(f) = E_L^1(f)$ is defined to be $\bigwedge_{j \in L} k_j f$, $E_L^0(f) := f$, and for $i \geq 1$, $E_L^i(f) := E_L(E_L^{i-1}(f))$. If there is no ambiguity, E will stand for E_J.

A formula $f \in \mathscr{L}(X,J)$ is common knowledge in a subset of formulas $A \subseteq \mathscr{L}(X,J)$ if $E^n f \in A$ for every $n < \infty$.

Throughout this paper, the multi-agent epistemic logic $S5$ will be assumed. For a discussion of the $S5$ logic, see Cresswell and Hughes (1968); and for the multi-agent variation, see Halpern and Moses (1992) and also Bacharach, et al, (1997).

A set of formulas $\mathscr{A} \subseteq \mathscr{L}(X,J)$ is called *complete* if for every formula $f \in \mathscr{L}(X,J)$ either $f \in \mathscr{A}$ or $\neg f \in \mathscr{A}$. A set of formulas is called *consistent* if no finite subset of this set leads to a logical contradiction, meaning a deduction of f and $\neg f$ for some formula f. We define

$$\Omega(X,J) := \{S \subseteq \mathscr{L}(X,J) \mid S \text{ is complete and consistent}\}.$$

Any consistent set of formulas can be extended to a complete and consistent set of formulas, a property we call the *Extension Property*, proven by applying Lindenbaum's Lemma. A *tautology* of $\Omega(X,J)$ is a formula f in $\mathscr{L}(X,J)$ such that f is contained in every member of $\Omega(X,J)$. A formula is *possible* if its negation is not a tautology.

For every agent $j \in J$ we define its *knowledge partition* $\mathscr{Q}^j(X,J)$ to be the partition of $\Omega(X,J)$ generated by the inverse images of the function $\beta^j : \Omega \to 2^{\mathscr{L}(X,J)}$, the set of subsets of $\mathscr{L}(X,J)$, defined by $\beta^j(z) := \{f \in \mathscr{L}(X,J) \mid k_j f \in z\}$. We will write $\mathscr{Q}^j$ if there is no ambiguity. A *possibility set* is defined to be a member of $\mathscr{Q}^j$ for some $j \in J$. Recall the definitions of *cell* and *centered* from the introduction.

The following central lemma is in Simon (1999), but all the components of the proof can be found in other papers (Lemma 4.1 of Halpern and Moses 1992, Aumann 1999):

Lemma A. *For any cell C of $\Omega(X,J)$ $\{f \in \mathscr{L}(X,J) \mid f$ is common knowledge in z for some $z \in C\} = \{f \in \mathscr{L}(X,J) \mid f$ is common knowledge in z for all $z \in C\}$ $= \{f \in \mathscr{L}(X,J) \mid f \in z$ for all $z \in C\}$.*

We define a topology for Ω, the same as in Samet (1990). For every $f \in \mathscr{L}$ define $\alpha(f) := \{z \in \Omega \mid f \in z\}$. Let $\{\alpha(f) \mid f \in \mathscr{L}\}$ be the base of open sets of Ω. (The set of open sets is defined to be the arbitrary unions of base elements. A topology is defined by the fact that $\alpha(f) \cap \alpha(g) = \alpha(f \wedge g)$). The topology of a subset A of Ω will be the relative topology for which the open sets of A are $\{A \cap O \mid O$ is an open set of $\Omega\}$. For any subset $D \subseteq \Omega$, $\bar{D}$ will stand for the closure of D. A dense subset of D is a subset F such that every open set that intersects D also intersects F. An isolated point of a set D is a point $x \in D$ such that there exists a open set O with $\{x\} = O \cap D$.

Due to Lemma A, we have a map F from the meet partition $\mathscr{Q}$ to subsets of formulas defined by $F(C) := \{f \mid f$ is common knowledge in any (equivalently all) members of $C\}$.

For any subset of formulas $T \subseteq \mathscr{L}$ define $\underline{Ck}(T) := \{f \in \mathscr{L} \mid$ there exists an $i < \infty$ and a finite set $T' \subseteq T$ with $(\bigwedge_{t \in T'} E^i(t)) \to f$ a tautology$\}$. We define $\mathscr{T}(X,J) = \{\underline{Ck}(T) \mid T \subseteq \mathscr{L}(X,J)\} \backslash \{\mathscr{L}(X,J)\}$, and we say that T generates $\underline{Ck}(T)$. If there is no ambiguity, we can write simply $\mathscr{T}$. $\underline{Ck}(T)$ is the set of formulas whose common knowledge is implied by the common knowledge of the formulas in T.

An $S \in \mathcal{T}$ is finitely generated if there exists a finite subset $T \subseteq S$ such that $\underline{Ck}(T) = S$. For every set of formulas $T \subseteq \mathcal{L}$ define the set

$$\mathbf{Ck}(T) := \{z \in \Omega \mid \text{every member of } T \text{ is common knowledge in } z\}.$$

For any $T \subseteq \mathcal{L}$, $\mathbf{Ck}(T)$ is a closed set, since the $\mathbf{Ck}(T)$ is the intersection of the sets $\alpha(E_K^l f)$ for all $l < \infty$, finite $K \subseteq J$, and all formulas f in T.

2.2. Semantic models

For this paper a semantic model is a quintuple $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ where J is a set of agents, for each $j \in J$ $\mathcal{P}^j$ is a partition of the set S, X is a set of primitive propositions, and $\psi : X \to 2^S$ is a map from X to the subsets of S, such that for every $x \in X$ the set $\psi(x)$ is interpreted to be the subset of S where x is true. We define a map $\alpha^{\mathcal{K}} : \mathcal{L}(X, J) \to 2^S$ inductively on the structure of the formulas in the following way:

Case 1 $f = x \in X$: $\alpha^{\mathcal{K}}(x) := \psi(x)$.

Case 2 $f = \neg g$: $\alpha^{\mathcal{K}}(f) := S \backslash \alpha^{\mathcal{K}}(g)$,

Case 3 $f = g \wedge h$: $\alpha^{\mathcal{K}}(f) := \alpha^{\mathcal{K}}(g) \cap \alpha^{\mathcal{K}}(h)$,

Case 4 $f = k_j(g)$: $\alpha^{\mathcal{K}}(f) := \{s \mid s \in P \in \mathcal{P}^j \Rightarrow P \subseteq \alpha^{\mathcal{K}}(g)\}$.

We define a map $\phi^{\mathcal{K}} : S \to \Omega(X, J)$ (see Fagin, Halpern, and Vardi 1991) by

$$\phi^{\mathcal{K}}(s) := \{f \in \mathcal{L}(X, J)) \mid s \in \alpha^{\mathcal{K}}(f)\}.$$

We are justified in using again the notation α for the following reason. Consider the map $\bar{\psi} : X \to 2^{\Omega}$ defined by $\bar{\psi}(x) := \{z \in \Omega \mid x \in z\}$. We have a semantic model $\Omega = (\Omega; J; \mathcal{Q}^1, \ldots, \mathcal{Q}^n; X; \bar{\psi})$. (Due to its canonical nature, we index this semantic model with Ω.)

Theorem. *For every $f \in \mathcal{L}(X, J)$, f is a theorem of the multi-agent S5 logic if and only if f is a tautology. Furthermore, $\phi^{\Omega}(z) = z$ for every $z \in \Omega$.*

For a proof of the first part of this theorem, see Halpern and Moses (1992) and Cresswell and Hughes (1968), and for how the second part follows from the first part see Aumann (1999). We will call this result the "Completeness Theorem."

For a semantic model $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$, if $s \in \alpha^{\mathcal{K}}(f)$, or equivalently $f \in \phi^{\mathcal{K}}(s)$, we say that f is true at s with respect to $\mathcal{K}$. We say that f is valid with respect to the semantic model $\mathcal{K}$ if f is true at s with respect to $\mathcal{K}$ for every $s \in S$. The semantic model is *connected* if the meet partition $\bigwedge_{j \in J} \mathcal{P}^j$ is a singleton (equal to $\{S\}$). We define a *connected component* of a semantic model to be a member of this meet partition. Two points $s, s' \in S$ are *adjacent* if they share some member of $\mathcal{P}^j$ for some $j \in J$. We define the *adjacency distance* between any two points s and s' in S as $\min\{d \mid$ there is a sequence $s = s_0, \ldots, s_d = s'$, a function $a : \{1, \ldots, d\} \to J$ and sequence of sets $D_1 \in \mathscr{P}^{a(1)}, \dots, D_d \in \mathscr{P}^{a(d)}$ such that for all $1 \le i \le d$ s_i and s_{i-1} both belong to $D_i\}$, with zero distance between any point and itself and infinite distance if there is no such sequence from s to s'.

2.3. Canonical finite models

We define the *depth* of a formula inductively on the structure of the formulas. If $x \in X$, then $\text{depth}(x) := 0$. If $f = \neg g$ then $\text{depth}(f) := \text{depth}(g)$; if $f = g \wedge h$ then $\text{depth}(f) := \max(\text{depth}(g), \text{depth}(h))$; and if $f = k_j(g)$ then $\text{depth}(f) := \text{depth}(g) + 1$.

