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Spanning network games*

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Abstract. We study fundamental properties of monotone network enterprises which contain public vertices and have positive and negative costs on edges and vertices. Among the properties studied are the nonemptiness of the core, characterization of nonredundant core constraints, ease of computation of the core and the nucleolus, and cases of decomposition of the core and the nucleolus.

Key words: Games, cooperative games, networks, core, nucleolus, decomposition, minimum cost spanning tree games

1. Introduction

Monotone spanning network enterprises were introduced in Megiddo [1978a] and Granot and Huberman [1981]. They are distinguished from other (not necessarily monotone) enterprises (Bird [1976], Granot and Huberman [1981]), by permitting coalitions to use vertices occupied by other players in order to connect themselves to the root. Such enterprises generalize airport games (e.g., Littlechild [1974]) and tree games (Megiddo [1978b]).

In this paper we extend the class of monotone enterprises to include *public vertices,* namely vertices not occupied by players. We also allow costs on vertices, in addition to costs on edges. These costs can be negative, thus representing profits. Some of the results we obtain, however, are valid only if some restrictions are imposed on the network. One restriction that we often

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require is that an optimal network for the grand coalition exists that is a tree spanning all vertices.

We study fundamental properties of games induced by these enterprises, such as nonemptiness of the core and its efficient representation, representation of the nucleolus and possible decomposition of such games.

The paper is organized as follows: After providing the necessary notation and definitions (Section 2), we introduce the *reduced enterprises* and prove that there exists a commutative diagram relating the reduced enterprises with the corresponding reduced games (Section 3).

Unlike the classical enterprises, in our case the core may be empty. Thus, it is important to provide conditions that can be read from the network, guaranteeing that the core is not empty. Such conditions are given in Section 4 and are derived, for example, from the commutative diagram, mentioned above, and from the reduced game property of the core concept. We also exhibit an important class of such games whose cores are not empty.

Section 5 is devoted to the problem of computation and representation of the core and the nucleolus. it turns out that only relatively few coalitions need to be considered when computing these solutions and these coalitions can be "read" directly from the network. Analogous results exist in the case when a coalition is not allowed to use vertices occupied by players not in that coalition (see Granot and Huberman [1984]). A byproduct of the proof is the fact that the intersection of the kernel and the core is the nucleolus when the core is not empty, and it can be derived by solving equations of the type $s_{ij}(x) = s_{ji}(x)$, where (i, j) are pairs of "adjacent" players in an optimal network.

Finally, we provide in Section 6 an extension of a decomposition result, given by Granot and Huberman [1981] for monotone and nonmonotone game representations, to our wider class of monotone enterprises. Namely, it is shown that if an optimal network enterprise consists of several branches emanating from the root, then, under certain conditions, the core and the nucleolus are cartesian products, not of the branches, but of some modification of the games on these branches.

2. The spanning network enterprise and its game

We shall be concerned with a spanning network enterprise (SNE), or a spanning network, for short, defined by $\mathscr{E} = (V, E, a, b, N)$. Here, (V, E) is an undirected connected graph with a finite set of vertices V, containing a distinguished vertex v_0, called *the root*, and a set of edges E. The function $a : E \to \Re$ associates with each edge e a *cost* $a(e)$, interpreted, e.g., as the cost of constructing edge e. The function $b : V \to \Re$ associates with each vertex v a cost $b(v)$, interpreted, e.g., as the cost of constructing vertex v. The n-tuple $N := \{1, 2, \ldots, n\}$ is the set of *players* (agents). It is assumed that each player i is located at some vertex, denoted v^i. A vertex can be occupied by several players. It may be occupied by no player, in which case we call it a *public vertex*. We assume that a player is located at precisely one vertex.

Discussion. Several scenarios may fit a SNE. For example, the root is the location of a supplying center. The players, located at some places, want to be connected to the root.[1] In order to do so, they have to construct routes to the root, which may use some public and private locations.

From the above discussion it follows that the implementation of the enterprise should be conducted in accordance with the following rules:

(1) Eventually each player should be connected to the root.
(2) If an edge or a vertex is 'constructed', its cost must be covered by (some of) the players.
(3) Once an edge, or a vertex, is constructed, every player can "use" it; there is no need to pay for constructing it more than once.

We say that a coalition T is formed if its members agree to connect each one of them to the root. Obviously, the members of T will choose to connect themselves using least expensive (or most profitable) edges and vertices. They may even choose to be connected through vertices occupied by players not in T. Thus, we allow free riders. This is in contrast to other models (see, e.g., Bird [1976], Granot and Huberman [1981]), where members of a coalition T are not allowed to use vertices occupied by members of $N \setminus T$. However, we require that *the set of vertices and edges constructed by a formed coalition should be a connected subgraph of* (V, E). Accordingly, for every coalition T we define *the cost* $c(T)$ *of* T to be

$$c(T) := \text{minimal cost (maximal profit, if negative) of joining all members of } T \text{ to the root via a connected subgraph of } (V, E). \tag{2.1}$$

This definition applies also to the empty coalition. Note that an optimal subgraph for the empty coalition does not necessarily consist of the root alone. It may "behoove" the empty coalition to choose a subgraph that passes through other vertices, including those that are inhabited by players. This, indeed, will often happen in networks that represent reduced games (see Section 3). We adopted this convention, instead of another convention that puts the worth of the empty coalition as zero, because some of our theorems do not hold without cumbersome modifications.[2] Note that our convention[3] has the formal advantage that no exception is needed in (2.1) and it simplifies the definition of the reduced game. It should be clear, however, that it is done for mathematical simplicity only. It makes little sense, perhaps, to claim that "the coalition of nobody incurred costs, or gained profits".

The computation of $c(T)$ may be long and tedious, but it will not concern us here. Sufficient to say that there may be several optimal subgraphs for a coalition T. We choose one of them and henceforth denote it $G^T = (V^T, E^T)$. The results in this paper do not depend on the choice of the optimal graph; however, if several optimal graphs for the grand coalition are known, one may find it advantageous to choose one of them for the computation of the core and the nucleolus (see Sections 5 and 6).

Definition 2.1. *Let* $\mathscr{E} = (V, E, a, b, N)$ *be a spanning network enterprise. The cost game* $\Gamma_{\mathscr{E}} := (N; c)$, *where the cost function c is given by (2.1), is called the*

[1] Think about a cable TV enterprise.
[2] e.g., Theorem 2.2, Lemma 4.1 and Corollary 4.2.
[3] Also adopted in Nouweland et al. (1993)

corresponding spanning network game *(SNG), or* **the game representing the enterprise**.

Assumption. In this paper we assume that no player resides at the root. This entails no loss of generality, because if this is not the case one can add a costless edge connecting the root to an uninhabited new root and transfer the cost of the old root to the new root. This entails no change to the characteristic function (2.1).

Let $\mathscr{E}'$ be obtained from a SNE $\mathscr{E}$ by increasing the cost of the root by an amount d. Then, the cost of every subgraph (V', E'), where $v_0 \in V'$, is increased by d. Thus, for every coalition T, (V^T, E^T) can be chosen to be the same subgraph for both enterprises. We can therefore call $\mathscr{E}$ and $\mathscr{E}'$ *network-equivalent enterprises*. However, if $\Gamma_{\mathscr{E}} = (N; c)$ and $\Gamma_{\mathscr{E}'} = (N; c')$ then $c'(T) = c(T) + d$ for every T; therefore these games are *not* strategically equivalent. For example, $\Gamma_{\mathscr{E}}$ may have an empty core (Section 4), whereas $\Gamma_{\mathscr{E}'}$ will have a nonempty core if d is large enough. Thus, changing the cost of the root affects the nature of the enterprise very little, but changes significantly the corresponding game.

The following is a simple generalization of Granot and Huberman [1981].

Theorem 2.2. *A SNG is monotonic; i.e., if $S \subset T \subseteq N$ then $c(S) \leq c(T)$.*

Proof. $c(T)$ is the cost of (V^T, E^T). The subgraph (V^T, E^T) connects every member of T to the root; therefore, it connects every member of S to the root. Its cost is therefore not smaller than $c(S)$. ■

A SNE was first discussed by Claus and Kleitman [1973], where each vertex other than the root was occupied by a player, $a \geq 0$ and $b = 0$. Bird [1976], Claus and Granot [1976] and Granot and Huberman [1981] constructed the corresponding SNG. A special case is *the airport game*, first studied by Littlechild and Owen [1973], Littlechild [1974] and Littlechild and Thompson [1977]. In this case the graph is a chain; namely, a sequence of edges that form a line. Littlechild and Owen [1977] introduced negative cost at the root. Megiddo [1978b] studied a tree enterprise and Granot and Huberman [1981] coined and studied the distinction between *(nonmonotone) minimum cost spanning tree games*, in which a coalition can use only those vertices occupied by its members, and *monotone minimum cost spanning tree games*, where a coalition can use other vertices, as in this paper. Public vertices essentially first appeared in Megiddo [1978a].

3. The reduced SNE and its game

Following Davis and Maschler [1965], for a given cost game $(N; c)$, a given nonempty coalition S and a given preimputation[4] x, we call $(S; \hat{c}^x_S)$ **the re-**

[4] I.e., a cost vector x in $\Re^N$, satisfying $x(N) = c(N)$.

duced game on S at x, if

$$\hat{c}^x_S(T) = \begin{cases} x(S), & \text{when } T = S, \\ \min\{c(T \cup Q) - x(Q) : Q \subseteq S^c\}, & \text{if } T \subseteq S. \end{cases} \tag{3.1}$$

Here, $S^c := N \backslash S$ and $x(Q) := \sum_{i \in Q} x_i$ if $Q \neq \emptyset$ and $x(\emptyset) = 0$.

Remark. Had we defined $c(\emptyset)$ to be always zero, as is sometimes done, we would then have to require that also $\hat{c}^x_S(\emptyset) = 0$, in order that the reduced game will always be a game. Such a requirement is indeed made in the literature in different contexts.

Formula (3.1) can be simplified if x belongs to the core $\mathscr{C}(N; c)$; i.e., if $x(N) = c(N)$ and $x(S) \leq c(S)$ for every coalition S.

Lemma 3.1. *If $x \in \mathscr{C}(N; c)$ and S is a nonempty coalition then*

$$\hat{c}^x_S(T) = \min\{c(T \cup Q) - x(Q) : Q \subseteq S^c\}, \quad \textit{all } T \subseteq S. \tag{3.2}$$

Proof. For $Q \subseteq S^c$, $c(S \cup Q) - x(Q) \geq x(S \cup Q) - x(Q) = x(S)$, because x is a core point. On the other hand, $c(S \cup S^c) - x(S^c) = x(N) - x(S^c) = x(S)$, so $\min\{c(S \cup Q) - x(Q) : Q \subseteq S^c\} = c(S \cup S^c) - x(S^c) = x(S)$ and (3.1) collapses to (3.2). ∎

Having defined a reduced SNG, we now define a reduced SNE and study the connection between the two concepts.

Definition 3.2. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE. Let S be a coalition and let $x = (x_1, x_2, \ldots, x_n)$ be a preimputation.* **The reduced spanning network on *S* at *x***, *denoted $\mathscr{E}^x_S$, is a SNE obtained from $\mathscr{E}$ by removing the players in S^c from the network enterprise and by subtracting x_i, for each removed player i, from the cost at vertex v^i, where player i is located. In symbols:*

$$\mathscr{E}^x_S = (V, E, a, \tilde{b}, S), \tag{3.3}$$

where

$$\tilde{b}(v) = b(v) - x(S^c_v), \quad \text{all } v \in V. \tag{3.4}$$

Here, $S^c_v :=$ the set of players in S^c that are located in v.

Interpretation. We regard x as a proposed cost allocation for the enterprise $\mathscr{E}$. The players in S tell the removed players in S^c: "Leave the place, go have some beer (or coffee), but let us keep your payment.[5] If a coalition T, $T \subseteq S$ needs some of your locations, it will connect them to the root, using your payments towards covering the costs of constructing its optimal subgraph. If none of the coalitions that eventually form benefits from someone's money, he can take it back as he will not be connected to the root."

[5] We owe this vivid description to Stef Tijs (oral communication).

The following theorem shows a remarkable relation between the reduced SNE and the corresponding reduced SNG.

Theorem 3.3. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE. Let $\Gamma_{\mathscr{E}} := (N; c)$ be its SNG. Let x be a core point, $x \in \mathscr{C}(N; c)$, and let S be a nonempty coalition. Denote by $(S; \hat{c}_S^x)$ the reduced game on S at x and let $\mathscr{E}_S^x := (V, E, a, \tilde{b}, S)$ be the reduced SNE on S at x. Let $(S; \tilde{c}_S^x)$ be the game corresponding to $\mathscr{E}_S^x$. With the above notation, $\hat{c}_S^x = \tilde{c}_S^x$.*

In symbols, Theorem 3.3 can be phrased as follows:

Theorem 3.3′. *With the notation of Theorem 3.3, the following commutative diagram holds for $x \in \mathscr{C}(N; c)$:*

$$\begin{array}{ccc} (V, E, a, b, N) & \rightarrow & (N; c) \\ \downarrow & & \downarrow \\ (V, E, a, \tilde{b}, S) & \rightarrow & (S; \tilde{c}_S^x), \end{array}$$

where horizontal arrows denote transition from an enterprise to its corresponding game and vertical arrows denote reduction on S at x.

