OPTIMAL MULTIVARIATE STOPPING RULES

DAVID ASSAF,∗∗∗ ∗ AND ESTER SAMUEL-CAHN, ∗
The Hebrew University of Jerusalem

Abstract

For fixed \(i\) let \(X(i) = (X_1(i), \ldots, X_d(i))\) be a \(d\)-dimensional random vector with some known joint distribution. Here \(i\) should be considered a time variable. Let \(X(i), i = 1, \ldots, n\) be a sequence of \(n\) independent vectors, where \(n\) is the total horizon. In many examples \(X_j(i)\) can be thought of as the return to partner \(j\), when there are \(d\geq 2\) partners, and one stops with the \(i\)th observation. If the \(j\)th partner alone could decide on a (random) stopping rule \(t\), his goal would be to maximize \(E X_j(t)\) over all possible stopping rules \(t \leq n\). In the present 'multivariate' setup the \(d\) partners must however cooperate and stop at the same stopping time \(t\), so as to maximize some agreed function \(h(\cdot)\) of the individual expected returns. The goal is thus to find a stopping rule \(t^*\) for which

\[h(E X_1(t^*), \ldots, E X_d(t^*)) = h(E X(t))\]

is maximized. For continuous and monotone \(h\) we describe the class of optimal stopping rules \(t^*\). With some additional symmetry assumptions we show that the optimal rule is one which (also) maximizes \(EZ_t\) where

\[Z_i = \sum_{d=1}^d X_j(i)\]

and hence has a particularly simple structure. Examples are included, and the results are extended both to the infinite horizon case and to the case when \(X(1), \ldots, X(n)\) are dependent. Asymptotic comparisons between the present problem of finding \(\sup h(E X(t))\) and the 'classical' problem of finding \(\sup E h(X(t))\) are given. Comparisons between the optimal return to the statistician and to a 'prophet' are also included. In the present context a 'prophet' is someone who can base his (random) choice \(g\) on the full sequence \(X(1), \ldots, X(n)\), with corresponding return \(\sup h(E X(g))\).

Keywords: Cooperative stopping rule; finite and infinite horizon; maximization of functions of expected value; optimal choice set; prophet inequalities

AMS 1991 Subject Classification: Primary 60G40

1. Introduction

Consider a firm which holds \(d\geq 2\) different stocks, and is seeking a loan against these holdings. The firm pledges to sell the stocks within a given time period, say a year, all stocks to be sold at one time. The loan granted will be some known function \(h(\cdot)\) of \(d\) variables, here of the expected values of the stocks, when sold. We shall formalize this model. Let \(X(i) = (X_1(i), \ldots, X_d(i))\) be the (random) value of the \(d\) stocks at time \(i\), where \(i = 1, \ldots, n\). We assume that the joint distribution of all variables is known, and that all have finite expectations. \(n\) is the total horizon. The time of selling, \(t\), is a random stopping time \(t \leq n\), and the decision to sell at time \(i\) can clearly depend only on the values observed in the past and at present, i.e.
x^n_i = \sum_{i=1}^{n} x^i_i I(t = i), \quad \text{and we denote by} \quad E_x(t) = (E_1 x_1(t), \ldots, E_d x_d(t)), \quad \text{i.e. the vector of the expected values of the} \quad d \quad \text{stocks at the time sold.}

If \( T_n \) denotes the set of all stopping rules for which \( P(t \leq n) = 1 \), and the goal is to maximize the loan, we seek \( \sup_{t \in T_n} h(E_x(t)) \). (1.1)

The present paper is devoted to solving (1.1), that is, to finding the value in (1.1), showing that under mild conditions it is actually attainable, and describing the rules for which it is attained.

It should be noted that the described problem is entirely different from that of finding a rule \( t \leq n \) for which \( \sup_{t \in T_n} E h(X(t)) \) is attained. The latter problem entails nothing new, since if we set \( Y_i = h(X_1^i), \ldots, X_d^i) \), then (1.2) is equivalent to seeking an optimal stopping rule for the 'classical' one dimensional problem \( \sup_{t \in T_n} E Y_t \) and can be solved by backward induction. To distinguish (1.1) from (1.2) we call the former the 'multivariate stopping problem', as in the title of this paper. There is no direct backward induction technique for the multidimensional problem. Sometimes the term 'cooperative stopping problem' may be more suitable (see Assaf and Samuel-Cahn (1998)).

The following is an example. A doctors' office with \( d \geq 2 \) partners plans to hire a receptionist. \( n \) candidates apply for the job, and are interviewed sequentially. The candidate must be told immediately after the interview whether he or she is hired. The utility of the \( i \)th candidate is a random vector \( X_i = (X_1^i, \ldots, X_d^i) \) with known joint distribution, where \( X_j^i \) is the utility attached by the \( j \)th doctor to this candidate, \( j = 1, \ldots, d \). The \( j \)th doctor's ideal would be to find a stopping rule so as to maximize \( E x_j(t) \). (In this particular example the \( d \) variables will typically be dependent.) Since all doctors will engage the same receptionist, the \( j \)th doctor must, however, cooperate with the others, and they agree to maximize some known agreed function \( h \) of each of their expected utilities, upon stopping. Thus (1.1) describes their agreed goal.

In Section 2 we find the general solution to the maximization problem (1.1) for the case where the vectors \( X_i, i = 1, \ldots, n \) are independent. In the presence of a symmetric structure the solution becomes surprisingly simple and is given explicitly. The optimal rule for this case is a 'one-dimensional' rule which maximizes \( E Z t \), where \( Z_i = \sum_{j=1}^{d} X_j^i \), \( i = 1, \ldots, n \). In Section 3 examples are given, and Section 4 is devoted to additional remarks and generalizations. These include asymptotic comparisons, the infinite horizon problem and generalizations to the dependent case. The last section is devoted to 'prophet inequalities' in this multivariate setting.