For every $0 \le i < \infty$ we define $\mathscr{L}_i := \{f \in \mathscr{L} \mid \text{depth}(f) \le i\}$ and define Ω_i to be the set of maximally consistent subsets of $\mathscr{L}_i$. If there may be ambiguity, we will write $\Omega_i(X, J)$. We must perceive an Ω_i in two ways, as a semantic model in its own right and as a canonical projective image of Ω inducing a partition of Ω through inverse images. We define $\pi_i : \Omega \to \Omega_i$ to be the canonical projection $\pi_i(z) := z \cap \mathscr{L}_i$. Due to an application of Lindenbaum's Lemma, the maps π_i are surjective. For any semantic model $\mathscr{K} = (S; J; (\mathscr{P}^j \mid j \in J); X; \psi)$ and $i \ge 0$ we define $\phi_i^{\mathscr{K}} : S \to \Omega_i(X, J)$ by $\phi_i^{\mathscr{K}}(s) := \phi^{\mathscr{K}}(s) \cap \mathscr{L}_i(X, J) = \pi_i(\phi^{\mathscr{K}}(s))$.

For every $0 \le i < \infty$ we consider the semantic model $\underline{\Omega_i} = (\Omega_i; J; (\overline{\mathscr{F}}_i^j \mid j \in J); X; \overline{\psi}_i)$, where $\overline{\psi}_i = \pi_i \circ \overline{\psi}$ and for $i > 0$ the partition $\overline{\mathscr{F}}_i^j$ of Ω_i is induced by the inverse images of the function $\beta_i^j : \Omega_i \to 2^{\mathscr{L}_{i-1}(X,J)}$ defined by $\beta_i^j(w) := \{f \in \mathscr{L}_{i-1}(X, J) \mid k_j(f) \in w\}$. We define $\overline{\mathscr{F}}_0^j = \{\Omega_0\}$ for every $j \in J$.

Now we consider Ω_i again as a canonical projective image. $\mathscr{G}_i$ is defined to be the partition of Ω induced by the inverse images of π_i, $\mathscr{G}_i := \{\pi_i^{-1}(w) \mid w \in \Omega_i\}$. By the definition of Ω, the join partition $\bigvee_{i=1}^{\infty} \mathscr{G}_i$ is the discrete partition of Ω, meaning that it consists of singletons. Let $\mathscr{F}_i^j$ be the partition on Ω, coarser than $\mathscr{G}_i$, defined by $\mathscr{F}_i^j := \{\pi_i^{-1}(B) \mid B \in \overline{\mathscr{F}}_i^j\}$. From the definitions of the Ω_i and the $\mathscr{F}_i^j$ it follows that $\bigvee_{i=0}^{\infty} \mathscr{F}_i^j = \mathscr{Q}^j$.

An *i-atom* (or just atom) is a member of Ω_i.

Since X and J are finite, there are several important properties of the semantic models Ω_i, all of which are used in this paper.

(i) Ω_i is finite for every $0 \le i < \infty$. (For a more general statement, see Lismont and Mongin 1995.)

(ii) For every $w \in \Omega_i$ we can define a formula $f(w)$ of depth i or less such that $\alpha^{\Omega_i}(f(w)) = \{w\}$, meaning that the formula $f(w)$ is true with respect to Ω_i only at $w \in \Omega_i$. This follows from the finiteness of Ω_i. For any subset $A \subseteq \Omega_i$ define $f(A) := \bigvee_{w \in A} f(w)$, a formula that is true with respect to Ω_i only in the subset A.

(iii) It is easy to extend a member of Ω_i to a member of Ω_{i+1}. Fix $0 \le i < \infty$ and $w \in \Omega_i$. For every $j \in J$ define $\overline{F}_i^j$ by $w \in \overline{F}_i^j \in \overline{\mathscr{F}}_i^j$. If $(M_i^j \mid j \in J)$ are subsets of $(\overline{F}_i^j \mid j \in J)$, respectively, such that

1) $w \in M_i^j$ for every $j \in J$, and
2) for every $B \in \mathscr{G}_{i-1}$ $\overline{F}_i^j \cap \pi_i(B) \ne \emptyset$ implies that $M_i^j \cap \pi_i(B) \ne \emptyset$,

then there is a unique $v \in \Omega_{i+1}$ such that $\pi_i \circ \pi_{i+1}^{-1}(v) = w$ and for every $u \in \Omega_i$ $\neg k_j \neg f(u) \in v$ if and only if $u \in M_i^j$. Furthermore, this is the only way to extend a member of Ω_i to a member of Ω_{i+1}; this is Lemma 4.2 of Fagin, Halpern, and Vardi (1991). For any $i \geq 0$ and $v \in \Omega_k$ with $k > i$ we define $M_i^j(v) := \{u \in \Omega_i \mid \neg k_j \neg f(u) \in v\}$. Notice that if $w \in F \in \overline{\mathscr{F}}_i^j$ then $M_{i-1}^j(w)$ is equal to $\pi_{i-1} \circ \pi_i^{-1}(F)$, which could be a proper subset of the member of $\overline{\mathscr{F}}_{i-1}^j$ that contains $\pi_{i-1} \circ \pi_i^{-1}(w)$.

(iv) For every formula $f \in \mathscr{L}_i$ and $l \geq i$ $\pi_l^{-1}(\alpha^{\Omega_l}(f)) = \alpha^{\Omega}(f)$. This follows from (iii) and the Completeness Theorem. (See also Lemma 2.5 in Fagin, Halpern, and Vardi 1991.)

(v) As a semantic model, every Ω_i is connected. This was proven first by Fagin, Halpern, and Vardi (1991) and it can be proven in several ways (for example from Proposition 1 of Simon, 1999).

3. Finite generation

Before proving Theorem 1 we must show for every connected semantic model $\mathscr{K}$ that the image in Ω by $\phi^{\mathscr{K}}$ can be approximated by finite cells.

Following the definition in Fagin, Halpern, and Vardi (1991) for "closed," for $i > 0$ we define a non-empty subset $A \subseteq \Omega_i(X, J)$ to be *semantically closed* if for every $j \in J$, every $B \in \mathscr{G}_{i-1}$ and every $w \in A$ if $\pi_i^{-1}(w) \subseteq F \in \mathscr{F}_i^j$ and $F \cap B \neq \emptyset$ then $F \cap B \cap \pi_i^{-1}(A) \neq \emptyset$. Any non-empty subset of Ω_0 is allowed to be semantically closed.

Two semantic models $\mathscr{K} = (S; J; (\mathscr{P}^j \mid j \in J); X; \psi)$ and $\mathscr{K}' = (S'; J; (\hat{\mathscr{P}}^j \mid j \in J); X; \psi')$ are *strongly isomorphic* if there is a bijection $\gamma : S \to S'$ such that for every $x \in X$ $\gamma(\psi(x)) = \psi'(x)$ and for every pair $x, x^* \in X$ and every $j \in J$, x and x^* share the same member of $\mathscr{P}^j$ if and only if $\gamma(x)$ and $\gamma(x^*)$ share the same member of $\hat{\mathscr{P}}^j$. (In this paper we do not consider isomorphisms involving permutations of X and J.) Given a subset $A \subseteq S$, we define the semantic model $\mathscr{V}^{\mathscr{K}}(A) := (A; J; (\mathscr{P}^j|_A \mid j \in J); X; \psi|_A)$ where for all $j \in J$ $\mathscr{P}^j|_A := \{F \cap A \mid F \cap A \neq \emptyset \text{ and } F \in \mathscr{P}^j\}$ and for all $x \in X$ $\psi|_A(x) = \psi(x) \cap A$. If there is no ambiguity concerning the semantic model $\mathscr{K}$, we can replace $\mathscr{V}^{\mathscr{K}}(A)$ by $\mathscr{V}(A)$. For any semantically closed set $A \subseteq \Omega_i$, by $\mathscr{V}(A)$ we mean $\mathscr{V}^{\Omega_i}(A)$

Lemma 1.