Proof. Let (V', E') be an arbitrary connected subgraph, connecting all members of a subset T of S to the root. Denote by $a(E')$ and by $b(V')$ the total cost needed to construct E' and V', respectively. By (3.4), the cost to construct (V', E') in the reduced enterprise is equal to $a(E') + b(V') - x(Q)$, where Q is the set of players residing in $S^c \cap V'$. This expression is greater than or equal to $c(T \cup Q) - x(Q)$, which is greater than or equal to $\hat{c}_S^x(T)$, because in the original SNE the members of $T \cup Q$ are indeed connected to the root via (V', E'). If (V', E') is an *optimal* subgraph for T in the reduced enterprise, then $a(E') + b(V') - x(Q) = \tilde{c}_S^x(T)$. Thus, $\tilde{c}_S^x(T) \geq \hat{c}_S^x(T)$. Conversely, let Q_0 be a subset of S^c such that $\hat{c}_S^x(T) = c(T \cup Q_0) - x(Q_0)$. Let $(V^{T \cup Q_0}, E^{T \cup Q_0})$ be an optimal subgraph for $T \cup Q_0$ in the original SNE. This subgraph connects all members of $T \cup Q_0$, and, in particular, all members of T to the root. Being optimal means that $a(E^{T \cup Q_0}) + b(V^{T \cup Q_0}) = c(T \cup Q_0)$. The cost of the same subgraph in the reduced enterprise is $c(T \cup Q_0) - x(Q_0)$, by the definition of the reduced enterprise. This amount is greater than or equal to $\tilde{c}_S^x(T)$. We have now proved that $\tilde{c}_S^x(T) \leq \hat{c}_S^x(T)$. ■

For x in the core of the original game, Theorem 3.3 offers a natural interpretation of the reduced game: The reduced game is the game of the reduced enterprise. One can get a similar interpretation, somewhat more complicated, if x is not in the core. In this case, though, one has to extend even further the class of games considered. We shall not pursue here this direction.

Definition 3.4. *A solution concept φ is said to satisfy the* **reduced game property**, *or* **consistency**, *if for every coalition S and every solution point x, the projection $x^S := (x_i)_{i \in S}$ belongs to $\varphi(S; \hat{c}_S^x)$.*

Discussion. The reduced game property can be used to evaluate a solution concept on intuitive grounds. We interpret $(S; \hat{c}_S^x)$ as the game members of S

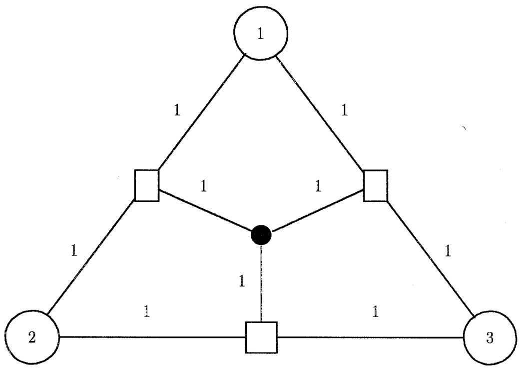

Fig. 1. A SNE whose derived SNG has an empty core

are facing, given that x in the solution φ is contemplated. If, for all nonempty coalitions S, $x^S \in \varphi(S; \hat{c}_S^x)$, then the players in S, who are supposed to believe in φ, will not move away from x^S, because it is in the solution of their own game. So, a consistent solution has this kind of stability. If, on the other hand, for some S, $x^S \notin \varphi(S; \hat{c}_S^x)$, then the players in S may criticize φ as not being a reasonable solution concept. They could argue that if φ is a good solution concept, then x is an appropriate outcome; so x^S should be an appropriate outcome in $(S; \hat{c}_S^x)$. But if they really believe in φ, x^S cannot be an appropriate outcome, because in their own game, $(S; \hat{c}_S^x)$, $x^s \notin \varphi(S; \hat{c}_S^x)$. Thus, by the above contradiction, φ is not a good solution concept after all.

The above argument is valid only if indeed it can be claimed that in some sense the players in S evaluate "their own game" as $(S; \hat{c}_S^x)$. Such a claim is enhanced if there exists a physical meaning to the reduced game, directly related to the situation. Definition 3.2 and Theorem 3.3 provide such a meaning.

It is well known that the core, the prekernel, the prenucleolus and the prebargaining set satisfy the reduced game property.[6] (See Sobolev [1975], Aumann and Drèze [1974], Peleg [1985], [1986], [1992]).[7]

4. The core of a SNE

It is well known (Granot and Huberman [1981], Megiddo [1978a]) that the core of a monotone minimum cost spanning tree game is not empty if the underlining network has nonnegative costs and every vertex is occupied by a player. It is also known that there are SNGs with an empty core. Megiddo [1978a] furnished one example in which the network (V, E) consists of a union of Steiner trees, the vertex costs are all zero and the edge costs are the euclidean distances among the vertices. A simpler example, provided by Tamir [1991], is reproduced in Figure 1. In this figure there are public vertices and only edges have costs.

[6] "Pre" denotes that the underlying space is the space of preimputations.

[7] These facts are true even if we allow $\hat{c}_S^x(\emptyset)$ to be different from zero, as we do in this paper. Indeed, the definitions of the prekernel and the prebargaining set do not involve the empty coalition. The excess of the empty coalition remains constant and does not affect the prenucleolus. If x is a core point, then $\hat{c}_S^x(\emptyset) \geq 0$ (see (3.1)); therefore $\hat{c}_S^x(\emptyset) - x(\emptyset) \geq 0$ automatically.

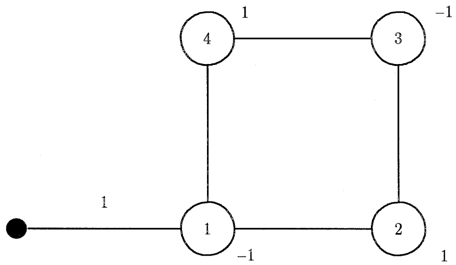

Fig. 2. A SNE without public vertices whose corresponding SNG has an empty core

Figure 2 offers a 4-person SNE with an empty core without public vertices, but some costs are negative.[8] It is easy to verify that the cores of these games are indeed empty.

The above examples raise the question of identifying classes of SNEs with nonempty cores, other than minimum cost spanning tree games. Some of these will be derived in this section. We start with some elementary lemmas.

Lemma 4.1. *Let* $(N; c)$ *be a cost game satisfying* $c(N \backslash \{j\}) \leq c(N)$ *for some player* j. *Under this condition,* $x_j \geq 0$ *for every core point* x. *In particular, the core of a SNG is a subset of* $\Re_+^N$.

Proof. If x is a core point then $x(N) - x_j = x(N \backslash \{j\}) \leq c(N \backslash \{j\}) \leq c(N) = x(N)$. Thus, $x_j \geq 0$. The rest of the proof follows from the monotonicity of the SNG (Theorem 2.2). ■

Corollary 4.2. *The core of a SNG is empty if* c *takes some negative values.*

Proof. Lemma 4.1, if $c(T) < 0$ for some nonempty coalition T. If $c(\emptyset) < 0$, the core is empty because $x(\emptyset) = 0$. ■

The next lemma shows that players occupying a vertex v can share their total cost in a core point any way they wish, without leaving the core. Denote by N^v the set of players occupying a vertex v.

Lemma 4.3. *Let* x *be a core point in a SNG* $(N; c)$ *corresponding to a SNE* (V, E, a, b, N). *Let* y *be a nonnegative imputation which coincides with* x *at all coordinates indexed by players not located at* v. *Then* y *is also a core point.*

Proof. For every subset T of players, $x(T \cup N^v) = y(T \cup N^v)$. If $T \cap N^v = \emptyset$ then $y(T) = x(T) \leq c(T)$. If $T \cap N^v \neq \emptyset$ then, by (2.1), $c(T) = c(T \cup N^v)$; hence, since x is a core point, $y(T \cup N^v) \leq c(T \cup N^v) = c(T)$, because $x(T \cup N^v) \leq c(T \cup N^v)$. Consequently, $y(T) \leq y(T \cup N^v) \leq c(T)$, because $y \geq 0$. ■

The next theorem extends Bird [1976], Granot and Huberman [1981] and Megiddo [1978a] to our SNEs. It pertains to SNEs, where it is known that

[8] In this figure, edges and vertices with no cost attached carry zero cost.

there exists an optimal subgraph $G^N = (V^N, E^N)$ *for* N *which is a tree*. This case occurs, for example, if $a \geq 0$, because one can delete edges in a cycle without losing connectivity and without increasing the cost of the remaining subgraph.

First, we introduce the following notation.

Notation 4.4. Let $G^N = (V^N, E^N)$ be an optimal network which is a tree. For each v in V^N, we denote by e_v the unique edge in E^N, which emanates from v and is on the unique path between v_0 and v in G^N.

Theorem 4.5. *Let* $\mathscr{E} = (V, E, a, b, N)$ *be a SNE without public vertices for which it is known that an optimal subgraph* $G^N = (V^N, E^N)$ *for* N *exists, and it is a tree.*[9] *If* $a(e_v) + b(v) \geq 0$ *for every* v *in* V *and* $b(v_0) \geq 0$ *then the core of* $\Gamma_{\mathscr{E}}$ *is not empty. Specifically,* $x = (x_1, x_2, \ldots, x_n)$ *is a core point if* $x \geq 0$*, the occupants of* v*, for each* v *in* V*, share* $a(e_v) + b(v)$ *and, in addition, each player contributes a nonnegative amount towards covering the cost* $b(v_0)$*.*

Proof.[10] It is sufficient to prove the theorem under the additional assumption that $b(v_0) = 0$. Indeed, if x is a core point when $b(v_0) = 0$ and we increase $b(v_0)$, and at the same time share the increase among the players in nonnegative amounts, we derive a core point to the new enterprise (see the discussion prior to Theorem 2.2).

Clearly x is a preimputation. Suppose it is not in the core, then there is a coalition R with $x(R) > c(R)$. Let (V^R, E^R) be an optimal subgraph for R and let T be the set of players occupying V^R. Then, $T \supseteq R$ and $a(E^R) + b(V^R) = c(R) = c(T)$. Therefore, since $x \geq 0$, $x(T) > c(T)$. It follows that

$$c(N) = x(N) = x(N \backslash T) + x(T) > x(N \backslash T) + c(T). \tag{4.1}$$

Consider now the subgraph (V, E'), where $E' = E^R \cup \{e_v : v \in V \backslash V^R\}$. Note that

$$e_v \in E^N \cap E', \text{ whenever } v \in V \backslash V^R, \tag{4.2}$$

and

$$e_v \notin E^R, \text{ whenever } v \in V \backslash V^R. \tag{4.3}$$

The cost to construct (V, E') is therefore equal to

$$\begin{aligned} b(V) &+ a(E^R) + \sum\{a(e_v) : v \in V \backslash V^R\} \\ &= b(V^R) + a(E^R) + b(V \backslash V^R) + \sum\{a(e_v) : v \in V \backslash V^R\} \\ &= c(R) + b(V \backslash V^R) + \sum\{a(e_v) : v \in V \backslash V^R\} \\ &= c(R) + \sum\{b(v) : v \in V \backslash V^R\} + \sum\{a(e_v) : v \in V \backslash V^R\} \\ &= c(R) + x(N \backslash T) = c(T) + x(N \backslash T). \end{aligned}$$

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[9] Of course, then $V^N = V$.

[10] The proof is similar to that given in Granot and Huberman [1981].

By (4.1), $b(V) + a(E') < c(N)$ and we arrive at a contradiction if we show that (V, E') is a connected subgraph. Indeed, every player in T is connected to the root via (V^R, E^R) and for every player not in T there is a path via (V^N, E^N) towards the root, by (4.2), which either reaches the root or reaches a vertex in V^R. ■

We call the payoff vector, described in Theorem 4.5, when, in addition, all residents at each vertex share their payments equally, **the tree vector for the enterprise**. An important usage of this payoff vector will be made at the end of this section.

Theorem 4.5 dealt only with games without public vertices. In the next theorem we show that, whenever we know core points of such games, we can derive core points for games that contain public vertices.

Theorem 4.6. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE containing public vertices. Let $\mathscr{E}^* := (V, E, a, b, N^*)$ be a SNE, obtained from $\mathscr{E}$ by placing an additional player at each public vertex. Let $x = \{x_i\}_{i \in N^*}$ be a core point of $\Gamma_{\mathscr{E}^*}$. Under these conditions the projection x^N of x into $\Re^N$ is a core point of $\Gamma_{\tilde{\mathscr{E}}}$, where $\tilde{\mathscr{E}} := (V, E, a, \tilde{b}, N)$. Here,*

$$\tilde{b}(v) = b(v) - \sum\{x_i : i \in N^v \cap (N^* \backslash N)\}, \tag{4.4}$$

where N^v is the set of players occupying vertex v.

Proof. $\tilde{\mathscr{E}}$ is the reduction of $\mathscr{E}^*$ to N at x; therefore $\Gamma_{\tilde{\mathscr{E}}}$ is the reduced game of $\Gamma_{\mathscr{E}^*}$ (Theorem 3.3). Consequently, x^N is a core point of $\Gamma_{\tilde{\mathscr{E}}}$, because the core satisfies the reduced game property (Definition 3.4). ■

Example 4.7. Consider the game of Figure 1. Add players 4, 5, 6 at the public vertices. By Theorem 4.5, the tree vector $x = (1, 1, 1, 1, 1, 1)$ is a core point of the new game. Reducing this enterprise to the set $\{1, 2, 3\}$ we obtain a SNE $\tilde{\mathscr{E}}$ which is identical to the one in Figure 1, except that there is a cost of -1 at each public vertex. The enterprise $\tilde{\mathscr{E}}$ has $(1, 1, 1)$ as its core point.

In the previous analysis we started with known core points and by the process of reduction we obtained games with fewer players and different costs, for which we could assert that the core is not empty. We now proceed in an opposite direction. We start with games with known core points and then try to *add* players and change costs so as to get other games with nonempty cores. Specifically, we start with an arbitrary SNE $\mathscr{E} = (V, E, a, b, N)$ having x as a core point. We then increase the costs at some vertices and add players at these vertices, hoping that they will pay the additional costs. Do we get games with nonempty cores by this procedure? The answer is negative, as the following example shows.