(a) *If $A \subseteq \Omega_i$ is semantically closed then for every $w \in A$ $\phi_i^{\mathscr{V}(A)}(w) = w$.*

(b) *If a semantic model $\mathscr{K} = (S; J; (\mathscr{P}^j \mid j \in J); X; \psi)$ is connected then $\mathscr{V}^{\Omega}(\phi^{\mathscr{K}}(S))$ and $\mathscr{V}^{\Omega_i}(\phi_i^{\mathscr{K}}(S))$ are connected for all $i < \infty$ and $\phi^{\mathscr{K}}(S)$ is contained and dense in a cell of Ω. Furthermore, the image $\phi_i^{\mathscr{K}}(S)$ in Ω_i is semantically closed.*

(c) *If C is a finite cell of Ω and i is large enough so that $\pi_{i-1} : C \to \Omega_{i-1}$ is injective, then $\mathscr{V}(\pi_i(C))$ is strongly isomorphic to $\mathscr{V}(C)$, with $\pi_i|_C$ providing the corresponding bijection. (The proof is similar but with a different conclusion to that of Theorem 4.23 of Fagin, Halpern, and Vardi, 1991.)*

(d) *If $A \subseteq \Omega_i$ is semantically closed and $\mathscr{V}(A)$ is connected, then there is a finite cell C of Ω such that $\phi^{\mathscr{V}(A)}(A) = C$ and furthermore $\mathscr{V}(C)$ and $\mathscr{V}(A)$ are strongly isomorphic by the bijection $\pi_i|_C : C \to A$ or its inverse $\phi^{\mathscr{V}(A)} : A \to C$.*

Proof: (a) It suffices by Property (iv) to prove for every formula $f \in \mathscr{L}_i$ that f is true at $w \in A$ with respect to the semantic model $\mathscr{V}(A)$ if and only if f is true at w with respect to the semantic model Ω_i. We proceed by induction on the structure of formulas. If the depth of f is zero, the claim follows from the definition of $\bar{\psi}|_A$. Likewise if the claim is true for f and g then it is true for either $\neg f$ or $f \wedge g$ directly from the definitions of $\alpha^{\mathscr{V}(A)}$ and α^{Ω_i}. Since every member of Ω_i is complete, it suffices to show for every $f \in \mathscr{L}_{i-1}$ for which the claim is established that $k_j f$ (respectively $\neg k_j f$) true at w with respect to Ω_i implies that $k_j f$ (respectively $\neg k_j f$) is true at w with respect to $\mathscr{V}(A)$.

Let us assume that $w \in A$, depth$(f) < i$, and $k_j f$ is true at the atom w with respect to the semantic model Ω_i. Let $w \in F \in \overline{\mathscr{F}_i^j}|_A$ and $w \in F' \in \overline{\mathscr{F}_i^j}$, so that $F' \subseteq \alpha^{\Omega_i}(f)$. It follows that $F = A \cap F' \subseteq \alpha^{\Omega_i}(f) \cap A = \alpha^{\mathscr{V}(A)}(f)$, the last equality from the induction hypothesis.

On the other hand, assume that $w \in A$, depth$(f) < i$, and $\neg k_j f$ is true at the atom w with respect to Ω_i. That means if $w \in F' \in \overline{\mathscr{F}_i^j}$ then $F' \cap \alpha^{\Omega_i}(\neg f) \neq \emptyset$. Let $B \in \mathscr{G}_{i-1}$ be such that $\neg f$ is true in B and $\pi_i(B) \cap F' \neq \emptyset$. By the semantically closed property of A there is some atom $w' \in A$ with $w' \in \pi_i(B) \cap F'$. Letting $F := A \cap F' \in \overline{\mathscr{F}_i^j}|_A$, the induction hypothesis implies that $\emptyset \neq A \cap \pi_i(B) \subseteq \alpha^{\mathscr{V}(A)}(\neg f)$, hence $F \cap \alpha^{\mathscr{V}(A)}(\neg f) \neq \emptyset$ and $\neg k_j \neg(\neg f)$ and $\neg k_j(f)$ are true at w with respect to the semantic model $\mathscr{V}(A)$.

(b) The proof of the first part is straightforward; dense containment is precisely the statement of Lemma 5b of Simon (1999). Now the rest follows by Lemma 5a of Simon (1999) and Property (iii), (or explicitely by Proposition 4.20 of Fagin, Halpern, and Vardi 1991, which states that for any cell C the subset of i-atoms $\pi_i(C)$ is semantically closed. Lemma 5a of Simon (1999) states that every member of $\mathscr{P}^j$ maps to a dense subset of some member of $\mathscr{Q}^j$.)

(c) For all $l < \infty$ and $c \in C$, $\pi_l(c) = \phi_l^{\Omega}(c)$ holds by the Completeness Theorem. Let $A = \pi_i(C) = \phi_i^{\Omega}(C)$. Because of the definition of $\bar{\psi}|_A$ it suffices to show that for all pairs $c, c' \in C$ the atoms $\pi_i(c)$ and $\pi_i(c')$ belong to the same member of $\overline{\mathscr{F}_i^j}|_A$ if and only if c and c' belong to the same member of $\mathscr{Q}^j$. We consider the non-trivial implication. For $m \geq i$ let $A(m)$ be $\pi_m(C) = \phi_m^{\Omega}(C)$, so that $A(i) = A$. We suppose, for the sake of contradiction, that $\pi_i(c)$ and $\pi_i(c')$ belong to the same member of $\mathscr{F}_i^j|_A$, meaning also that c and c' belong to the same member of $\mathscr{F}_i^j$, but c and c' don't belong to the same member of $\mathscr{Q}^j$. Since $\mathscr{Q}^j = \bigvee_{i=0}^{\infty} \mathscr{F}_i^j$, we must assume that there is a maximal l with the property that $\pi_l(c)$ and $\pi_l(c')$ do belong to the same member of $\overline{\mathscr{F}_l^j}|_{A(l)}$, and let F be this member of $\overline{\mathscr{F}_l^j}|_{A(l)}$. Let F' be the member of $\overline{\mathscr{F}^j}_{l+1}|_{A(l+1)}$ containing $\pi_{l+1}(c)$ but not $\pi_{l+1}(c')$ and let G be the member of $\mathscr{G}_{l-1}$ containing c'. $\pi_l(G)$ and F have a non-empty intersection (they both contain $\pi_l(c')$), so by Property (iii) $\pi_{l+1}(G)$ and $F^* \in \mathscr{F}_{l+1}^j$ have a non-empty intersection, where $F' = F^* \cap A(l+1)$. Because $A(l+1)$ is a semantically closed set by (b), there must be a $c^* \in C$ with $c^* \in G \in \mathscr{G}_{l-1}$ and $\pi_{l+1}(c^*) \in F'$. But since $l \geq i$ and $\pi_{i-1} : C \to \Omega_{i-1}$ is an injection, c^* must be c', a contradiction.

(d) By (b), $\phi^{\mathscr{V}(A)}(A)$ is a dense subset of a cell C. But $\phi^{\mathscr{V}(A)}(A)$ is finite, hence closed, and therefore equals C.

As before, for $l \geq i$ define $A(l) := \phi_l^{\mathscr{V}(A)}(A) = \pi_l(\phi^{\mathscr{V}(A)}(A)) = \pi_l(C)$. By (c), for $l > i$ we have $\mathscr{V}(A(l))$ strongly isomorphic to $\mathscr{V}(C)$ using the map $\pi_l|_C$. By (a), $\phi_{i+1}^{\mathscr{V}(A)} : A \to A(i+1)$ is bijective and $\pi_i \circ \pi_{i+1}^{-1}|_{A(i+1)}$ is the inverse. It suffices to show an isomorphism between $\mathscr{V}(A)$ and $\mathscr{V}(A(i+1))$ using the map $\phi_{i+1}^{\mathscr{V}(A)} : A \to A(i+1)$. This follows directly from the definition of ϕ and that the partition $\mathscr{F}_{i+1}^j$ is finer than $\mathscr{F}_i^j$.

Lastly, if $c \in \phi^{\mathscr{V}(A)}(A) = C$ then $\phi^{\Omega}(c) = \phi^{\mathscr{V}(C)}(c) = \phi^{\mathscr{V}(A(l))}(\pi_l(c))$ for all $l \geq i$, the last equality by the isomorphism just proved. The Completeness Theorem implies that $c = \phi^{\Omega}(c)$, meaning that $c = \phi^{\mathscr{V}(A(l))}(\pi_l(c))$ for all $l \geq i$. q.e.d.

Theorem 1. *Let C be any cell and let S and T be sets of formulas such that* $S = \underline{Ck}(T)$, $S = F(C)$, *and* $|T| < \infty$. *Then the following are equivalent:*

(1) *C is a finite set,*
(2) $C = \mathbf{Ck}(T) = \mathbf{Ck}(S)$,
(3) *S is maximal in* $\mathscr{T}$,
(4) *S is maximal among all the members of* $\mathscr{T}$ *generated by finite sets of formulas,*
(5) *C is centered, meaning that* $F^{-1}(S) = \{C\}$.

Proof: We can assume that $T = \{g\}$ for a single formula g. Let d be the depth of g.

(1) ⇒ (2): Being finite, C is a closed set. Since C is dense in $\mathbf{Ck}(\{g\})$ it follows that $C = \mathbf{Ck}(\{g\})$.

(2) ⇒ (3), (3) ⇒ (4), (2) ⇒ (5): All three implications are obvious.

Due to Lemma 1b, for every $i < \infty$ $\pi_i(\mathbf{Ck}(\{g\}))$ is semantically closed and $\mathscr{V}(\pi_i(\mathbf{Ck}(\{g\}))$ is connected.

(4) ⇒ ((1) and (2)): Consider $A := \pi_d(\mathbf{Ck}(\{g\}))$. Since $\text{depth}(g) = d$ Property (iv) implies that $\pi_d^{-1}(A) \subseteq \alpha^{\Omega}(g)$. For $l \geq d$ define $A(l) := \pi_l(\phi^{\mathscr{V}(A)}(A)) = \phi_l^{\mathscr{V}(A)}(A)$. By Lemma 1a we have that $\pi_l^{-1}(A(l)) \subseteq \pi_d^{-1}(A) \subseteq \alpha^{\Omega}(g)$ and therefore by Property (iv) $f(A(l)) \to f(A)$ and $f(A) \to g$ are tautologies for all $l \geq d$. By Lemma A and Property (iv) we have for all $l \geq d$ that $f(A(l))$ is valid with respect to $\mathscr{V}(A(l))$ and $\phi^{\mathscr{V}(A)}(A) = \phi^{\mathscr{V}(A(l))}(A(l)) \subseteq \mathbf{Ck}(\{f(A(l))\}) \subseteq \mathbf{Ck}(S) = \mathbf{Ck}(\{g\})$, the first equality by Lemma 1d. It follows from (4) that $\mathbf{Ck}(S) = \mathbf{Ck}(\{f(A(l))\})$ for all $l \geq d$. By Lemma 1d there is a finite cell C' of Ω such that $C' = \phi^{\mathscr{V}(A)}(A)$. But then it follows that $C' \subseteq \mathbf{Ck}(S) = \bigcap_{l=d}^{\infty} \mathbf{Ck}(\{f(A(l))\}) \subseteq \bigcap_{l=d}^{\infty} \alpha^{\Omega}(f(A(l)) = \bigcap_{l=d}^{\infty} \pi_l^{-1}(A(l)) = \bigcap_{l=d}^{\infty} \pi_l^{-1}(\phi_l^{\mathscr{V}(A)}(A)) = \phi^{\mathscr{V}(A)}(A) = C'$, and we must conclude that $C' = C$.

(5) ⇒ (1) By the proof of Theorem 1 of Simon (1999) C contains a non-empty open set of $\mathbf{Ck}(\{g\})$. (The proof of Theorem 1 of Simon 1999 is the argument that a cell C is centered if and only if it contains a non-empty set that is open relative to the closure of C.) Without loss of generality, we assume that this non-empty open set is $W^* \cap \mathbf{Ck}(\{g\})$ for some $W^* \in \mathscr{G}_i$ and $i \geq d$. Consider the set $A := \pi_i(\mathbf{Ck}(\{g\}))$. As in the proof of (4) ⇒ (1), we have that $f(A)$ is valid with respect to $\mathscr{V}(A)$ and we can conclude from $i \geq d$ that $f(A) \to g$ is a tautology and $\phi^{\mathscr{V}(A)}(A) \subseteq \mathbf{Ck}(\{g\})$. Since $W^* \cap \pi_i^{-1}(A) \supseteq W^* \cap \mathbf{Ck}(\{g\}) \neq \emptyset$ we conclude by Lemma 1a that $\phi^{\mathscr{V}(A)}(A)$ has a non-empty intersection with W^*. But since $\phi^{\mathscr{V}(A)}(A) \subseteq \mathbf{Ck}(\{g\})$ and C contains the set $\mathbf{Ck}(\{g\}) \cap W^*$, we must conclude that $C \cap \phi^{\mathscr{V}(A)}(A) \neq \emptyset$. But by Lemma 1d $\phi^{\mathscr{V}(A)}(A)$ is a cell, hence $C = \phi^{\mathscr{V}(A)}(A)$, and furthermore $\phi^{\mathscr{V}(A)}(A)$ is finite. q.e.d.

4. A continuum of dense cells

The goal of this section is to prove the following:

Theorem 2. *If there are at least two agents then the number of cells dense in Ω has the cardinality of the continuum.*

Before proving Theorem 2, we describe the main ideas behind the proof.

We want to construct uncountably many distinct dense cells through the uncountably many possibilities in $\{0,1\}^{\infty}$. As was shown by Fagin, Halpern, and Vardi (1991), infinite repetitions of what they called the *no-information extension* (applied to the members of any Ω_i) generate membership in dense cells. But one does not need to repeat the no-information extension at every stage to generate membership in a dense cell. One can alternate the no-information extension with long periods where the mutual knowledge of some formulas is confirmed to arbitrarily long depths.

We consider the formulas $g_1, g_2, \ldots$ whose common knowledge define the semantic models $\Omega_1, \Omega_2, \ldots$. At every level i we choose between the no-information extension and an increasing depth of mutual knowledge of the formula g_k for some $k \leq i$; (k is determined to be the lowest number for which this is possible). The latter is performed by the canonical map of Ω_k into Ω_{i+1}. An *alienated extension* is an alternation between these two processes which involves an infinite subsequence of stages where the no-information extension is applied. We call it alienated because there is a process of building common knowledge that is interrupted repeatedly.

In our construction, we take a subsequence $n_1, n_2, \ldots$ such that the distance $n_i - n_{i-1}$ increases as i increases. For any point in Ω_{n_i} we make a choice whether to confirm the mutual knowledge of the formula g_{n_i} until the n_{i+1} stage (meaning up to the depth of $n_{i+1} - n_i$) or to perform the no-information extension repeatedly until the n_{i+1} stage.

If two points belong to the same cell, there is some finite sequence of adjacent pairs connecting them (meaning that for every pair of adjacent points in the sequence there is some agent with identical knowledge at this pair). If for one of these points the mutual knowledge of some formula f is confirmed to depth n (meaning that $E^n(f)$ is valid at this point) then $E^{n-k}(f)$ is also true for all other points in the same cell within an adjacency distance of k. In contrast, the no-information extension applied from stage $n_i + 1$ to stage $n_i + 2$ destroys the first level of mutual knowledge of the formula g_{n_i}. We conclude that any two points x, y that belong to the same cell and result from alienated extensions generated by our subsequence $n_1, n_2, \ldots$ must have involved the same extension choices after some finite level, namely any level n_i where $n_{i+1} - n_i$ exceeds the adjacency distance between them. Therefore infinitely repeated differences in our extension choices generate membership in distinct dense cells.

Let $0 \leq i < \infty$, fix $w \in \Omega_i$, and let $\bar{F}_i^j$ be the member of $\bar{\mathscr{F}}_i^j$ containing w. Define $p_{i+1}(w)$ to be that unique member of Ω_{i+1} such that $\pi_i(p_{i+1}(w)) = w$ and $M_i^j(p_{i+1}(w)) = \bar{F}_i^j$ for every $j \in J$ (see the definition of M_i^j at the end of Section 2.3). The map $p_{i+1} : \Omega_i \to \Omega_{i+1}$ was called the "no-information" extension in Fagin, Halpern, and Vardi (1991).

Let $2_{\infty}^{\mathbf{N_0}}$ be the set of subsets of the whole numbers $\mathbf{N_0} = \{0, 1, 2, \ldots\}$ with infinite cardinality ($S \in 2_{\infty}^{\mathbf{N_0}}$ implies $|S| = \infty$). For every pair $i, k \in S$ with $k \geq i$ we will define a map $p_k^S : \Omega_i \to \Omega_k$. If $i \in S \in 2_{\infty}^{\mathbf{N_0}}$ define $n_S(i) := \inf\{k \in S \mid k > i\}$, the first member of S strictly larger than i. If $i \in S$ and $w \in \Omega_i$ then define $p_{n_S(i)}^S(w) := \phi_{n_S(i)}^{\Omega_i}(w)$ and define $p_i^S(w) := w \cdot p_{n_S(i)}^S(w)$ is an extension of w because of Lemma 1, meaning that $\pi_i \circ \pi_{n_S(i)}^{-1}(p_{n_S(i)}^S(w)) = w$. For every $k \in S$

and $w \in \Omega_i$ with $k \geq i \in S$ and $p_k^S(w) \in \Omega_k$ already defined, define $p_{ns(k)}^S(w)$ to be $p_{ns(k)}^S(p_k^S(w))$. Lastly, for all $i \in S \in 2_\infty^{\mathbf{N}_0}$ and $w \in \Omega_i$ define $p^S : \Omega_i \to \Omega$ by

$$p^S(w) := \bigcap_{l \in S, l > i} \pi_l^{-1}(p_l^S(w)).$$

For any $i \in S \in 2_\infty^{\mathbf{N}_0}$ and $w \in \Omega_i$ we call $p^S(w)$ the alienated extension of w with respect to S.