Example 4.8. Consider the 3-person SNE of Figure 3. In the derived SNG, $c(S) = 0$ for every coalition, so that $(0, 0, 0)$ is a core point. Placing a player 4 at the public vertex and increasing the cost at that vertex by any nonnega-

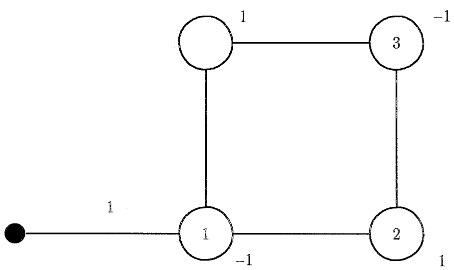

Fig. 3. A 3-person SNE with a nonempty core that becomes empty when a public vertex becomes occupied

tive amount δ, and we obtain a SNE $\mathscr{E}^*$ with an empty core (compare Figure 2).[11,12]

Note that in Example 4.8, $c^*(1234) - c(123) = \delta + 1$ whereas $b^*(V) - b(V) = \delta$. Thus, the increase in b is different from the increase in the worth of the grand coalition. If this were not the case we could have claimed that the new game has a nonempty core:

Theorem 4.9. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE with a nonempty core. Let $x \in \mathscr{C}(\Gamma_{\mathscr{E}})$. Let b^* be another cost function on V satisfying $b^* \geq b$. Place an additional player at each vertex v for which $b^*(v) > b$. Denote by K the set of additional players and let $N^* = N \cup K$. Let $(N^*; c^*)$ be the SNG that corresponds to the new SNE $\mathscr{E}^*$. If*

$$c^*(N^*) - c(N) = b^*(V) - b(V), \tag{4.5}$$

then $\mathscr{E}^*$ has a nonempty core. In particular,

$$x^* := (\{x_i\}_{i \in N}, \{b^*(v^k) - b(v^k)\}_{k \in K}) \tag{4.6}$$

is a core point of $\mathscr{E}^*$.

Proof. For a coalition T, denote by v^T the set of vertices occupied by the members of T. With this notation $x^*(N^*) = x(N) + (b^* - b)(v^K) = c(N) + (b^* - b)\ (V) = c^*(N^*)$, so x^* is a preimputation. Let S^* be a coalition in $\Gamma_{\mathscr{E}^*}$, then $S^* = S \cup T^*$, where $S \subseteq N$ and $T^* \subseteq K$. It follows that $x^*(S^*) = x(S) + (b^* - b)(v^{T^*}) \leq c(S) + (b^* - b)(v^{T^*}) = \min\{a(\overline{E}) + b(\overline{V}) : \overline{V} \supseteq v^S \cup \{v_0\}$ and $(\overline{V}, \overline{E})$ is a connected subgraph$\} + (b^* - b)(v^{T^*}) \leq \min\{a(\overline{E}) + b(\overline{V}) : \overline{V} \supseteq v^S \cup \{v_0\} \cup v^{T^*}$ and $(\overline{V}, \overline{E})$ is a connected subgraph$\} + (b^* - b)(v^{T^*}) = c^*(S \cup T^*)$. These inequalities follow from the fact that any subgraph in $\Gamma_{\mathscr{E}^*}$ that connects $S \cup T^*$ to the root must connect S to the root. We have proved that $x^*(S^*) \leq c^*(S^*)$, so x^* is indeed a core point of $\mathscr{E}^*$. ■

Example 4.10. Consider the enterprise $\mathscr{E}^*$ given in Figure 4. We want to find out if its game has a nonempty core. To do so we remove all the players

[11] In this figure, costs of edges are zero, whenever not shown.
[12] The core of this game is empty for any positive cost placed at the vertex occupied by Player 4.

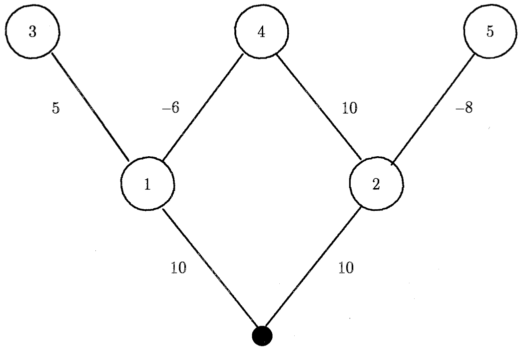

Fig. 4. Does this game have a nonempty core?

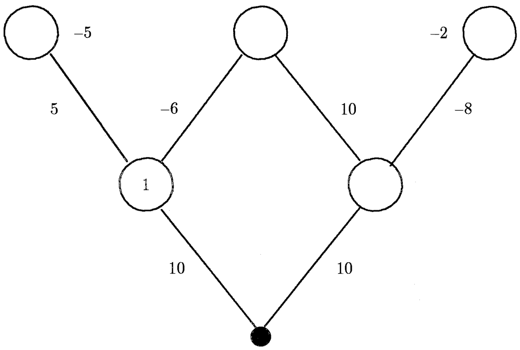

Fig. 5. A derived 1-person enterprise with a nonempty core

except player 1, thus making their locations public. We also add negative costs at the endpoints so as to make it *barely* profitable for player 1 to absorb all the vertices.[13] We obtain the game of Figure 5. Denote by $\mathscr{E}$ this 1-person enterprise and note that the relation between $\mathscr{E}$ and $\mathscr{E}^*$ satisfies the conditions of Theorem 4.9 (including (4.5)). The enterprise $\mathscr{E}$ has a nonempty core, namely $\{(4)\}$. Consequently, by Theorem 4.9, $(4, 0, 5, 0, 2)$ is a core point of the original game.

Tamir's example (Figure 1) shows that allowing for public vertices may yield enterprises with empty cores. In his example, however, the graph G^N does not pass through all the vertices of the enterprise. We were hoping that if we impose this requirement (which is obviously met if there are no public vertices), and require, say, that all costs are nonnegative, we shall be able to extend the nonemptiness result of Bird [1976], Granot and Huberman [1981] and Megiddo [1978a]. This, unfortunately, is not the case as was pointed to us by H. Reijnierse,[14] who created an example based on a construction of Kuipers [1994]:

[13] This is done in order to achieve condition (4.5).
[14] Written communication.

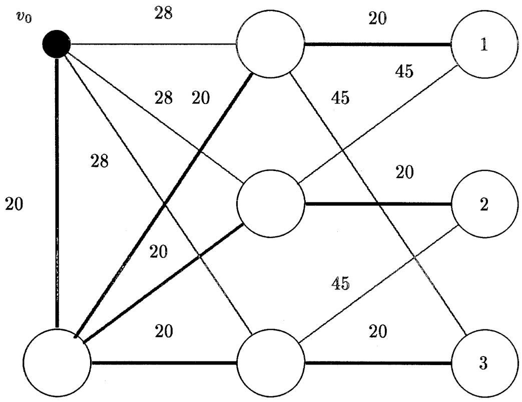

Fig. 6. A network with an empty core, even though G^N passes through all the vertices.

Example 4.11. Consider the network of Figure 6. It contains the root v_0 and four public vertices. Three vertices are occupied by players 1, 2, 3, respectively. The costs of the edges are given in the figure. The vertices have zero costs. The graph of G^N is denoted by bold lines. Its cost is 140. Note that it passes through all the vertices. Every 2-person coalition costs 93. This is enough to prove that the core is empty, as $2 \cdot c(123) > c(12) + c(13) + c(23)$.

Looking again at Figure 6, we notice that G^N contains adjacent public vertices. We subsequently show that if there exists an optimal network G^N which is a tree that spans all the vertices, such that no two public vertices are adjacent therein, then the core is not empty. Before presenting this result, we need the following Lemma which demonstrates that the assumption that there exists an optimal network, G^N, which is a tree spanning all vertices implies that there exists a G^S which is a tree for every S.

Lemma 4.12. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE for which it is known that $G^N = (V^N, E^N)$ is a tree that spans all the vertices. Then, G^S can be assumed to be a tree for every S, $S \subseteq N$.*

Proof. Every edge in $E \backslash E^N$ has a nonnegative cost, because, otherwise it could be added to G^N to generate a cheaper network. Let $G^S = (V^S, E^S)$ be an optimal network for S. If G^S contains a cycle, then at least one edge in this cycle does not belong to G^N, because G^N is a tree. The cost of such an edge is nonnegative and it could therefore be removed from G^S to bring about a network spanning V^S, whose cost is not higher.[15] Continuing to eliminate edges as long as there are cycles, we finally obtain a tree spanning V^S, whose cost is not higher than the cost of G^S. ■

Theorem 4.13. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE. Suppose, that there exists an optimal network G^N which is a tree that spans all the vertices and such that no two public vertices are adjacent therein. Suppose, further, that $a(v) + b(e_v) \geq 0$ for all v in V.*[16] *Under these conditions the core of $\Gamma_{\mathscr{E}}$ is not empty.*

[15] It could remain the same only if the cost of the deleted edge was zero.
[16] See Notation 4.4 for the definition of e_v.

Proof. Without loss of generality we can assume that all the edges in the complete graph on V, which are missing in (V, E) are not really missing therein. Rather, they have been added to (V, E), each with a large enough cost,[17] M, so that G^N remains an optimal tree for the grand coalition. Indeed, any core point in this modified game will remain a core point in the original game, where these new edges cannot be used and thus their cost is infinity.

The proof will be carried out in several steps. As in Theorem 4.5, we can assume that $b(v_0) = 0$.

Step 1. Place a *virtual player* at each public vertex other than the root. This will create a new SNE, $\mathscr{E}^* = (V, E, a, b, N^*)$, where $N^* = N \cup \{\text{virtual players}\}$, generating a game[18] $\Gamma_{\mathscr{E}^*} := (N^*; c)$.

Step 2. Show that $\mathscr{C}(\Gamma_{\mathscr{E}^*})$ contains a payoff vector $(x_i)_{i \in N^*}$, in which all virtual players pay zero. The execution of this step will take place subsequently.

Step 3. Consider the reduced game representing $\mathscr{E}^*$ on N at x. The core has the reduced game property, therefore $(x_i)_{i \in N}$ belongs to the core of the reduced game. However, by Theorem 3.3, the reduced game represents precisely the original enterprise, because the players that were removed paid zero at x. This shows that $(x_i)_{i \in N} \in \mathscr{C}(\Gamma_{\mathscr{E}})$, which concludes the proof once Step 2 is established.

Step 4. Define a new cost function $\tilde{c} : 2^{N^*} \to \Re \cup \{\infty\}$, by

$$\tilde{c}(S) := \text{least cost needed to connect all members of } S \text{ to the root, without passing through vertices not inhabited by members of } S. \tag{4.7}$$

Note that $\tilde{c}(\emptyset) = 0$, because $b(v_0) = 0$.
The game $\tilde{\Gamma}_{\mathscr{E}^*} := (N^*; \tilde{c})$ is called **the nonmonotonic representation of** $\mathscr{E}^*$.
Note that for $S \subseteq N^*$,

$$c(S) = \min\{\tilde{c}(R) : R \supseteq S\}, \tag{4.8}$$

from which it follows that

$$\mathscr{C}(\Gamma_{\mathscr{E}^*}) = \mathscr{C}(\tilde{\Gamma}_{\mathscr{E}^*}) \cap \Re_+^{N^*}. \tag{4.9}$$

Thus, the proof needed in Step 2 will be established once we show that $\mathscr{C}(\tilde{\Gamma}_{\mathscr{E}^*})$ contains a nonnegative payoff vector in which each virtual player pays zero. This will be established in the subsequent steps.

Note that the optimal network G^N in $\Gamma_{\mathscr{E}}$ is also an optimal network, denoted[19] $\tilde{G}^{N^*}$, in $\tilde{\Gamma}_{\mathscr{E}^*}$, because we assumed that G^N spans all vertices, including the public vertices of $\mathscr{E}$.

[17] E.g., $M > c(N)$.
[18] We use the same letter c as for the original $(N; c)$, because the original one is merely a restriction of this cost function to N.
[19] Note that $G^N = G^{N^*} = \tilde{G}^{N^*}$, because G^N was assumed to span all nodes, including the public nodes.

Step 5. Consider again $\mathscr{E}^* = (V, E, a, b, N^*)$. By construction, it has no public vertices. Also each original public vertex is inhabited by exactly one player. Let i be the virtual player, who resides at vertex v^i. Denote by $F(i)$ the set of vertices v adjacent to v^i and such that v^i lies between v_0 and v. By removing the edges (v^i, v), for $v \in F(i)$, we obtain $|F(i)| + 1$ disjoint connected components of $\tilde{G}^{N^*}$, rooted at $v \in F(i)$ and at v_0. Denote the last subtree by $R(v_0)$ and any other, rooted at v in $F(i)$, by $R(v)$.

Denote $F(i) = \{v_1, v_2, \ldots, v_k\}$ and let M_1 be a number larger than the cost of any edge (including the added edges whose cost is M). It is known (see, e.g., Tarjan (1979)) that if the cost of an edge e belonging to an optimal spanning tree T increases sufficiently, so as to render the tree non-optimal, then a new optimal tree can be found by adding just one new edge to T, to replace e.