An alienated extension involves an infinite number of no-information extensions. For all $0 \leq i < \infty$ and $w \in \Omega_i$ it follows that $\phi_{i+1}^{\Omega_i}(w) = p_{i+1}(w)$, meaning also that $p^{\mathbf{N}_0}$ is the infinite repetition of the no-information extension. This is because if $w \in F \in \overline{\mathscr{F}}_i^j$ and v is any member of F then $\phi_{i+1}^{\Omega_i}(v)$ and $\phi_{i+1}^{\Omega_i}(w)$ share the same member of $\overline{\mathscr{F}}_{i+1}^j$, implying that $\neg k_j \neg f(v) \in \phi_{i+1}^{\Omega_i}(w)$. We can conclude for every $j \in J$ that $M_{i+1}^j(\phi_i^{\Omega_i}(w)) = F$. However, as we will see from Lemma 5, $\phi_{i+2}^{\Omega_i}(w)$ does not equal $p_{i+2} \circ p_{i+1}(w)$ for any $w \in \Omega_i$.

Lemma 2. *If $S \in 2_\infty^{\mathbf{N}_0}$ and there are at least two agents, all alienated extensions with respect to S share the same cell of Ω.*

Proof: If w and w' are members of Ω_i such that both are contained in the same member of $\overline{\mathscr{F}}_i^j$, then from induction and the definition of $\phi^{\Omega_i} p^S(w)$ and $p^S(w')$ are both contained in the same member of $\mathscr{Q}^j$.

Now, given any $i, k \in S$ and $B \in \mathscr{G}_i$ and $D \in \mathscr{G}_k$, the adjacency distance between $p^S(\pi_i(B))$ and $p^S(\pi_k(D))$ in Ω is no more than the adjacency distance between $p_{\max(i,k)}^S(\pi_i(B))$ and $p_{\max(i,k)}^S(\pi_k(D))$ in $\Omega_{\max(i,k)}$. The rest follows by the connectedness of Ω_i for every i. □

Notice by Lemma A that the cell containing all the alienated extensions with respect to some $S \in 2_\infty^{\mathbf{N}_0}$ must intersect $\alpha^\Omega(f)$ for all possible formulas $f \in \mathscr{L}$, which means that only the tautologies of the logic are held in common knowledge in the cell.

Define the formula $g_i := f(\phi_{i+1}^{\Omega_i}(\Omega_i))$ of depth $i+1$, the formula true with respect to Ω_{i+1} only in the image $\phi_{i+1}^{\Omega_i}(\Omega_i)$. As we will see, the common knowledge of g_i is closely linked to the semantic model Ω_i (see also Theorem 4.23 of Fagin, Halpern, and Vardi 1991).

Lemma 3. *If $i \in S \in 2_\infty^{\mathbf{N}_0}$, and $i+1, i+2, \ldots, i+l \notin S$ for some $l \geq 1$, then $p^S(\Omega_i) \subseteq \alpha^\Omega(E^l(g_i))$ and g_i is common knowledge in the semantic model Ω_i.*

Proof: By Lemma 1d, $\phi^{\Omega_i}(\Omega_i)$ is a cell. Because $\phi^{\Omega_i}(\Omega_i) \subseteq \alpha^\Omega(g_i)$, Property (iv) and Lemma A imply that g_i is common knowledge in the cell $\phi^{\Omega_i}(\Omega_i)$. If $E^l(g_i)$, a formula of depth $i+l+1$, were not true at some point of $\phi_{i+l+1}^{\Omega_i}(\Omega_i)$ then also by Property (iv) we must have that g_i is not common knowledge at some point of $\phi^{\Omega_i}(\Omega_i)$, a contradiction.

Additionally, by Lemma 1d we have that g_i is common knowledge in the semantic model Ω_i. □

The proof of the following lemma is omitted because it is a direct corollary of Property (iii).

Lemma 4. *With at least two agents, the following is true for every $j \in J$ and $0 \leq i < \infty$:*

a) *if $w \in \Omega_i$, $F \in \mathscr{F}_i^j$ and $w \in \pi_i(F) \in \overline{\mathscr{F}}_i^j$ then the number of members of $\mathscr{F}_{i+1}^j$ contained in F with a non-empty intersection with $\pi_i^{-1}(w)$ is at least 2, and*
b) *if $w \in \Omega_i$ and $\pi_i^{-1}(w) \cap F \neq \emptyset$ for some $F \in \mathscr{F}_{i+1}^j$ then the number of elements of Ω_{i+1} that are in $\pi_{i+1}(\pi_i^{-1}(w)) \cap \pi_{i+1}(F)$ is at least 2.*

Lemma 5. *If there are at least two agents and $i+1$ is in $S \in 2^{\mathbf{N_0}}_\infty$, then Eg_i is not true at any point of $p^S(\Omega_i)$.*

Proof: Because $\phi_{i+2}^{\Omega_{i+1}}(w) = p_{i+2}(w)$ for any $w \in \Omega_{i+1}$, it suffices to show for any $j \in J$ that $k_j(g_i)$ is not true at $p_{i+2}(w)$ for any $w \in \Omega_{i+1}$. Let $w \in \pi_{i+1}(F) \in \overline{\mathscr{F}}_{i+1}^j$ for some $F \in \mathscr{F}_{i+1}^j$. For every $v \in \Omega_i$ there is only one member of Ω_{i+1} in the subset $\pi_{i+1}(\pi_i^{-1}(v))$ where g_i is true. From now on let $v = \pi_i \circ \pi_{i+1}^{-1}(w)$. By Lemma 4 there is more than one $u \in \Omega_{i+1}$ with $u \in \pi_{i+1}(F) \cap \pi_{i+1}(\pi_i^{-1}(v))$. The $F' \in \mathscr{F}_{i+2}^j$ containing $\pi_{i+2}^{-1}(p_{i+2}(w))$ must have a non-empty intersection with $\pi_{i+1}^{-1}(u)$ for all $u \in \Omega_{i+1}$ with $u \in \pi_{i+1}(F)$, and therefore F' is not contained in $\alpha^\Omega(g_i)$. □

Proof of Theorem 2: Define a map $\beta : 2^{\mathbf{N_0}} \to 2^{\mathbf{N_0}}_\infty$ by $\beta(S) := \{0, 1, 2, 4, 8, \ldots\} \cup \{2^i + 1, \ldots, 2^{i+1} - 1 \mid i \in S\}$. Define an equivalence relation on $2^{\mathbf{N_0}}$ by $S \sim T$ if and only if there exists an $m \in \mathbf{N_0}$ such that $S \backslash \{0, 1, 2, \ldots m\} = T \backslash \{0, 1, 2, \ldots, m\}$. The co-sets of this equivalence relation have the cardinality of the continuum.

Due to Lemma 2, it suffices to show for some $w \in \Omega_0$ that if S and T are both subsets of $\mathbf{N_0}$ with $S \not\sim T$ then $p^{\beta(S)}(w)$ does not share the same cell as $p^{\beta(T)}(w)$. For the sake of contradiction, let us suppose that the adjacency distance in Ω between $p^{\beta(S)}(w)$ and $p^{\beta(T)}(w)$ equals a finite number $l < \infty$. Because $S \not\sim T$ there exists an $i > \log_2((l+2))$ such that $i \in S$ and $i \notin T$, or vice versa. By symmetry, let us assume that $i \in S$ and $i \notin T$. Lemma 3 applied to $p^{\beta(T)}(w)$ implies that $p^{\beta(T)}(w) \in \alpha^\Omega(E^{l+1} g_{2^i})$. But because the adjacency-distance between $p^{\beta(S)}(w)$ and $p^{\beta(T)}(w)$ is l we have that $p^{\beta(S)}(w) \in \alpha^\Omega(E(g_{2^i}))$, a contradiction to Lemma 5. q.e.d.

5. A special observing agent

Theorem 1 and Theorem 2 were made possible by our ability to get beyond the maximal depth of the finite set of generating formulas. Now we will construct infinite sequences of generating formulas to find cells that do not obey the statement of Theorem 1 when the assumption of finite generation is removed. We do this by introducing a partition of $\Omega(X, \{1, 2\})$, creating a new semantic model $\mathscr{K}$ by assigning the new partition to a new third agent, and then looking at the image of $\phi^{\mathscr{K}}$ in $\Omega(X, \{1, 2, 3\})$.

There are several problems with this approach. First, we would like to relate our partition of $\Omega(X, \{1, 2\})$ to a set of generating formulas for a member of $\mathscr{T}(X, \{1, 2, 3\})$. More seriously, an arbitrary partition of $\Omega(X, \{1, 2\})$ for a third agent will not in general yield a continuous or an open map from $\Omega(X, \{1, 2\})$ to $\Omega(X, \{1, 2, 3\})$. We would like to use the topology of $\Omega(X,\{1,2\})$ in our analysis of the image in $\Omega(X,\{1,2,3\})$. Related to this, we need to identify a *cell* rather than just a semantic model with the desired property. These problems are overcome, however, by Theorem 3.

To illustrate this last problem with techniques of the fourth section, take the set $A = \{p^{\mathbf{N}_0}(w) \mid i \geq 0, w \in \Omega_i\} \subseteq \Omega$. It is not difficult to show that $\phi^{\mathscr{V}(A)}(a) = a$ for all $a \in A$. $\mathscr{V}(A)$ is a countable and connected semantic model that maps canonically into an un-centered cell. But this is no example of a countable un-centered cell.