Let $T^1 = (V, E^1)$ be an optimal network for the grand coalition in[20] $\tilde{\Gamma}_{\mathscr{E}^*}$, where $\mathscr{E}^*$ is modified by increasing the cost of (v_i, v_1) to M_1. It follows that T^1 is a tree and $E^1 = (E^N \backslash \{(v_i, v_1)\}) \cup u_1$, where u_1 is a new edge not in $\tilde{G}^{N^*}$. Similarly, if we increase (v_i, v_2) to M_1, an optimal tree $T^2 = (V, E^2)$ is obtained from T^1 by deleting (v_i, v_2) and adding a new edge u_2. Continuing in this fashion, we finally reach an optimal tree T^k for the grand coalition in $\tilde{\Gamma}_{\mathscr{E}^*}$, where $\mathscr{E}^*$ is modified by increasing the costs of (v_i, v_j), $j = 1, 2, \ldots, k$, to M_1. This tree is obtained from $\tilde{G}^{N^*}$ by deleting all the edges (v_i, v_j), $j = 1, 2, \ldots, k$, and adding the new and distinct edges $u_1, u_2, \ldots, u_k$. Since v_i is a leaf in T^k, it follows that by deleting v_i and the edge e_{v_i} from T^k, we obtain an optimal tree $\tilde{G}^{N^* \backslash \{i\}}$ for $N^* \backslash \{i\}$ in the original game $\tilde{\Gamma}_{\mathscr{E}^*}$. Indeed, it is a tree of least cost in which members of $N^* \backslash \{i\}$ reside, that is not allowed to pass through v_i. This tree has the following properties:

(1) $R(v_0)$ and $R(v)$, $v \in F(i)$, are subgraphs of $\tilde{G}^{N^* \backslash \{i\}}$,
(2) exactly one edge, denoted $\bar{e}_v$, emanates from each $R(v)$, in the direction of the root. It connects $R(v)$ either to another $R(v')$, or to $R(v_0)$. Denote its cost by d_v; i.e., $d_v = a(\bar{e}_v)$. (See Figure 7, where the various $\bar{e}_v$'s are indicated.)

Note that $d_v \geq a(e_v)$, where $e_v := (v^i, v)$; otherwise, G^{N^*} would not be an optimal tree for N^*.

Let x be an arbitrary imputation for $\tilde{\Gamma}_{\mathscr{E}^*}$, satisfying the following requirements:

(i) Each one of the inhabitants at the same vertex pays the same amount.
(ii) Inhabitants at a vertex v in $F(i)$ pay together the cost of their vertex plus the cost $a(e_v)$.
(iii) Player i, the single resident in v^i, pays $b(v^i) + a(e_{v^i})$.

Next, we extend the *weak demand operation*, introduced by Granot and Huberman [1984] for a SNE in which $b = 0$ and a single resident at each vertex. Formally, for an imputation x in $\tilde{\Gamma}_{\mathscr{E}^*}$, satisfying (i), (ii) and (iii), the imputation derived from x by a weak demand operation performed by i, de-

[20] Which is, of course optimal also in $\Gamma_{\mathscr{E}^*}$.

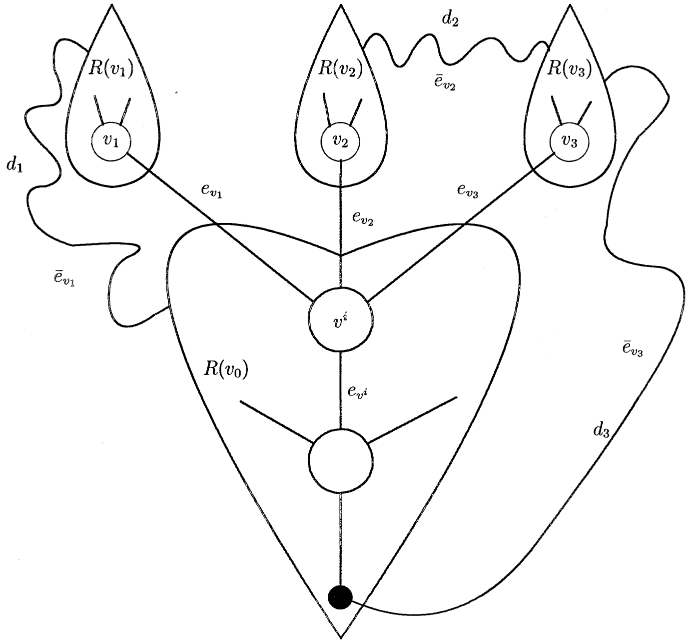

Fig. 7. The graphs of $\tilde{G}^{N^*}$ and $\tilde{G}^{N^*}\setminus\{i\}$

noted $wd^i(x)$, is given by[21]

$$wd^i_j(x) = \begin{cases} \dfrac{b(v)+d_v}{|N^v|}, & \text{if } j \in v \in F(i), \\ b(v^i) + a(e_{v^i}) - \sum_{v\in F(i)}[d_v - a(e_v)], & \text{if } j = i, \\ x_j, & \text{otherwise.} \end{cases} \tag{4.10}$$

A weak demand operation by player i, increases the payments allocated to each member of each v in $F(i)$ by $[d_v - a(e_v)]/|N^v|$, while reducing player i's payment appropriately.

We outline the next few steps: Starting from the tree-vector[22] t for the game $\tilde{\Gamma}_{\mathscr{E}^*}$, we perform recursively weak demand operations, each coupled with an appropriate convex combination with the previous payoff vector. We thus arrive at vectors $z^1 = wd^{i_1}(t)$, $\hat{z}^1 = \alpha_1 z^1 + (1-\alpha_1)t$, $z^2 = wd^{i_2}(\hat{z}^1)$, $\hat{z}^2 = \alpha_2 z^2 + (1-\alpha_2)\hat{z}^1$, $z^3 = wd^{i_3}(\hat{z}^2)$, $\hat{z}^3 = \alpha_3 z^3 + (1-\alpha_3)\hat{z}^2, \ldots$, where $i_1, i_2, i_3 \ldots$ are the virtual players added in Step 1.

We achieve it in such a way that $\hat{z}^k \in \mathscr{C}(\tilde{\Gamma}_{\mathscr{E}^*})$, $\hat{z}^k \geq 0$, and more and more virtual players' coordinates $\hat{z}^k_i$ vanish as k gets larger. Eventually we arrive at a vector satisfying all the requirements of Step 4, which was required to complete Step 2. This will therefore conclude our proof. Note that the order in

[21] $|A|$ denotes the number of elements in a set A.
[22] For the terminology, see the section after the proof of Theorem 4.5.

which we choose $i_1, i_2, \ldots$ is not important, because the vertices containing the virtual players are not adjacent.

Weak demand operations, performed consecutively, *starting from the tree-vector*, bring about core points, as was shown in Granot and Huberman [1984]. However, they do not necessarily yield core points if the starting core point is not the tree-vector. A counter example is given in the same paper. To show that combining the weak demand operations with convex combinations does not spoil the core inclusion requires a proof. This is provided in Steps 6 and 7 below.

Step 6. Let $\tilde{\Gamma}_{\mathscr{E}^*}$ be the game associated with an *arbitrary* $\mathscr{E}$ that satisfies the conditions of Theorem 4.13. Starting from the tree-vector for an optimal network for N^*, we assume by induction that up to $k-1$ consecutive weak demand operations coupled with appropriate convex combinations, lead to a nonnegative core point for $\tilde{\Gamma}_{\mathscr{E}^*}$, whose coordinates (if any) at virtual players for which these operations were already performed, vanish. $k = 1, 2, \ldots$. This is true for $k = 1$, because the tree-vector is a nonnegative core point.

We first apply this hypothesis to our given game $\tilde{\Gamma}_{\mathscr{E}^*}$. Thus, starting from the tree-vector t, we obtain a payoff vector $\hat{z}^{k-1}$, $k = 1, 2, \ldots$, which is a nonnegative core point of $\tilde{\Gamma}_{\mathscr{E}^*}$, whose coordinates (if any) at virtual players for which we already performed weak demand operations, coupled with convex combinations, vanish. $\hat{z}^0 := t$ satisfies this assumption.

We now perform an additional weak demand operation to obtain $z^k = wd^i(\hat{z}^{k-1})$, where i is a virtual player not treated earlier. The payoff vector $\hat{z}^{k-1}$ satisfies requirements (ii) and (iii), because the "hat operations", done so far, did not change the payments of player i and players residing at vertices of $F(i)$, and t also satisfies these requirements. Also (i) is satisfied, because t satisfies (i) and neither weak demand operations nor convex combinations alter the property that players at each vertex pay equal amounts.

Clearly, $z_v^k \geq 0$ for $v \neq i$, because, compared to $\hat{z}^{k-1}$, all payments at these coordinates did not decrease. The payment z_i^k, however, is nonpositive! Indeed, if $z_i^k > 0$, then the players in N could have connected themselves to the root in the original enterprise $\mathscr{E}$, using the tree $\tilde{G}^{N^* \setminus \{i\}}$. By (4.10), that would have been cheaper than the cost of G^N.

Consequently, an appropriate convex combination $\alpha z^k + (1-\alpha)\hat{z}^{k-1} =: \hat{z}^k$ exists for which $\hat{z}_i^k = 0$. Coordinates for players not residing in v^i and $F(i)$ do not change and remain nonnegative. Those of them that vanished earlier, vanish also at $\hat{z}^k$. In order to show that $\hat{z}^k$ is a core point of $\tilde{\Gamma}_{\mathscr{E}^*}$, it remains to prove that z^k is a core point of the same game. This proof, which is similar to the proof in Granot and Huberman [1984], who demonstrated that a sequence of weak demand operations would yield a core point if the starting vector is the tree-vector, is described below.

For a coalition S containing neither i nor residents of vertices in $F(i)$, $z^k(S) = \hat{z}^{k-1}(S) \leq \tilde{c}(S)$. For a coalition S containing i, $z^k(S) \leq \hat{z}^{k-1}(S) \leq \tilde{c}(S)$, because the decrease in the i-th coordinate compensates any possible increases in coordinates of $S \cap N(F(i))$, where $N(F(i))$ denotes the set of players residing at $F(i)$. The remaining case will be dealt with in the next step.

Step 7. Construct an auxiliary enterprise $\bar{\mathscr{E}}^* = (V, E, \bar{a}, b, N^*)$ that differs from $\mathscr{E}^*$ only in some costs of edges as follows: If a component $R(v)$, $v \in F(i)$, was connected directly to $R(v_0)$ in $\tilde{G}^{N^* \setminus \{i\}}$, change the cost of edge (v^i, v) to become

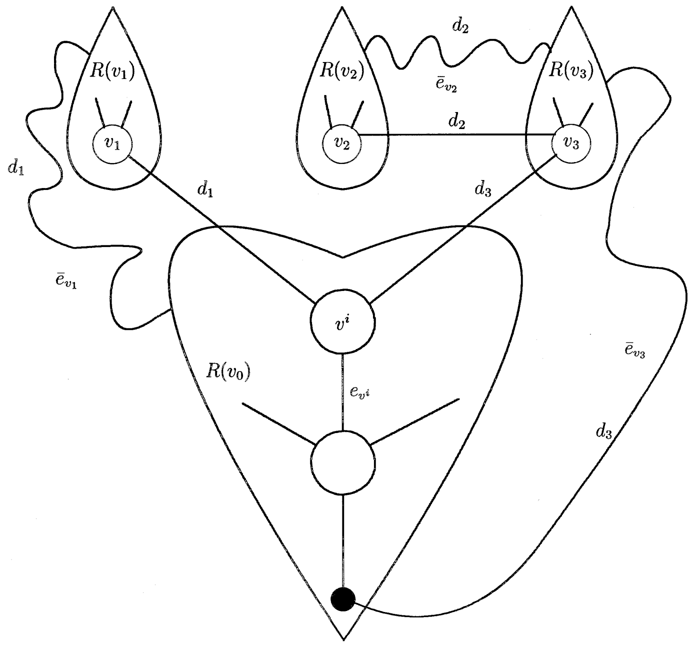

Fig. 8. The auxiliary enterprise in which the edges of the optimal tree are drawn in straight lines.

d_v. For other components $R(v)$, $v \in F(i)$, change the cost of (v^i, v) to be M_1, i.e., effectively deleted from the optimal tree. If $R(v)$ was connected in $\tilde{G}^{N^* \setminus \{i\}}$ to another component $R(v')$, in the direction of the root, where $v' \in F(i)$, change the cost of edge (v, v') to become d_v. (Compare Figure 7 with Figure 8, where these changes are indicated). These modifications imply that an optimal network $\bar{G}^{N^*}$, for N^* in $\bar{\mathscr{E}}^*$, can be obtained from G^N by replacing edges (v^i, v) by edges (v, v') for those $R(v)$'s not connected directly to $R(v_0)$ in $\tilde{G}^{N^* \setminus \{i\}}$. The cost of $\bar{G}^{N^*}$ now becomes equal to the cost of $\tilde{G}^{N^* \setminus \{i\}} + a(e_{v^i}) + b(v^i)$. Notice that the tree-vector associated with $\bar{G}^{N^*}$ in $\bar{\mathscr{E}}^*$, denoted $\bar{t}$, differs from t only at coordinates j for players residing in vertices of $F(i)$. For a player j residing in v, $v \in F(i)$, $\bar{t}_j = (b(v) + d_v)/|N^v|$.

Let us denote by $\bar{u}^{k-1}$ the nonnegative core vector for $\tilde{\Gamma}_{\bar{\mathscr{E}}^*}$, derived by performing in $\bar{G}^{N^*}(k-1)$ weak demand operations followed by convex combinations, recursively, for players $j = i_1, i_2, \ldots, i_{k-1}$, starting from the tree-vector $\bar{t}$. In $\bar{u}^{k-1}$, players $i_1, i_2, \ldots, i_{k-1}$ pay zero. This is true because of our induction hypothesis. Notice that $\bar{u}^v$ and $\hat{z}^v$, $v = 1, 2, \ldots, k-1$, are equal at all coordinates except coordinates j, $j \in F(i)$. However, $\bar{u}^{k-1}$ and z^k are equal at all coordinates except coordinate i. This is due to the changes made while passing from $\tilde{G}^{N^*}$ to $\tilde{\bar{G}}^{N^*}$.