For any $n \geq 2$, we define a *consistent sequence of partitions* of $\Omega = \Omega(X,J)$ to be a sequence of partitions $(\mathscr{P}_0, \mathscr{P}_1, \ldots)$ of Ω such that

1) for every $0 \leq i < \infty$ $\mathscr{P}_i$ is equal to or coarser than $\mathscr{G}_i$,
2) for every $0 < i < \infty$ $\mathscr{P}_i$ is equal to or finer than $\mathscr{P}_{i-1}$, and
3) for every $0 < i < \infty$ if $P_i \in \mathscr{P}_i$, $P_{i-1} \in \mathscr{P}_{i-1}$ and $P_i \subseteq P_{i-1}$ then $P_i \cap B \neq \emptyset$ for every $B \in \mathscr{G}_{i-1}$ with $B \subseteq P_{i-1}$.

For any consistent sequence of partitions $\mathscr{B} = (\mathscr{P}_i \mid 0 \leq i < \infty)$ of $\Omega(X,J)$ define a semantic model

$$\mathscr{K}(\mathscr{B}) = (\Omega(X,J); (\mathscr{Q}^j \mid j \in J), \mathscr{P}_\infty; X; \bar{\psi})$$

where the partition $\mathscr{P}_\infty$ for the $|J|+1$st agent is the limit of the partitions $\mathscr{P}_i$, (meaning that z and z' share the same member of $\mathscr{P}_\infty$ if and only if they share the same member of $\mathscr{P}_i$ for every $i < \infty$), and $\bar{\psi}$ and the $\mathscr{Q}^j$ are the same used to define the semantic model $\Omega(X,J)$. Let χ stand for the $|J|+1$st agent. For every $i < \infty$ and $w \in \Omega_i$ define $P_i(w)$ to be the member of $\mathscr{P}_i$ containing $\pi_i^{-1}(w)$ and for any $z \in \Omega$ define $P_i(z)$ to be member of $\mathscr{P}_i$ containing z. For every $0 \leq i < \infty$ define a formula $h(\mathscr{P}_i) \subseteq \mathscr{L}(X, J \cup \{\chi\})$ of depth $i+1$ by

$$h(\mathscr{P}_i) := \bigwedge_{w \in \Omega_i} \left(f(w) \rightarrow \left(\bigwedge_{v \in \pi_i(P_i(w))} \neg k_\chi \neg f(v) \bigwedge_{v \notin \pi_i(P_i(w))} k_\chi \neg f(v) \right) \right).$$

Define the set of formulas $T(\mathscr{B}) := \{h(\mathscr{P}_0), h(\mathscr{P}_1), \ldots\}$.

Theorem 3. *The semantic model* $\mathbf{Ck}(T(\mathscr{B}))$*, as a closed subset of* $\Omega(X, J \cup \{\chi\})$*, is strongly isomorphic to* $\mathscr{K}(\mathscr{B})$ *using the bijection* $\Gamma : \mathbf{Ck}(T(\mathscr{B})) \rightarrow \Omega(X,J)$ *defined by* $\Gamma(z) := \{f \in \mathscr{L}(X,J) \mid f \in z\} = z \cap \mathscr{L}(X,J)$*. Furthermore, the map* Γ *induces a homeomorphism between* $\Omega(X,J)$ *and* $\mathbf{Ck}(T(\mathscr{B}))$*, and the inverse of* Γ *is* $\phi^{\mathscr{K}(\mathscr{B})}$.

Proof: First we show for every i that $h(\mathscr{P}_i)$ is valid with respect to $\mathscr{K}(\mathscr{B})$. Let z be any member of $\Omega(X,J)$ and let $z \in P \in \mathscr{P}_\infty$. Due to Property (iv) it suffices to show that for every $B \in \mathscr{G}_i$ with $B \subseteq P_i(z)$ it follows that $P \cap B \neq \emptyset$ and for every $B \in \mathscr{G}_i$ with $B \cap P_i(z) = \emptyset$ it follows that $P \cap B = \emptyset$.

The latter follows from the fact that $\mathscr{P}_\infty$ is finer than or equal to $\mathscr{P}_i$. For the former, Condition 3 implies the existence of a nested decreasing sequence of non-empty compact sets $B_k \in \mathscr{G}_k$, $k \geq i$, with $B_i = B$ and $B_k \subseteq P_k(z)$ for every $k \geq i$. The limit $z' := \bigcap_{k=i}^{\infty} B_k$ will share the same member of $\mathscr{P}_\infty$ with z. Therefore we conclude that $\phi^{\mathscr{K}(\mathscr{B})}$ maps $\Omega(X,J)$ to $\mathbf{Ck}(T(\mathscr{B}))$.

Notice from the definition of $\mathcal{K}(\mathcal{B})$ that if $f \in \mathcal{L}(X, J)$ then $f \in z \in \Omega(X, J)$ if and only if $f \in \phi^{\mathcal{K}(\mathcal{B})}(z)$. This implies that the map $\Gamma \circ \phi^{\mathcal{K}(\mathcal{B})}$ is the identity on $\Omega(X, J)$.

If $z \in \Omega(X, J)$ then there exists at least one i-atom $v \in \Omega_i(X, J \cup \{\chi\})$ with $f(\pi_i(z)) \wedge E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge E_{J\cup\{\chi\}}(h(\mathcal{P}_{i-2})) \wedge h(\mathcal{P}_{i-1}) \in v$, namely $v = \phi_i^{\mathcal{K}(\mathcal{B})}(z)$. The following claim is equivalent to the claim that for every $i < \infty$ and $z \in \Omega(X, J)$ $f(\phi_i^{\mathcal{K}(\mathcal{B})}(z)) \leftrightarrow (f(\pi_i(z)) \wedge E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-1})) \in \mathcal{L}(X, J \cup \{\chi\})$ is a tautology.

Claim: If $z \in \Omega(X, J)$ then $v = \phi_i^{\mathcal{K}(\mathcal{B})}(z)$ is the only atom in $\Omega_i(X, J \cup \{\chi\})$ containing $f(\pi_i(z)) \wedge E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-1})$.

We prove the claim by induction on i. If $i = 0$ then $\Omega_0(X, J) = \Omega_0(X, J \cup \{\chi\})$ and $f(\phi_0^{\mathcal{K}(\mathcal{B})}(z)) = f(\pi_0(z))$. Otherwise, if $i \geq 1$, any two different atoms v and v' in $\Omega_i(X, J \cup \{\chi\})$ both containing $f(\pi_i(z))$ must differ on the containment of $k_j(f(u))$ for some $j \in J \cup \{\chi\}$ and some $u \in \Omega_{i-1}(X, J \cup \{\chi\})$. If $\neg(E^{i-2}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-2})) \in u \in \Omega_{i-1}(X, J \cup \{\chi\})$, then $E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-1})$ being true at v and v' would mean that both v and v' contain $k_j(\neg f(u))$. So for the rest of the proof we suppose for the sake of contradiction that there are two atoms v and v' in $\Omega_i(X, J \cup \{\chi\})$ both containing $f(\pi_i(z)) \wedge E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-1})$ and $k_j f(u) \in v$ and $\neg k_j f(u) \in v'$ for some $u \in \Omega_{i-1}(X, J \cup \{\chi\})$ with $E^{i-2}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-2}) \in u$. For $k < \infty$ define $\gamma_k : \Omega_k(X, J \cup \{\chi\}) \to \Omega_k(X, J)$ by $\gamma_k(v) := v \cap \mathcal{L}_k(X, J)$. By the induction hypothesis we must assume that $u = \phi_{i-1}^{\mathcal{K}(\mathcal{B})}(z^*)$ for some $z^* \in \pi^{-1}_{i-1}(\gamma_{i-1}(u))$ and that $f(u) \leftrightarrow (f(\gamma_{i-1}(u)) \wedge E^{i-2}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-2}))$ is a tautology.

Case 1; $j \in J$: Since either $f(\pi_i(z)) \to k_j(f(\gamma_{i-1}(u))$ is a tautology or $f(\pi_i(z)) \to \neg k_j(f(\gamma_{i-1}(u))$ is a tautology we have also that either $(f(\pi_i(z)) \wedge E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-1})) \to k_j f(u)$ or $(f(\pi_i(z)) \wedge E^{i-1}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-1})) \to \neg k_j f(u)$ is a tautology, a contradiction.

Case 2, $j = \chi$: By the induction hypothesis and the validity of $h(\mathcal{P}_{i-1})$ with respect to $\mathcal{V}(\mathbf{Ck}(T(\mathcal{B})))$, the atom $\pi_{i-1}(z) = \gamma_{i-1}(\pi_{i-1} \circ \pi_i^{-1}(v)) = \gamma_{i-1}(\pi_{i-1} \circ \pi_i^{-1}(v'))$ has already determined whether or not $k_\chi(f(\gamma_{i-1}(u))$ is in v or v', and equivalently for $k_\chi(f(\gamma_{i-1}(u)) \wedge E^{i-2}_{J\cup\{\chi\}}(h(\mathcal{P}_0)) \wedge \cdots \wedge h(\mathcal{P}_{i-2}))$ and for $k_\chi(f(u))$. The claim is proven.