Comparing $(N^*; \tilde{\bar{c}})$, the nonmonotonic representation of $\bar{\mathscr{E}}^*$, with $(N^*; \tilde{c})$, the nonmonotonic representation of $\mathscr{E}^*$, one realizes that $\tilde{\bar{c}}(S) \leq \tilde{c}(S)$ for every coalition not containing i. Indeed, such coalition cannot make use of edges (v^i, v), $v \in F(i)$ and $\bar{a}(e) \leq a(e)$ for all $e \in E \setminus \{(v^i, v) : v \in F(i)\}$. As previously observed, $z^k := wd^i(\hat{z}^{k-1})$ and $\bar{u}^{k-1}$ differ only at coordinate i. Therefore, for coalition S not containing player i,

$$z^k(S) = \bar{u}^{k-1}(S) \leq \bar{\bar{c}}(S) \leq \tilde{c}(S). \tag{4.11}$$

This completes the proof that $z^k \in \mathscr{C}(\tilde{\Gamma}_{\mathscr{E}^*})$ and concludes the proof of Theorem 4.13. ■

5. Efficient representation of the core and the nucleolus of spanning network games

The core of an n-person game is determined by a system of 2^n inequalities. In this section we show that, for a large class of SNEs, much fewer inequalities need be considered. This class is characterized by the existence of an optimal network G^N which is a tree that spans all the vertices. By Lemma 4.12 it follows that we can assume, and we will assume, that G^S is a tree for every subset of N. Again, we assume that $b(v_0) = 0$.

Let W be a subset of V. We denote by $(W, \hat{E}(W))$ the set of vertices W and edges $\hat{E}(W)$, such that $\hat{E}(W) = \{e \in E^N : e = (i,j) \text{ and } \{i,j\} \cap W \neq \varnothing\}$.

We shall say that $(W, \hat{E}(W))$ is a connected component of G^N if

(i) $v_0 \notin W \subseteq V$,
(ii) $(W, \hat{E}(W))$ is connected; that is, between any two vertices of W there exists a path consisting of edges in $\hat{E}(W)$.

Note that in general $(W, \hat{E}(W))$ is not a subgraph, because it may contain edges with some endpoints not belonging to W. However, $(V^N \backslash W, E^N \backslash \hat{E}(W))$ is a subgraph of (V^N, E^N). If it is connected, we say that $(W, \hat{E}(W))$ is **an extreme component of** G^N; otherwise, we call it **an interior component of** G^N.

The characterization of coalitions whose corresponding core constraints can safely be omitted is quite complicated and requires some preparations.

Let $G^N = (V^N, E^N)$ be an optimal tree for N in a SNE $\mathscr{E} = (V, E, a, b, N)$. For a nonempty coalition S, let $G^S = (V^S, E^S)$ be an optimal tree for S. In general, $N(V^S)$—the set of players residing in V^S—may contain players not belonging to S. Also, some edges in E^S are also edges in E^N and some are not.

Remove from G^N the root and the vertices in which S resides. The resulting network, $(V^N \backslash (V(S) \cup \{v_0\}), E^N)$ will consist of several (maximal) connected components. Denote by $P^S := \{C^1, C^2, \ldots, C^{k(S)}\}$ the set consisting of those components *that contain players*. Here, $V(S)$ denotes the set of vertices occupied by members of S. We also denote by $V(C)$ and by $\hat{E}(C)$ the vertex set and the edge set of the component C, respectively, and by $N(C)$ – the set of players that reside in $V(C)$.

Example 5.1. Figure 9 demonstrates our analysis. Here, G^N is drawn in straight line segments. $S = \{1, 2, 8, 9, 11, 12\}$ and G^S is drawn in wavy lines. Note that it spans also a, b and 15.

$$\begin{aligned} P^S = \{&(\{3\}, \hat{E}(\{3\})), (\{4, b, 5, 6\}, \hat{E}(\{4, b, 5, 6\}), (\{7\}, \hat{E}(\{7\})), \\ &(\{c, 10\}, \hat{E}(\{c, 10\})), (\{14\}, \hat{E}(\{14\})), (\{15\}, \hat{E}(\{15\})), \\ &(\{d, 16\}, \hat{E}(\{d, 16\}))\}. \end{aligned} \tag{5.1}$$

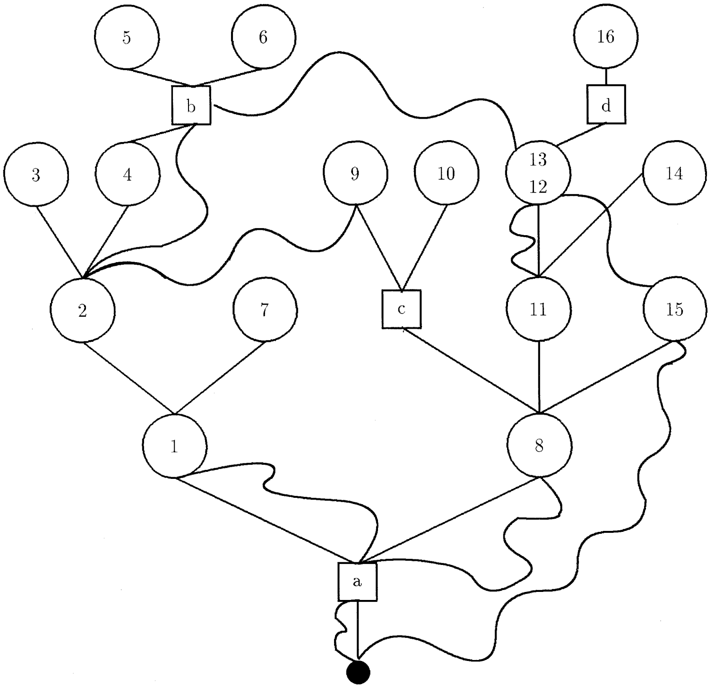

Fig. 9. G^N and G^S

We identified names of vertices with names of players residing in them. Public vertices are denoted by low case letters.

We are interested in those coalitions for which $|P^S| = 1$. Denote the set of such coalitions by $\mathscr{S}(G^N)$. It turns out that only these coalitions and N need be considered in order to determine the core of the enterprise:

Theorem 5.2. *Let (V, E, a, b, N) be a SNE, for which there exists an optimal network $G^N := (V^N, E^N)$, which is a tree that spans all the vertices V; i.e., $V = V^N$. A nonnegative cost allocation vector x belongs to the core of the enterprise if x satisfies the core constraints for the coalitions in $\mathscr{S}(G^N)$.*

A similar result was proved in Granot and Huberman [1984] for the non-monotonic representation of a SNE, in which all vertices are occupied by a single player, costs on vertices are zero and the edges-costs are nonnegative. Other proofs of this theorem, for the same enterprises considered by Granot and Huberman [1984], were given also by Aarts and Driessen [1992] and by Kuipers [1994]. The proof of the present theorem follows the spirit of the proof in Granot and Huberman [1984]. It requires some notation and two lemmas:

For a set of vertices W and a set of edges E we write $k(W, E)$ for short, instead of $b(W) + a(E)$, and call it **the cost of** (W, E). As before, we denote by $N(W)$ the set of players in W.

Lemma 5.3. *Let* $G^N = (V^N, E^N)$ *be an optimal subgraph for* N *in a SNE* (V, E, a, b, N) *and let* $(W, \hat{E}(W))$ *be an extreme component of* G^N. *For any cost allocation vector* x, *satisfying the core constraint for the coalition*[23] $N \backslash N(W)$ *in the spanning network game* $(N; c)$,

$$x(N(W)) \geq k(W, \hat{E}(W)). \tag{5.2}$$

Proof. Clearly $x(N) = k(V^N, E^N)$, because x is a cost allocation and (V^N, E^N) is optimal for N. If (5.2) is violated then

$$x(N \backslash N(W)) > k(V^N \backslash W, E^N \backslash \hat{E}(W)). \tag{5.3}$$

Now, $(V^N \backslash W, E^N \backslash \hat{E}(W))$ is a connected subgraph, which contains the root, because its complement $(W, \hat{E}(W))$ is an extreme connected component of G^N (see (i) and (ii) above); therefore, it is a feasible network for $N \backslash N(W)$. Thus, $c(N \backslash N(W)) \leq k(V^N \backslash W, E^N \backslash \hat{E}(W)) < x(N \backslash N(W))$, contradicting the fact that x satisfies the core constraint for the coalition $N \backslash N(W)$. ∎

The extension of this lemma is somewhat more complicated if $(W, \hat{E}(W))$ is not extreme.

Lemma 5.4. *Let* $G^N = (V^N, E^N)$ *be an optimal subgraph for* N *in a SNE* (V, E, a, b, N), *and let* $(W, \hat{E}(W))$ *be an interior connected component of* G^N. *Choose an arbitrary set of vertices and edges* (Q, E_Q), *disjoint from the complement* $(V^N \backslash W, E^N \backslash \hat{E}(W))$, *such that its union with this complement is connected.*[24] *For any cost allocation vector* x, *satisfying the core constraint for* $N \backslash N(W)$ *in the spanning network game* $(N; c)$,

$$x(N(W)) \geq k(W, \hat{E}(W)) - k(Q, E_Q). \tag{5.4}$$

Proof. Again, $x(N) = k(V^N, E^N)$. If (5.4) is violated then

$$\begin{aligned} x(N \backslash N(W)) &> k(V^N \backslash W, E^N \backslash \hat{E}(W)) + k(Q, E_Q) \\ &= k(((V^N \backslash W) \cup Q, (E^N \backslash \hat{E}(W)) \cup E_Q)). \end{aligned} \tag{5.5}$$

The argument of k on the right side is a connected graph, which contains the root; therefore, it is a feasible network for $(N \backslash N(W)) \cup N(Q)$. Thus, by the monotonicity of c, $x(N \backslash N(W)) > c((N \backslash N(W)) \cup N(Q)) \geq c(N \backslash N(W))$, contrary to the fact that x satisfies the core constraint for the coalition $N \backslash N(W)$. ∎

Proof of Theorem 5.2. Assume first that the core is not empty. Let x be a nonnegative cost allocation vector satisfying the core constraints for the coalitions in $\mathscr{S}(G^N)$ and let S be an arbitrary coalition not in $\mathscr{S}(G^N)$. If $|P^S| = 0$, then $c(S) = c(N)$ and the excess of S at x is nonnegative, because x is nonnegative.

[23] Namely, the excess of this coalition is nonpositive.
[24] One possible choice, of course, is $(Q, E_Q) = (W, \hat{E}(W))$.

The idea of the rest of the proof is this: for an arbitrary coalition S, not in $\mathscr{S}(G^N)$, we replace S by larger and larger coalitions with smaller and smaller excesses[25] at x, which induce fewer and fewer components in P^S, until we arrive at a coalition in $\mathscr{S}(G^N)$, whose excess is known to be nonnegative. This shows that the excess of the original S at x is nonnegative; namely, x satisfies the core constraint for coalition S.

Step 1. For each coalition S, it is sufficient to consider $S' := N(V^S)$ instead of S. Indeed,[26] $c(S') = c(S)$ and $x(S') \geq x(S)$ because $x \geq 0$, so certainly $c(S') - x(S') \leq c(S) - x(S)$. We shall henceforth assume that $S = N(V^S)$.

Step 2. Let C be an extreme component of G^N, which is a member of P^S. Since C is connected, it follows that $N \backslash N(C)$ is a coalition in $\mathscr{S}(G^N)$.

So, $N \backslash N(C) \in \mathscr{S}(G^N)$ and satisfies the core constraint. Therefore, by Lemma 5.3, $x(N(C)) \geq k(C)$. It follows that

$$\begin{aligned} c(S) - x(S) &\geq c(S) - x(S) + k(C) - x(N(C)) \\ &\geq c(S \cup N(C)) - x(S \cup N(C)), \end{aligned} \tag{5.6}$$

where the last inequality holds, because $(V^S \cup V(C), E^S \cup E(C))$ is feasible for $S \cup N(C)$ and $k(G^S) = c(S)$. We have shown that we can assume that S is such that all members of P^S are only interior components.

Step 3. Let S be an arbitrary coalition not in $\mathscr{S}(G^N)$, such that $|P^S| \geq 2$ and $N(V^S) = S$. Let C be an interior component of G^N, which is a member of P^S. We claim the existence of a network of vertices and edges (Q, E_Q) such that $Q = \emptyset$, E_Q is disjoint from C that satisfy:

(i) $((V^N \backslash V(C)) \cup Q,\ (E^N \backslash \hat{E}(C)) \cup E_Q)$ is connected.
(ii) $((V^S \backslash Q) \cup V(C),\ (E^S \backslash E_Q) \cup \hat{E}(C))$ is a feasible network for $S' := S \cup N(C)$.

If this claim is established then we can prove that $S \cup N(C)$ has a smaller excess than S at x, as follows: The coalition $N \backslash N(C)$ induces a single component, namely $C = (V(C), \hat{E}(C))$, so x satisfies the core constraint for this coalition. Consequently, by item (i), and Lemma 5.4,

$$x(N(C)) \geq k(C) - k(Q, E_Q). \tag{5.7}$$

Thus, $c(S) - x(S) = k(G^S) - x(S) + x(N(C)) - x(N(C)) \geq k(G^S) + k(C) - k(Q, E_Q) - x(S) - x(N(C)) \geq c(S \cup N(C)) - x(N(C)) - x(S) = c(S \cup N(C)) - x(S \cup N(C))$. The last inequality follows from item (ii) above.

Thus, it is sufficient to show that $S \cup N(C)$ has a nonnegative excess. In this fashion we can add more and more connected components until we reach a coalition in $\mathscr{S}(G^N)$, whose excess is known to be nonnegative.

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[25] By the *excess of a coalition T* at x we mean $c(T) - \sum_{i \in T} x_i$.
[26] $S' \supseteq S$ implies $c(S') \geq c(S)$ (Theorem 2.2). On the other hand, $c(S') \leq c(S)$, because G^S is feasible for S'.

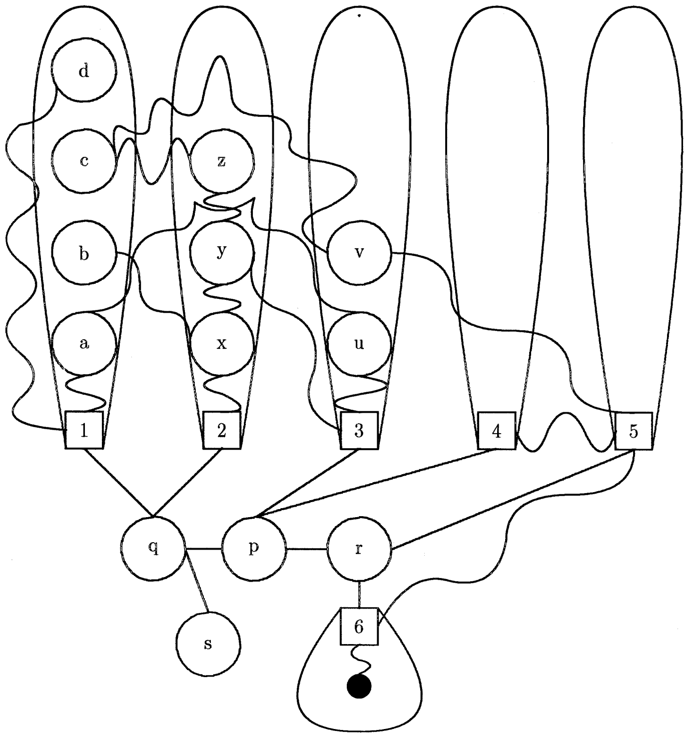

Fig. 10. G^N AND G^S

Step 4. We now construct the required set $(Q, E_Q) \equiv (\emptyset, E)$. It will turn out that $E \subseteq E^S$.

Call the boundary points[27] of C that also belong to V^S **distinguished vertices** of G^S, and enumerate them $1, 2, \ldots, k$. Let $G^N \backslash C$ denote the graph G^N after the removal of C. In $G^N \backslash C$ there are k connected subgraphs, $k-1$ of which are subtrees of G^N rooted at distinguished vertices and one which connects a distinguished vertex to the root. We shall refer to the distinguished vertices, which were enumerated $1, 2, \ldots, k$, as **the roots of the subgraphs**.

We now operate on two graphs and modify them successively. One graph is G^S and the other graph, called H, consists at first of k isolated vertices that represent the various subgraphs of G^N. They are also enumerated $1, 2, \ldots, k$, in correspondence with the enumeration of the distinguished vertices in G^S. Note that G^S is disjoint from C, because $S = N(V^S)$.

Figure 10 shows in a schematic way the graph G^N, that contains an interior component $C = (\{p, q, r, s\}, \hat{E}(\{p, q, r, s\})$, whose edges are drawn in straight lines. (Some vertices designated by letters contain players and some are public.) The distinguished vertices are marked by square boxes. Each of them contains players and they are enumerated $1, 2, \ldots, k$. Other edges of the enterprise, as well as some vertices not in S which are contained in the various subgraphs are not indicated. Observe that the connected component of $G^N \backslash C$ containing

[27] Namely, endpoints of edges in C that do not belong to C.

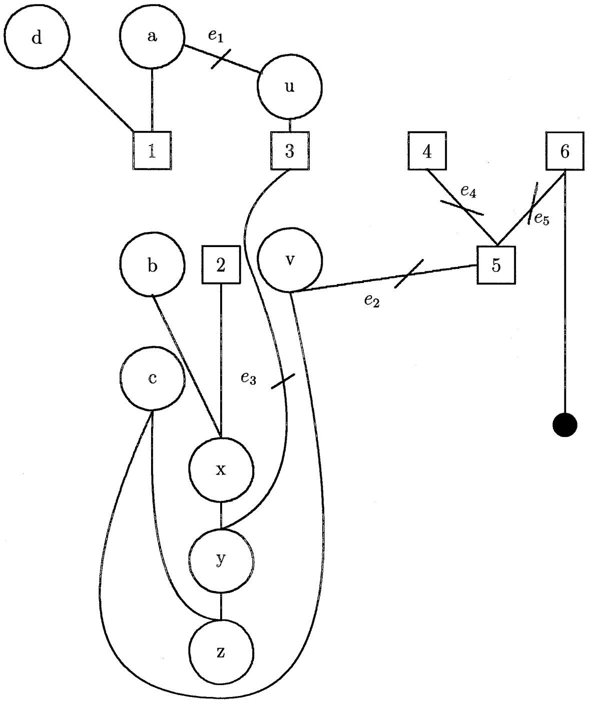

Fig. 11. G^S alone

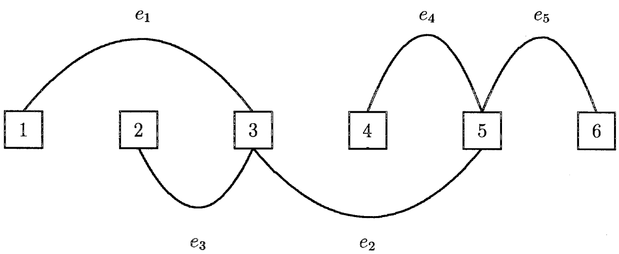

Fig. 12. The final H graph

vertex 6 also contains the root. Vertices that belong to the same subtree of G^N, rooted at a distinguished vertex, are drawn vertically and are encircled in Figure 10. In addition, G^S is drawn in wavy lines, where $S = \{1, a, b, c, d, 2, x, y, z, 3, u, v, 4, 5, 6\}$.

Figure 11 shows G^S alone, after some manipulations that will be described subsequently. Figure 12 shows the graph H after all necessary manipulations. The distinguished vertices of G^S are drawn as squares. Their numbers are $1, 2, \ldots, 6$.

We continue now in steps, modifying gradually G^S and H. The general idea is this: We delete $k - 1$ edges from G^S, each one of which is on the path between two distinguished vertices in G^S, which correspond to two subgraphs of $G^N \backslash C$. Each deleted edge in G^S connects two components of $G^N \backslash C$. When we delete the edge, we add in H an edge connecting the vertices that represent these components. We say that a deleted edge in G^S *gives rise* to the added edge in H. We also say that the edge added in H is *is the image* of the deleted edge in G^S.

We want to achieve the above in such a way that there will be no cycles in H , and that at the same time any two adjacent distinguished vertices in G^S will eventually be separated by exactly one deleted edge.

Figures 11 and 12 show one such example. The deleted edges in G^S are marked $e_1, \ldots, e_5$. Their "images" are drawn in Figure 12. Note that our task has been achieved in this example. If we can always achieve this task then the proof continues as follows:

We claim that $(Q, E_Q) := (\emptyset, E_Q)$, where $E_Q = \{e_1, e_2, \ldots, e_{k-1}\}$ is the requested set described in Step 3.

Proof of item (ii) in Step 3. The graph therein is precisely the graph G^S from which the edges of E were deleted and C was added. It is feasible for S. Indeed, every edge in E_Q lies on the path between two distinguished vertices in S, and it is the only edge on the unique path between these two distinguished vertices that was removed from E^S (i.e., contained in E_Q). Thus, the removal of E_Q from E^S decomposed G^S to k connected subgraphs, each one of which containing precisely one distinguished vertex. The addition of C to $(V^S, E^S \backslash E_Q)$ connects all these distinguished vertices, and since one of them is connected to the origin, we conclude that $(V^S \cup V(C), (E^S \backslash E_Q) \cup \hat{E}(C))$ is feasible for S. ■

Proof of item (i) in Step 3. In H we have k vertices and $k-1$ edges and it has no cycle. So, H is a tree, spanning all its vertices. In terms of the original tree that means that all the connected components of $G^N \backslash C$ are connected to each other via edges in E_Q that were added to $G^N \backslash C$. Since the root is contained in $G^N \backslash C$, G^N, from which C was removed and E_Q added, is connected. ■

Step 5. We have to exhibit a process of eliminating edges that will achieve the task described in Step 4. This will be done in stages. The general idea is this: Call a distinguished vertex in G^S *a leaf* if it does not lie on the path in G^S between the root and another distinguished vertex. (In Figure 11 the leaves are vertices 1, 2, and 4). At each stage we separate one leaf from the rest of the tree, by deleting a suitable edge. We choose an edge in such a way that its image in H does not overlap a previously drawn edge, nor does it form a cycle with previously drawn edges.

Stage i, $i = 1, 2, \ldots, k-1$. We assume that up until now we have deleted $i-1$ edges from G^S, added $i-1$ different edges to H and plucked $i-1$ leaves from G^S. We also assume that so far there are no cycles in H. Pick a leaf in the remaining part of G^S, starting, say, at a distinguished vertex r. Along the path from this leaf to an adjacent distinguished vertex find the first edge joining two connected components in $G^N \backslash C$, which contain distinguished vertices, say, p and q, so that H neither contains already the edge (p, q), nor the addition of (p, q) creates a cycle in H with existing edges (we shall later show that such an edge always exists) and whose deletion separates the leaf from the remaining distinguished vertices in G^S. (We shall later show that this is always possible.) Give the deleted edge the name e_r and add its image (p,q) to H, giving it the same name. Pluck the leaf that was separated from the rest of G^S. The process continues until $i = k - 1$. Figures 11 and 12 demonstrate this procedure.

Step 6. We have to show that the process is well defined. For this we have to show the existence of a leaf and an adjacent distinguished vertex, and the existence of an edge on the path in G^S between that leaf to its adjacent distinguished vertex, satisfying the above requirements. We assume that this was the case up to Stage i and we are now at the beginning of Stage i. The following lemma will prove useful.

Lemma 5.5. *Let K be a connected component of H, drawn until Stage i. This component K contains precisely one vertex q, whose corresponding edge e_q was not created yet in H.*

Proof. K is connected and contains no cycles. Thus, it contains, say, m vertices and $m - 1$ edges. By construction, every edge e_r in K is connected to vertex r either because vertex r is an endpoint of e_r, or, because edges drawn previously connect e_r to vertex r in H. Thus K must contain all vertices having the indices of the edges as names, and since there are $m - 1$ different edges, there are $m - 1$ different vertices connected to them and precisely one vertex q remains whose corresponding edge e_q was not constructed yet. ■

Continuation of Step 6. We now prove that at Stage i, along the path in G^S between *any* pair of adjacent distinguished vertices, different from the pairs treated previously, there is an edge whose image in H does not overlap a previously drawn edge nor does it form a cycle with previously drawn edges. Indeed, if that were not the case for a pair of adjacent vertices q_1 and q_2, then there would exist a component K in H, drawn before Stage i, that contains both q_1 and q_2. However, neither q_1 nor q_2 created any edge before Step i, because adjacent distinguished vertices are chosen after leaves were plucked. Yet, both belong to K. This contradicts Lemma 5.5.

Finally, we have to show that there always exists a leaf and an adjacent distinguished vertex such that when the edge on the path between them is deleted in accordance with the requirements, that leaf is separated out. This is certainly true if the leaf has a single distinguished adjacent vertex, so we can continue and pluck leaves until each remaining leaf has more than one adjacent leaf. (Figure 13 is such an example). Proceed as follows: Define **a junction** between two adjacent leaves to be the furthest vertex from the root which is common to the paths between the root and these vertices in G^S. (The vertices a and b are such junctions in Figure 13.) Consider a pair whose junction is *highest* in the sense that it does not lie on a path between another junction and the root. Call this vertex a (as in Figure 13). Pick up a pair of leaves whose junction is this vertex. Say the leaves are p and q (as in Figure 13). Starting at p in the direction of q, eliminate an edge on the path between p and q according to the above procedure. If this edge, call it e, is between p and a you have succeeded to separate vertex p out. If it is between q and a go in the reverse direction; namely starting from q in the direction of p, along the path between q and p. This time the edge you delete will be e, or an earlier edge. Indeed, you already know that e joins two disjoint components in G^N and its image does

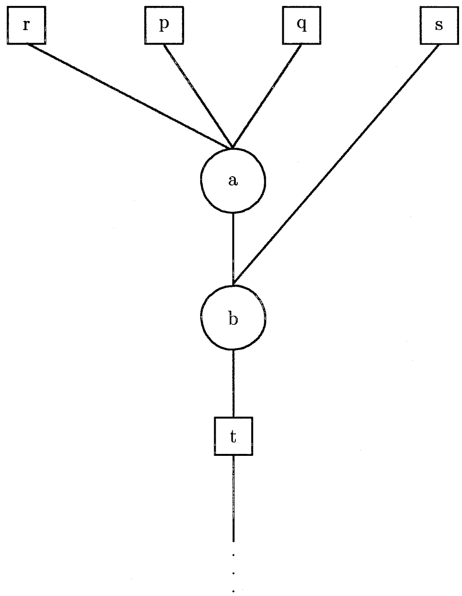

Fig. 13. A possible remain of G^S

not overlap an existing edge in H, nor does it form a cycle with the existing edges. You have succeeded to separate leaf q out.

We have proved that the process of eliminating an edge, as described above, separates either leaf p or leaf q out. Thus we completed the proof that the process is well defined, thereby finishing the proof of Theorem 5.2 for the case where the core is not empty.