By the claim, Γ is injective. With $\Gamma \circ \phi^{\mathcal{K}(\mathcal{B})}$ the identity on $\Omega(X, J)$, $\phi^{\mathcal{K}(\mathcal{B})}$ and Γ are inverses. The claim implies for every $w \in \Omega_i(X, J)$ that $\phi^{\mathcal{K}(\mathcal{B})}(\pi_i^{-1}(w))$ equals $\alpha^{\Omega(X, J\cup\{\chi\})}(f(w)) \cap \mathbf{Ck}(T(\mathcal{B}))$, an open set of $\mathbf{Ck}(T(\mathcal{B}))$. With the above, showing the isomorphism and the homeomorphism is now straightforward. q.e.d.

5.1. Un-centered maximal common knowledge

Now we are able to construct an example with three agents for which $S \in \mathcal{T}$ is maximal and $F^{-1}(S)$ is uncountable.

Example 1. For $\Omega = \Omega(X, \{1,2\})$ we define a consistent sequence of partitions. For all $i \geq 2$ define the subset $A_i := \{\phi_i^{\Omega_{2^k}}(v) \mid 2^k < i \leq 2^{k+1} \text{ and } v \in \Omega_{2^k}\} \subseteq \Omega_i$. We define the $\mathscr{P}_i$ in the following way: $\mathscr{P}_0 = \mathscr{P}_1 = \{\Omega\}$, if $i \geq 2$ then $\mathscr{P}_i = \mathscr{P}_{i-1} \vee \{\pi_i^{-1}(A_i), \pi_i^{-1}(\Omega_i \backslash A_i)\}$.

By Lemma 4, for all $2^k \leq i < 2^{k+1}$, $v \in \Omega_{2^k}$ and $w = \phi_i^{\Omega_{2^k}}(v)$ there is an extension of w in A_{i+1} and another in $\Omega_{i+1} \backslash A_{i+1}$, which means that both $\pi_i^{-1}(A_{i+1})$ and $\pi_i^{-1}(\Omega_{i+1} \backslash A_{i+1})$ have a non-empty intersection with all the members $\mathscr{G}_i$ in $P_i(w)$. Otherwise, if $w \in \Omega_i$ is not equal to $\phi_i^{\Omega_{2^k}}(v)$ for any $v \in \Omega_{2^k}$ with $2^k \leq i < 2^{k+1}$ then the member of $\mathscr{P}_i$ containing $\pi_i^{-1}(w)$ will also be a member of $\mathscr{P}_{i+1}$.

First we show that every member of $\mathscr{Q}^1 \wedge \mathscr{Q}^2 \wedge \mathscr{P}_\infty$ is dense in Ω. By Theorem 3 that would imply that $F^{-1}(\underline{Ck}(T(\mathscr{B})))$ is not empty and that $\underline{Ck}(T(\mathscr{B}))$ is maximal in $\mathscr{T}$.

Consider any $z \in \Omega$ and define $A(z) := \{i \mid \pi_i(z) \in A_i\} \subseteq \{2, 3, \ldots\}$. Define a new point $z' \in \Omega$ in the following way: Start with any $w_0 = \pi_0(z') \in \Omega_0$. We assume that $\pi_l(z')$ is defined for all $l < i$. If i is equal to 1 or $i \notin A(z)$ and $2^k + 1 < i \leq 2^{k+1}$ for some $k \geq 0$ then let $\pi_i(z')$ be $p_i(\pi_{i-1}(z'))$. If $i \in A(z)$ and $2^k < i \leq 2^{k+1}$ for some k then let $\pi_i(z')$ be $\phi_i^{\Omega_{2^k}}(\pi_{2^k}(z'))$. If $i \notin A(z)$, $2^k + 1 = i$ for some $k \geq 0$ and F^1 and F^2 are the members of $\overline{\mathscr{F}}_{i-1}^1$ and $\overline{\mathscr{F}}_{i-1}^2$ containing $\pi_{i-1}(z')$, respectively, then let $M_{i-1}^1(\pi_i(z'))$ be F^1 and let $M_{i-1}^2(\pi_i(z'))$ be any proper subset of F^2 containing $\pi_{i-1}(z')$ and satisfying (iii); (by Lemma 4 at least one exists). z' and z share the same member of $\mathscr{P}_\infty$. Let $Q = \{0\} \cup \{2^k \mid k = 0, 1, 2, \ldots\}$. Define $B(z') := \{i \mid i + 1 \in A(z')\} \backslash Q$. By induction on i we have that z' and $p^{\mathbf{N_0} \backslash B(z')}(w_0)$ share the same member of $\mathscr{F}_i^1$ for every $i < \infty$, and therefore they also share the same member of $\mathscr{Q}^1$. (z and $p^{\mathbf{N_0} \backslash B(z)}(w_0)$ do not in general share the same member of $\mathscr{P}_\infty$.) Since $Q \subseteq \mathbf{N_0} \backslash B(z')$ and $\mathscr{Q}^1 \wedge \mathscr{Q}^2 \wedge \mathscr{P}_\infty$ is coarser than or equal to $\mathscr{Q}^1 \wedge \mathscr{Q}^2$, by Lemma 2 $p^{\mathbf{N_0} \backslash B(z')}(w_0)$, and thus also z, belongs to a member of $\mathscr{Q}^1 \wedge \mathscr{Q}^2 \wedge \mathscr{P}_\infty$ dense in Ω.

Now we must show that all of Ω cannot belong to the same member of $\mathscr{Q}^1 \wedge \mathscr{Q}^2 \wedge \mathscr{P}_\infty$.

Lemma 6. *With regard to Example 1, for every $w \in \Omega_{2^i}$ and $0 \leq l \leq 2^i - 1$, $f(\phi_{2^i+l+1}^{\Omega_{2^i}}(w)) \to E_{\{1,2,3\}}^l(g_{2^i})$ is valid with respect to the semantic model $\mathscr{K}(\mathscr{B})$.*

Proof: We proceed by induction on l. If $l = 0$, then the statement is true from the definition of g_{2^i}. Let us assume that the claim is true for $l - 1 \geq 0$. It suffices to show for all $j \in \{1, 2, 3\}$ that $f(\phi_{2^i+l+1}^{\Omega_{2^i}}(w)) \to k_j E_{\{1,2,3\}}^{l-1}(g_{2^i})$ is valid with respect to the semantic model $\mathscr{K}(\mathscr{B})$. By Property (iv) and that $f(\phi_{2^i+l}^{\Omega_{2^i}}(\Omega_{2^i}))$ is common knowledge in $\phi^{\Omega_{2^i}}(\Omega_{2^i})$ if $j = 1, 2$ then $f(\phi_{2^i+l+1}^{\Omega_{2^i}}(w)) \to k_j f(\phi_{2^i+l}^{\Omega_{2^i}}(w)) \in \mathscr{L}(X, \{1,2\})$ is a tautology, so that we have the result by the induction hypothesis. For $j = 3$ it follows by the induction hypothesis and from the validity of the formula $h(\mathscr{P}_{2^i+l})$ with respect to $\mathscr{K}(\mathscr{B})$. q.e.d.

We can proceed exactly as in the proof of Theorem 1. The only difference is that we use Lemma 6 instead of Lemma 3 to arrive at a contradiction with Lemma 5. If we want only that the cardinality of $\mathscr{Q}^1 \wedge \mathscr{Q}^2 \wedge \mathscr{P}_\infty$ is uncountable, we can use a part of Proposition 2 of Simon (1999) which states that a compact cell has finite adjacency diameter. Supposing $m < \infty$ is the diameter of $\mathscr{K}(\mathscr{B})$ and choosing an i such that $2^i > m + 2$ and a $w \in \Omega_{2^i}$ we have from

Lemma 6 that g_{2^i} is true with respect to $\mathscr{K}(\mathscr{B})$ at all points within a distance of m from $\phi^{\Omega_{2^i}}_{2^i+m+1}(w)$, and therefore g_{2^i} is valid in $\mathscr{K}(\mathscr{B})$ and hence also in $\Omega(X,\{1,2\})$, a contradiction.

5.2. Increasing common knowledge

One could imagine that the correct analogue to Theorem 2 would be that enlarging the set of formulas held in common knowledge can never result in a switch from centered to non-centered. But it is possible to have two members S and S' of $\mathscr{T}$ with $F^{-1}(S)$ and $F^{-1}(S')$ not empty such that $S \subset S'$, $F^{-1}(S)$ is a singleton but $F^{-1}(S')$ is uncountable – such is the case with Example 2.