Step 7. If the core is empty, we claim that there does not exist a nonnegative cost allocation vector satisfying the constraints in $\mathscr{S}(G^N)$. Indeed, otherwise, if such a vector x existed, then, by the proof above, we would have concluded that x also belongs to the core of $\Gamma_{\mathscr{E}}$, contradicting the emptiness of the core. ■

The essence of the proof of Theorem 5.2 was to show that if a coalition S gives rise to several components then, for every cost allocation x that satisfies the constraints of the theorem, the excess of S does not increase if we add to S all members of a component that contains players. This result is handy also if one computes the nucleolus of the game.

In fact, with this result one can prove that under the conditions of Theorem 5.2, if the core is not empty,[28] the nucleolus of a SNE depends only[29] on the coalitions in $\mathscr{S}(G^N)$. We omit the proof since it is quite similar to a proof given in Granot and Huberman [1984, Theorem 5] even though it is given there for a different coalition function defined on the SNE.

[28] Example 4.11 shows that this requirement is not redundant. See also Theorem 4.13.

[29] Namely, to compute it, it is sufficient to successively minimize the excesses of these coalitions.

We find this result interesting, because in general the nucleolus is not a locus of the core; i.e., in general there exist games having the same core and different nucleoli (see Maschler, Peleg and Shapley [1979]). The above result shows that this cannot happen if the games are induced by SNEs as described above, and having nonempty cores.

We conclude this section by only stating the following results:

Theorem 5.6. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE and assume that there exists an optimal network $G^N = (V^N, E^N)$, which is a tree that spans all the vertices; i.e., $V^N = V$. Under these conditions, if the core is not empty,*[30] *then x is the nucleolus point iff $s_{pq}(x) = s_{qp}(x)$ for every pair of adjacent players p and q in G^N and $x_p = x_q$ whenever players p and q reside in the same vertex. Here, by adjacent we mean that the path between the two players in G^N contains only public vertices (if any). Also, $s_{pq}(x)$ is a short notation for* $\min\{c(S) - x(S) : S \ni p, S \not\ni q\}$.

Corollary 5.7. *Under the conditions of the previous theorem, if the core is not empty, the nucleolus of the corresponding game is the intersection of the core and the kernel of this game; i.e.,*

$$\mathscr{N}(\Gamma_{\mathscr{E}}) = \mathscr{C}(\Gamma_{\mathscr{E}}) \cap \mathscr{K}(\Gamma_{\mathscr{E}}).$$

The proofs are based on the previous results of this section and are similar to the proofs given in Granot and Huberman [1984]. Therefore they will be omitted.

6. Decomposition

It is well known (see e.g., Megiddo [1978b]) that if the SNE is a tree and the tree has several branches emanating from the root then the core/nucleolus is a cartesian product of the cores/nucleoli of the subgames defined on the branches. These results were generalized by Granot and Huberman [1981] to the core and the nucleolus of a monotonic and nonmonotonic representations of a SNE which is not necessarily a tree but G^N is a tree of the above type. It was shown there, under the assumption that $a \geq 0$, $b = 0$, no public vertices and a single player at each vertex, that the core/nucleolus is still a cartesian product – not of the cores/nucleoli of the subgames, but of some modification thereof.

In this section we show that the above decomposition results still hold for more general monotonic representation of a SNE. Our proofs in the case of the core are similar to those given in Granot and Huberman [1981], but our decomposition proof for the nucleolus is different.

Consider a SNE $\mathscr{E} = (V, E, a, b, N)$, in which there may be public vertices, several players residing in a vertex, and where it is assumed that $b(v_0) = 0$. In this section we also assume that $G^N = (V^N, E^N)$ is a tree that spans all the vertices and for all vertices of[31] V^N, $a(e_v) + b(v) \geq 0$.

[30] Example 4.11 shows that this requirement is not redundant.
[31] See Notation 4.4 for the definition of e_v.

We state the decomposition results when the tree, G^N, has two branches. The extension to more branches can be done recursively. Suppose that $G^N = (V^N, E^N)$ is a union of two subtrees[32] $G^{N_i} := (V^{N_i}, E^{N_i})$, $i = 1, 2$, where (N_1, N_2) is a partition of N, $N(V^{N_i}) = N_i$ and these subtrees have only the root in common.

We now modify $\mathscr{E}$ by replacing each edge-cost[33] a_{v,v_0} by

$$\begin{cases} \bar{a}_{v,v_0} = \min\{a_{v,v_0}, \min\{a_{v,w} : w \in V^{N_2}\}\}, & \text{if } v \in V^{N_1}, \\ \bar{a}_{v,v_0} = \min\{a_{v,v_0}, \min\{a_{v,w} : w \in V^{N_1}\}\}, & \text{if } v \in V^{N_2}. \end{cases} \tag{6.1}$$

All other edge-costs remain unchanged; that is, $\bar{a}_{v,w} = a_{v,w}$. Here, regard every nonexisting edge as an edge whose cost is ∞. We denote the modified enterprise by $\bar{\mathscr{E}}$. Let $(N; \bar{c})$ be its game, and let $(N_i; \bar{c}^i)$ be the restriction of $(N; \bar{c})$ to N_i, $i = 1, 2$.

Lemma 6.1. *The optimal graph $G^N = (V^N, E^N)$ is an optimal network also for the modified enterprise $\bar{\mathscr{E}}$.*

Proof. Let v be an arbitrary vertex for which $\bar{a}_{v,v_0} < a_{v,v_0}$, and let C denote the unique cycle[34] in $(V^N, E^N \cup \{(v, v_0)\})$. The proof will follow if we show that $\bar{a}_{v,v_0} \geq \bar{a}(e)$ for all $e \in C$, (see Ford and Fulkerson [1962]). Now, suppose, without loss of generality, that $v \in V^{N_1}$ and let $w \in V^{N_2}$ be such that $a_{v,w} = \bar{a}_{v,v_0}$. Consider the unique cycle C' in $(V^N, E^N \cup \{(v, w)\})$. Then, since G^N is optimal for $\mathscr{E}$ and $(v, w) \notin E^N$, $a_{v,w} \geq a(e)$ for all $e \in C'$. However, all edges in $C \backslash \{(v, v_0)\}$ are contained in C', and we conclude that $a_{v,w} = \bar{a}_{v,v_0} \geq \bar{a}(e)$ for all $e \in C$. ■

Let $\bar{\mathscr{E}}_i$, $i = 1, 2$, denote the subenterprise of $\bar{\mathscr{E}}$, determined by $(V^i, E^i, \bar{a}, b, N_i)$, where $V^i = V^{N_i}$, $E^i = \{(u, v) \in E : u, v \in V^i\}$ and $\bar{a}$ and b are restricted to E^i and V^i, respectively.

Corollary 6.2. *G^{N_i} is optimal for the subenterprise $\bar{\mathscr{E}}_i$, $i = 1, 2$.*

Proof. If G^{N_i} is not optimal for some i, then G^N is not optimal for $\bar{\mathscr{E}}$, which would contradict Lemma 6.1. ■

We have seen (Lemma 4.12) that the assumption that $G^N = (V^N, E^N)$ is a tree spanning all vertices implies that for every $S \subseteq N$, $G^S = (V^S, E^S)$ can be assumed to be a tree. Here we shall show that if, in addition, we assume that $a(e_v) + b(v) \geq 0$, then the characteristic function c is nonnegative.

Lemma 6.3. *Let $\mathscr{E} = (V, E, a, b, N)$ be a SNE and let $(N; c)$ be its monotonic representation. If G^N is a tree spanning all vertices and $a(e_v) + b(v) \geq 0$, for all v in V, then $c \geq 0$.*

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[32] There is no danger of confusion in this notation, because if G^N is optimal for N, then G^{N_i} is optimal for N_i.

[33] For the sake of better visibility, we omit the parentheses in the subscripts, when we denote edges.

[34] Edge (v, v_0) cannot belong to G^N, since, otherwise, there would have been a spanning tree cheaper than G^N.

Proof. Suppose that $c(S) < 0$ for some S. Consider the network $(V^N, \hat{E})$, where $\hat{E} = E^S \cup \{e_v : e_v \in E^N \backslash E^S\}$. Just as in the proof of Theorem 4.5, $(V, \hat{E})$ is a connected network spanning V, whose cost, $k(V^N, \hat{E})$, satisfies $k(V^N, \hat{E}) = c(S) + a(E^N \backslash E^S) + b(V^N \backslash V^S) < a(E^N \backslash E^S) + b(V^N \backslash V^S) \le a(E^N) + b(V^N) = c(N)$, because $a(e_v) + b(v) \ge 0$. This contradicts the optimality of G^N. ■

Corollary 6.4. *Under the conditions of Lemma 6.3, any connected subnetwork* $\hat{G} = (\hat{V}, \hat{E})$ *for which* $v_0 \in \hat{V}$, *satisfies* $k(\hat{G}) = a(\hat{E}) + b(\hat{V}) \ge 0$.

Proof. Let S be the set of players (possible empty) residing in $\hat{V}$, then $a(\hat{E}) + b(\hat{V}) \ge c(S) \ge 0$. ■

Lemma 6.5. *The game* $\Gamma_{\bar{\mathscr{E}}_i}$, *that represents the SNE* $\bar{\mathscr{E}}_i = (V^{N_i}, E^{N_i}, \bar{a}, b, N_i)$, *is precisely* $(N_i, \bar{c}^i)$.

Proof. We have to show that for each S in N_i, there is an optimal network $\bar{G}^S$, made of edges and vertices contained in (V^i, E^i). If $S = \emptyset$, it follows from Corollary 6.4 that the root is optimal for $\Gamma_{\bar{\mathscr{E}}}$ as well as for $\Gamma_{\bar{\mathscr{E}}_i}$, $i = 1, 2$. Let $S \neq \emptyset$ and suppose, without loss of generality, that $S \subseteq N_1$. By Lemma 4.12, we can assume that $\bar{G}^S$ is a tree. Now, for every edge (p, q) in $\bar{G}^S$, where $p \in V^{N_1} \backslash \{v_0\}$ and $q \in V^{N_2} \backslash \{v_0\}$, such that p (resp. q) is on the unique path between q (resp. p) and v_0 in $\bar{G}^S$ we can replace the edge (p, q) in $\bar{G}^S$ by an edge (q, v_0) (resp. (p, v_0)). By (6.1), $a_{p,q} \ge a_{v_0,p}$ and $a_{p,q} \ge a_{v_0,q}$. We then obtain a tree $\tilde{G}^S$ which is decomposable into two subtrees $\tilde{G}^{S_1} = (\tilde{V}^{S_1}, \tilde{E}^{S_1})$ and $\tilde{G}^{S_2} = (\tilde{V}^{S_2}, \tilde{E}^{S_2})$ such that $S \subseteq N(\tilde{V}^{S_1})$. They have only the root in common. Now, $k(\bar{G}^S) \ge k(\tilde{G}^S) = k(\tilde{G}^{S_1}) + k(\tilde{G}^{S_2}) \ge k(\tilde{G}^{S_1})$. The last inequality follows from Corollary 6.4. Thus, we obtained an optimal network $\tilde{G}^{S_1}$ for $S \subset N_1$, which is contained in (V^1, E^1), and this completes the proof. ■

Lemma 6.6. *The game* $(N; \bar{c})$ *is decomposable over* N_1 *and* N_2; *i.e., if* $S_1 \subseteq N_1$ *and* $S_2 \subseteq N_2$ *then* $\bar{c}(S_1 \cup S_2) = \bar{c}(S_1) + \bar{c}(S_2)$.

Proof. By Lemma 6.5, we can assume that an optimal network G^{S_i} in $\bar{\mathscr{E}}$ is contained in (V^i, E^i), $i = 1, 2$. Thus, the union of G^{S_1} and G^{S_2} is feasible for S_1 and S_2, and $\bar{c}(S_1) + \bar{c}(S_2) \ge \bar{c}(S_1 \cup S_2)$. Let $\bar{G}^{S_1 \cup S_2}$ be an optimal subgraph for $S_1 \cup S_2$ in the game $\Gamma_{\bar{\mathscr{E}}}$. For each edge (p, q) in this subgraph, where $p \in V^{N_1} \backslash \{v_0\}$ and $q \in V^{N_2} \backslash \{v_0\}$, such that p (resp. q) is on the unique path between v_0 and q (resp. p) in $\bar{G}^{S_1 \cup S_2}$, replace the edge (p, q) in $\bar{G}^{S_1 \cup S_2}$ by an edge (q, v_0) (resp. (p, v_0)). Note that by (6.1), $\bar{a}_{p,q} \ge \bar{a}_{v_0,q}$ and $\bar{a}_{p,q} \ge \bar{a}_{v_0,p}$. We obtain an enterprise $\tilde{G}^{S_1 \cup S_2}$ which is decomposable into two enterprises $\tilde{G}^{S_1}$ and $\tilde{G}^{S_2}$, over S_1 and S_2. They have only the root in common. Now, $\bar{c}(S_1 \cup S_2) = k(\bar{G}^{S_1 \cup S_2}) \ge k(\tilde{G}^{S_1 \cup S_2}) = k(\tilde{G}^{S_1}) + k(\tilde{G}^{S_2}) \ge \bar{c}(S_1) + \bar{c}(S_2)$. The last inequality follows from the fact that $\tilde{G}^{S_i}$ is feasible for S_i, $i = 2$. ■

Corollary 6.7. *For every* $S_1 \subseteq N_1$ *and* $S_2 \subseteq N_2$, $\bar{c}^1(S_1) + \bar{c}^2(S_2) \le c(S_1 \cup S_2)$.