Example 2. Let $X = \{x, y\}$ with $x \neq y$. Let Ω stand for $\Omega(\{x,y\},\{1,2\})$. Define the following consistent sequence $\mathscr{B}$ of partitions: $\mathscr{P}_0 = \{\alpha^{\Omega}(\neg x)\} \cup \{\{\pi_0^{-1}(w)\} \mid x \in w \in \Omega_0\}$, and for $i > 0$ $\mathscr{P}_i = \{\{\alpha^{\Omega}(\neg x)\} \cup \{\{\pi_i^{-1}(w)\} \mid x \in w \in \Omega_i\}$.

The limit partition is therefore $\mathscr{P}_\infty = \{\alpha^{\Omega}(\neg x)\} \cup \{\{z\} \mid x \in z \in \Omega\}$.

First, we claim that the set $\mathbf{Ck}(T(\mathscr{B})) \cap (\bigcup_{i=0}^{\infty} \alpha^{\Omega(X,\{1,2,3\})}(\neg E^i_{\{1,2\}}x))$ is a cell dense in $\mathbf{Ck}(T(\mathscr{B}))$. If $z \in \Omega$ and $\neg E^i_{\{1,2\}}x \in z$ then there is some $j = 1$ or $j = 2$ and some $z' \in \alpha^{\Omega}(\neg E^{i-1}_{\{1,2\}}x)$ that shares the same member of $\mathscr{Q}^j(X,\{1,2\})$ with z. By induction we have that z shares the same member of $\mathscr{Q}^1(X,\{1,2\}) \wedge \mathscr{Q}^2(X,\{1,2\})$ with some member of $\alpha^{\Omega}(\neg x)$ and all members of $\alpha(\neg x)$ share the same member of $\mathscr{P}_\infty$. Due to Theorem 3 and the density of the set $\bigcup_{i=0}^{\infty} \alpha^{\Omega}(\neg E^i_{\{1,2\}}x)$ in Ω it is left to show that no point of $\phi^{\mathscr{K}(\mathscr{B})}_{\infty}(\mathbf{Ck}(\{x\}))$ can belong to this same cell of $\mathbf{Ck}(T(\mathscr{B}))$, where $\mathbf{Ck}(\{x\})$ is a subset of $\Omega(X,\{1,2\})$. By the discretion of the partition $\mathscr{P}_\infty$ in $\alpha^{\mathscr{K}(\mathscr{B})}(x)$, x is also held in common knowledge in $\phi^{\mathscr{K}(\mathscr{B})}_{\infty}(\mathbf{Ck}(\{x\})) \subseteq \Omega(X,\{1,2,3\})$, which completes the first claim by Lemma 1.

Second, we claim that $\underline{Ck}(T(\mathscr{B}) \cup \{x\}) \in \mathscr{T}(X,\{1,2,3\})$ does not correspond to a centered cell and that $F^{-1}(\underline{Ck}(T(\mathscr{B}) \cup \{x\}))$ is not empty. Every subset A of Ω_0 is semantically closed, and also $\mathscr{V}(A)$ is connected since $\overline{\mathscr{F}}^j_0 = \{\Omega_0\}$ for every j. Therefore $F^{-1}(\underline{Ck}(\{x\}))$ is not empty by Theorem 4.22 of Fagin, Halpern, and Vardi, (1991), which states that if a subset $A \subseteq \Omega_i$ is semantically closed and $\mathscr{V}(A)$ is connected then $F^{-1}(\underline{Ck}(\{f(A)\}))$ is not empty. $\underline{Ck}(\{x\}) \subseteq \mathscr{L}(X,\{1,2\})$ does not correspond to a centered cell by Theorem 2, (since $\underline{Ck}(\{x,y\})$ strictly contains $\underline{Ck}(\{x\})$). The rest follows by Theorem 3 and the descretion of the partition $\mathscr{P}_\infty$ in $\alpha^{\mathscr{K}(\mathscr{B})}(x)$.

5.3. Expressive

Let F be a possibility set contained in a cell C. F is defined to be *expressive* (Maruta, 1997) if there exists some $i < \infty$ and an i-atom $u \in \Omega_i$ such that $F \supseteq \pi_i^{-1}(u) \cap C$. A cell is called expressive if it contains a some expressive $F \in \mathscr{Q}^j$ for some agent j. Because the F and $\pi_i^{-1}(u)$ are closed sets, the containment $F \supseteq \pi_i^{-1}(u) \cap C$ implies the containment $F \supseteq \pi_i^{-1}(u) \cap \overline{C}$, so that by the proof of Theorem 1 of Simon (1999) a expressive cell is centered.

In Samet (1990), the knowledge of agent j at some state is *finitely generated* if the set of formulas it knows is implied logically by a finite set of formulas (equivalently by one formula). In our context of looking at the closure of the cell containing some $z \in \Omega$ as the relevant subspace of Ω, this means that the set of formulas $\{k_j f \mid k_j f \in z\}$ is implied logically by the set $S \cup \{k_j g\}$ where S is the member of $\mathscr{T}$ held in common knowledge at z and g is some formula. If a possibility set $F \in \mathscr{Q}^j$ is expressive, it contains a non-empty set $\pi_i^{-1}(u) \cap C$ for some $u \in \Omega_i$, equivalently contains $\pi_i^{-1}(u) \cap \bar{C}$, therefore it is the only possibility set of agent j in $\bar{C}$ where $\neg k_j \neg f(u)$ is true, hence also where $k_j \neg k_j \neg f(u))$ is true. If the set of formulas $\{k_j f \mid k_j f \in z\}$ were not logically implied by the set $S \cup \{k_j \neg k_j \neg f(u))\}$ then by Lindenbaum's Lemma there would be another possibility set of agent j in $\bar{C}$ where $k_j \neg k_j \neg f(u)$ would be true. (See also Maruta, 1997). The converse, that in our context the finite generation of knowledge implies expressive, is the easier direction.

It is plausible to believe that a cell is expressive if and only if it is centered. However, the following is an example of a non-expressive compact cell with an adjacency radius of 2.

Example 3. Let $\Omega = \Omega(X, \{1,2\})$. We define a consistent sequence of partitions in the following way:

For every $0 < i < \infty$ define $A_i = \{p_i(w) \mid w \in \Omega_{i-1}\}$. Define $\mathscr{P}_0 = \{\Omega\}$ and $\mathscr{P}_i = \mathscr{P}_{i-1} \vee \{\pi_i^{-1}(A_i), \Omega \backslash \pi_i^{-1}(A_i)\}$.

Consider the member F^1 of $\mathscr{Q}^1(X, \{1,2\})$ containing $p^{\mathbf{N_0}}(w_0)$ for some fixed $w_0 \in \Omega_0$. For any $i < \infty$ consider the $F' \in \mathscr{F}_i^1$ satisfying $F^1 \subset F'$ and consider any $u \in \Omega_i$ with $u \in \pi_i(F') \in \bar{\mathscr{F}}_i^1$. Let F'' and F^* be the members of $\mathscr{F}_{i+1}^1$ and $\mathscr{F}_{i+1}^2$, respectively, defining $p_{i+1}(u)$. By Lemma 4 there will be a $v \in \Omega_{i+1}$ with $v \in \pi_{i+1}(\pi_i^{-1}(u))$, $v \in \pi_{i+1}(F'')$, but $v \notin \pi_{i+1}(F^*)$. By finite induction this means that for every subset $S \subseteq \mathbf{N}$ there is an nested sequence $C_i \in \mathscr{G}_i$, $i = 1, 2, \ldots,$ such that for every i $C_i \subseteq \pi_i^{-1}(\pi_i(F^1))$ but $\pi_i(C_i) = p_i(\pi_{i-1}(C_i))$ if and only if $i \in S$. By the compactness of F^1 this implies that $z = \bigcap_{i=1}^{\infty} C_i \in F^1$ and $z \in \pi_i^{-1}(A_i)$ if and only if $i \in S$. This means that F^1 intersections every member of $\mathscr{P}_\infty$ and the adjacency distance with respect to the semantic model $\mathscr{K}(\mathscr{B})$ between $p^{\mathbf{N_0}}(w_0)$ and every element of $\Omega(X, \{1,2\})$ is no more than 2. Every member of $\mathscr{Q}^j$ for $j = 1, 2$ is a meagre set of Ω. By Lemma 4, no member of $\mathscr{P}_\infty$ can contain an open set of Ω. Theorem 3 implies that $\phi_\infty^{\mathscr{K}(\mathscr{B})}(\Omega)$ is a centered cell that is not expressive. Because the cell has finite adjacency diameter, it is easy to show that it is also compact, (see Lemma 2 of Samet, 1990), and therefore the corresponding set of formulas held in common knowledge is maximal.

There is some sense in which the notion that "non-expressive implies uncentered" may be valid.

Question 1: If a cell C is centered and has no isolated points, what additional conditions are necessary, if any, for there to exist a semantic model $\mathscr{K}$ with uncountably many connected components that maps by $\phi^{\mathscr{K}}$ injectively into C?

For all three examples, we had the advantage of Theorem 3, which applies only if there are at least three agents. We have tried but failed to answer the following question.

Question 2: For which of the questions Examples 1, 2, and 3 were designed to answer can one find alternative examples with only two agents?

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