Proof. By Lemma 6.6, $\bar{c}^1(S_1) + \bar{c}^2(S_2) = \bar{c}(S_1 \cup S_2) \le c(S_1 \cup S_2)$, because $(N; \bar{c})$ was obtained from $(N; c)$ by reducing costs of some edges. ■

Lemma 6.8. *For $S_1 \subseteq N_1$ (resp., $S_2 \subseteq N_2$), $\bar{c}^1(S_1) + \bar{c}^2(N_2) = c(S_1 \cup N_2)$ (resp., $\bar{c}^1(N_1) + \bar{c}^2(S_2) = c(N_1 \cup S_2)$).*

Proof. By Corollary 6.7, $\bar{c}^1(S_1) + \bar{c}^2(N_2) \le c(S_1 \cup N_2)$. Consider now the enterprises $\bar{\mathscr{E}}_1$ and $\bar{\mathscr{E}}_2$. Let $\bar{G}_1^{S_1}$ and $\bar{G}_2^{N_2} = G^{N_2}$ be optimal networks for S_1 and N_2 in these enterprises, respectively (see Lemma 6.5). Connect the two networks by replacing each edge (v, v_0), $v \in \bar{G}_1^{S_1}$, such that $\bar{a}_{v,v_0} < a_{v,v_0}$ with edge (v, w), $w \in \bar{G}_2^{N_2}$, for which $\bar{a}_{v,v_0} = a_{v,w}$ (see (6.1)). Denote by $\tilde{G}$ the resulting graph.

Since $\bar{G}_1^{S_1}$ and $\bar{G}_2^{N_2}$ are feasible for S_1 and N_2, respectively, it follows that $\tilde{G}$ is feasible in $\mathscr{E}$ for $S_1 \cup N_2$. Thus, $\bar{c}^1(S_1) + \bar{c}^2(N_2) = k(\bar{G}_1^{S_1}) + k(\bar{G}_2^{N_2}) = k(\tilde{G}) \ge c(S_1 \cup N_2)$. We conclude that $\bar{c}^1(S_1) + \bar{c}^2(N_2) = c(S_1 \cup N_2)$. In a similar fashion we prove the other part of the lemma. ■

We are now ready for the main results of this section.

Theorem 6.9. *With the above notation, $\mathscr{C}((N; c)) = \times_{i=1}^2 \mathscr{C}((N_i; \bar{c}^i))$.*

Proof. Suppose first that $\mathscr{C}(N; c) \neq \varnothing$.

Let $x = (x^{N_1}, x^{N_2})$ be a core point of $(N; c)$. As such, it satisfies $x^{N_i}(N_i) = c(N_i)$, $i = 1, 2$. Moreover, for any S_1, $S_1 \subseteq N_1$,

$$x(S_1 \cup N_2) = x^{N_1}(S_1) + x^{N_2}(N_2) \le c(S_1 \cup N_2) = \bar{c}^1(S_1) + \bar{c}^2(N_2), \qquad (6.2)$$

by Lemma 6.8. Therefore, $x^{N_1}(S_1) \le \bar{c}^1(S_1)$. This proves that $x^{N_1} \in \mathscr{C}((N_1; \bar{c}^1))$. In a similar fashion, $x^{N_2} \in \mathscr{C}((N_2; \bar{c}^2))$. Note that consequently $\mathscr{C}((N_i; \bar{c}^i)) \neq \varnothing$, $i = 1, 2$.

Conversely, let $x^i \in \mathscr{C}((N_i; \bar{c}^i))$, $i = 1, 2$. Let $x = (x^1, x^2)$. Clearly, $x(N) = x^1(N_1) + x^2(N_2) = \bar{c}^1(N_1) + \bar{c}^2(N_2) = c(N_1 \cup N_2)$, by Lemma 6.8. Moreover, for any S, $S \subseteq N$, let $S = S_1 \cup S_2$, where $S_i \subseteq N_i$, $i = 1, 2$. Then, $x(S) = x^1(S_1) + x^2(S_2) \le \bar{c}^1(S_1) + \bar{c}^2(S_2) \le c(S_1 \cup S_2)$, by Corollary 6.7. This shows that $x \in \mathscr{C}((N; c))$.

If $\mathscr{C}((N; c)) = \varnothing$ then necessarily $\mathscr{C}((N_i; \bar{c}^i)) = \varnothing$ for $i = 1$, or for $i = 2$. Therefore, the theorem is satisfied also in this case. ■

We now turn to the nucleoli $\mathscr{N}$ and $\mathscr{N}_i$ of $\Gamma_{\mathscr{E}}$ and $\Gamma_{\bar{\mathscr{E}}_i}$, $i = 1, 2$. If the cores of these SNGs are not empty (Theorem 4.5), their nucleoli and prenucleoli coincide.[35] Our purpose is to show that in this case the nucleolus of $\Gamma_{\mathscr{E}}$ is a cartesian product of the nucleoli of $\Gamma_{\bar{\mathscr{E}}_i}$. To achieve this we need a few lemmas.

Lemma 6.10. *Let $x \in \mathscr{C}(N; c))$, let $x = (x^{N_1}, x^{N_2})$ and let $S = S_1 \cup S_2$, where $S_1 \subseteq N_1$, $S_2 \subseteq N_2$. With this notation,*

$$\begin{cases} \bar{c}^1(S_1) - x^{N_1}(S_1) = c(S_1 \cup N_2) - x(S_1 \cup N_2), \\ \bar{c}^2(S_2) - x^{N_2}(S_2) = c(S_2 \cup N_1) - x(S_2 \cup N_1) \end{cases} \qquad (6.3)$$

[35] It is proved in Maschler, Peleg and Shapley [1979] that the intersection of the kernel and prekernel with the core coincide. The nucleolus and the prenucleolus are obtained from these by appropriate lexicographic minimization. Therefore they too coincide.

and

$$\begin{cases} c(S_1 \cup S_2) - x(S_1 \cup S_2) \geq \bar{c}^1(S_1) - x^1(S_1), \\ c(S_1 \cup S_2) - x(S_1 \cup S_2) \geq \bar{c}^2(S_2) - x^2(S_2). \end{cases} \tag{6.4}$$

Proof. Since $x \in \mathscr{C}(N;c)$, it follows that $x(N_i) = \bar{c}^i(N_i)$, $i = 1, 2$. By Lemma 6.8, $c(S_1 \cup N_2) - x(S_1 \cup N_2) = \bar{c}^1(S_1) + \bar{c}^2(N_2) - x^{N_1}(S_1) - x^{N_2}(N_2) = \bar{c}^1(S_1) - x^{N_1}(S_1)$. The other part of (6.3) is proved in a similar fashion. By a similar reasoning one finds from Corollary 6.7 that $c(S_1 \cup S_2) - x(S_1 \cup S_2) \geq \bar{c}^1(S_1) - x^{N_1}(S_1) + \bar{c}^2(S_2) - x^{N_2}(S_2) \geq \bar{c}^i(S_i) - x^{N_i}(S_i), i = 1, 2,$ because $x^{N_i} \in \mathscr{C}(N_i; \bar{c}^i)$. ■

The following theorem was proved in Sobolev [1975], extending Kohlberg [1971].

Theorem 6.11. *A necessary and sufficient condition that a cost allocation x is the prenucleolus of a cost game $(N;c)$ is that for every real number α,*

$$\{S : c(S) - x(S) \leq \alpha\} \tag{6.5}$$

is a balanced collection, whenever it is not empty.

Theorem 6.12. *Suppose that $\mathscr{C}((N;c)) \neq \emptyset$. With the above notation, the nucleolus $\mathscr{N}((N;c))$ satisfies $\mathscr{N}((N;c)) = \times_{i=1}^{2}(\mathscr{N}_i((N_i;\bar{c}^i))$.*

Proof. Let x be a core point of $(N;c)$. Denote $x = (x^{N_1}, x^{N_2})$. Let α be a real number and consider the coalitions in the collection satisfying (6.5). By (6.3), if $S = S_1 \cup S_2$ is in this collection, where $S_i \subseteq N_i$, $i = 1, 2$, then $S_1 \cup N_2$ and $S_2 \cup N_1$ are also in this collection. Moreover, by (6.4),

$$S_i \in \{S \in N_i : \bar{c}^i(S) - x^{N_i}(S) \leq \alpha\}, \quad i = 1, 2. \tag{6.6}$$

Conversely, if S_i satisfies (6.6) for some i, then, by (6.3), $S_i \cup N_j$ satisfies (6.5), $i, j \in \{1, 2\}$, $i \neq j$. Thus, the restriction of the incidence matrix of the collection (6.5) to the columns that represent the players in N_i, becomes identical to the incidence matrix of the collection (6.6), after removing duplications and rows of 1's.

Let x be the nucleolus point of $(N;c)$, then the incidence matrix of (6.5) is balanced for every α. Its restriction to the columns corresponding to N_i, $i = 1, 2$, remains balanced. Indeed removing duplications and rows of 1's from the restricted matrix does not spoil balancedness. Consequently $x^{N_i} \in \mathscr{N}(N_i; \bar{c}^i)$, $i = 1, 2$.

The other direction follows from the fact that the nucleolus consists of a unique point. ■

Example 6.13. Consider the 3-person enterprise $\mathscr{E}$, given in Figure 14. Its cost function is:

$$\begin{aligned} &c(\emptyset) = 0, \quad c(1) = 4, \quad c(2) = 6, \quad c(3) = 2, \\ &c(12) = 7, \quad c(13) = 6, \quad c(23) = 7, \quad c(123) = 9. \end{aligned} \tag{6.7}$$

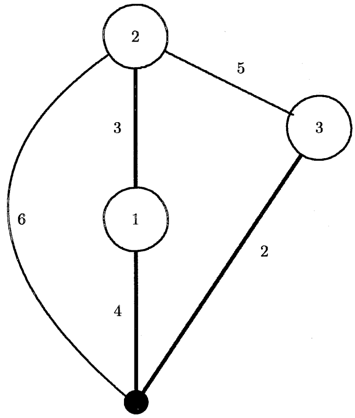

Fig. 14. The decomposition requires modification

The tree G^N is obtained by deleting the edges $(v_0, 2)$ and $(2, 3)$. It is indeed a union of two subtrees, having only the root in common. Its core/nucleolus are not the cartesian products of the cores/nucleoli of the subgames on $\{1, 2\}$ and $\{3\}$. Rather they are the cartesian product of the subgames on these players after we replace the cost of the edge $(v_0, 2)$ by 5. Indeed, the core of the original enterprise is the line segment $[(2, 5, 2), (4, 3, 2)]$ and the nucleolus point is $(3, 4, 2)$.

References

Aarts H, Driessen Th (1992) On the core-structure of minimum cost spanning tree games. Memorandum No. 1085, Faculty of Applied Mathematics, University of Twente, The Netherlands

Aumann RJ, Drèze JH (1974) Cooperative games with coalition structures. International Journal of Game Theory 3:217–237

Bird CG, (1976) On cost allocation for a spanning tree: A game theoretic approach. Networks 6: 335–350

Claus A, Granot D (1976) Game theory application to cost allocation for a spanning tree. Working paper no. 402, Faculty of Commerce and Business Administration, University of British Columbia, Vancouver

Claus A, Kleitman DJ (1973) Cost allocation for a spanning tree. Network 3:289–304

Davis M, Maschler M (1965) The kernel of a cooperative game. Naval Research Logistics Quarterly 12:223–259

Ford LR Jr, Fulkerson DR (1962) Flows in networks. Princeton University Press, Princeton, New Jersey

Granot D, Huberman G (1981) Minimum cost spanning tree games. Mathematical Programming 21:1–18

Granot D, Huberman G (1984) On the core and nucleolus of minimum cost spanning tree games. Mathematical Programming 29:323–347

Kohlberg E (1971) On the nucleolus of a characteristic function game. SIAM Journal of Applied Mathematics 20:62–66

Kuipers J (1994) Combinatorial methods in cooperative game theory. Ph.D. Dissertation, Rijksuniversiteit Limburg te Maastricht, The Netherlands

Littlechild SC (1974) A simple expression for the nucleolus in a special case. International Journal of Game Theory 3:21–29

Littlechild SC, Owen G (1973) A simple expression for the Shapley value in a special case. Management Science 20:370–372

Littlechild SC, Owen G (1977) A further note on the nucleolus of the ‘Airport Game’. International Journal of Game Theory 5:91–95

Littlechild SC, Thompson GF (1977) Aircraft landing fees: a game theory approach, The Bell Journal of Economics 8:186–204

Maschler M, Peleg B, Shapley LS (1979) Geometric properties of the kernel, nucleolus, and related solution concepts. Mathematics of Operations Research 4:303–338

Megiddo N (1978a) Cost allocation for Steiner trees. Networks 8:1–6

Megiddo N (1978b) Computational complexity and the game theory approach to cost allocation for a tree. Mathematics of Operations Research 3:189–196

Nouweland van den A, Tijs S, Maschler M (1993) Monotonic games are spanning network games. International Journal of Game Theory 21:419–427

Peleg B (1985) An axiomatization of the core of cooperative games without side payments. Journal of Mathematical Economics 14:203–214

Peleg B (1986/87) On the reduced game property and its converse. International Journal of Game Theory 15:187–200, Correction, International Journal of Game Theory 16:290

Peleg B (1992) Axiomatization of the core. Handbook of Game Theory with Economic Applications, In: Aumann RJ, Hart S (eds.), vol. 1, Elsevier Science Publishers B.V., North Holland, Amsterdam-London-New York-Tokyo, pp. 397–412

Sobolev AI (1975) The characterization of optimality principles in cooperative games by functional equations. In: Vorobjev NN (ed.) Mathematical Methods in the Social Sciences, Proceedings of a Seminar, Issue 6, Vilnius, Institute of Physics and Mathematics, Academy of Sciences of the Lithuanian SSR, pp. 94–151 (Russian, English Summary)

Tamir A (1991) On the core of network synthesis games. Mathematical Programming 50:123–135

Tarjan RE (1979) Applications of path compression in balanced trees. Journal of Association of Computing Machinary 26:690–715