Multivariate stopping rules have been considered in the literature earlier. Most of the papers have been in a game-theoretic setting. For example, Mamer (1987) establishes, under certain assumptions, the existence of a Nash equilibrium in non-randomized stopping times. In the setup described there, the players are not all required to stop at the same time. The approach of Yasuda et al. (1982) is closer to the approach taken here. In that paper, at every time instant \( i \), each partner (player) votes whether to stop, (1) or not, (0), after seeing \( X_i \) (and depending on
observed, and one has not stopped earlier. Non-randomized rules are obtained by letting
this replacement has taken place in all cases of independence, and thus omit the tilde, and write
maximize the value to society. The model is therefore related to ours, but different inasmuch
urable with respect to the probability with which one stops at the
increase function of the sum of the utilities of the
Let
\( X_1, \ldots, X_n \) be independent. Then \( S_n \) is compact.

Main results

Theorem 2.1. Let \( X_1, \ldots, X_n \) be independent.

For any \( \phi \) one has
\[
\sum_{i=1}^{\infty} E \left[ \sum_{j=1}^{n} \phi(X_j) \right] = \sum_{j=1}^{n} E \left[ \phi(X_j) \right].
\]

Denote by \( T_n \) the set of sequences \( (X_1, \ldots, X_n) \) of \( X_1, \ldots, X_n \).

Let \( \phi \) be a bounded, continuous function on \( T_n \).

By (2.1) we thus have
\[
\sum_{i=1}^{\infty} E \left[ \sum_{j=1}^{n} \phi(X_j) \right] = \sum_{j=1}^{n} E \left[ \phi(X_j) \right] = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{n} \phi(X_j) \right).
\]
essentially the weak compactness theorem.) Thus

Let $X$.

Theorem 2.3.

Let $\pi$.

Proof. Immediate by compactness.

i.e. the supremum is attained.

There are examples where one can show directly that the maximum is attained, without this

assume it is well defined for all $X$.

induction cannot be used, since $h$.

of this section is by means of the sequence of functions

But it follows by representation (2.2) and by the construction that this value must be

function $\phi$. Now consider the rule

$\tilde{T}_n$, which shows that

$\lambda$ is the value for the stopping rule $t \in T$. Again using (2.2), it is seen that its value is

for $0 \leq \lambda \leq t$.

Now consider the function $\phi$ for which we seek sup $\phi$.

$T_n$ such that

With this representation (2.1) is written as

$X^1$ and

$t \in T$.

\begin{align*}
\phi(x_1, \ldots, x_T) &= \sum_{i=1}^{T_n} \phi_i(x_i) \\
&= \sum_{i=1}^{T_n} \phi_i(x_i) \prod_{j=1, j \neq i}^{T_n} (1 - \phi_j(x_j)) \\
&= \sum_{i=1}^{T_n} \phi_i(x_i) \prod_{j=1, j \neq i}^{T_n} (1 - \phi_j(x_j)) \\
&\leq \sum_{i=1}^{T_n} \phi_i(x_i) \prod_{j=1, j \neq i}^{T_n} (1 - \phi_j(x_j)) \\
&= \phi(x_1, \ldots, x_T)
\end{align*}

$\phi_i(x_i) offset$
Theorem 2.4. Let $t \in \mathbb{R}^n$ be a boundary point of $\{Z_i \geq 0 \}$, then an optimal rule $t^*$ for the $Z$-sequence, in (2.3). On the 'arbitrary' part, we get $t^*$, as this rule stops as early as possible.

Let $\mathcal{H}$ denote the set of all $\mathcal{H}$-functions. For the multivariate case, which is described in the next theorem, however, $\mathcal{H}$ is upper semicontinuous and monotone in each coordinate.

There exists a vector of constants $c_0, c_1, \ldots, c_n \in \mathbb{R}$. The function $Z_t$ is of the form $Z_t = c_0 + c_1 t_1 + \cdots + c_n t_n$, where $t = (t_1, \ldots, t_n)$. This function is monotone increasing in some coordinates and monotone decreasing in the others.

Let $h \in \mathcal{H}$ be a non-randomized rule with $\phi = h$. Though any rule of the form (2.4) maximizes $\phi = h$, on the observed randomization depending on the $Z$-sequence cannot be ignored. Furthermore, on the set $\{Z_i \geq 0 \}$, this reflects on the form of $h$.

For the solution of the multivariate case, which is described in the next theorem, however, $\mathcal{H}$ is upper semicontinuous and monotone in each coordinate. The function $Z_t$ is of the form $Z_t = c_0 + c_1 t_1 + \cdots + c_n t_n$, where $t = (t_1, \ldots, t_n)$. This function is monotone increasing in some coordinates and monotone decreasing in the others.

Let $h^* \in \mathcal{H}$ be an optimal rule. Then $h^*$ stops as early as possible. A necessary and sufficient condition is that $h^*$ maximizes $\phi = h$.
Corollary 2.1. The exchangeability of each of the unique (see Example 3.2).

Proof. We next show that the inequality in (2.6) can be replaced by equality. This is achieved by noting that for any rule $X_j$ and let

$$EX_t \in \sum$$

through the sequence of all Schur concave permutations. But

$$EX_t \in \sum_{\pi_1}$$

with $EX_t \in \sum_{\pi_1}$, and thus this value must be

$$EX_t \in \sum_{\pi_1}$$

and

$$EX_t \in \sum_{\pi_1}$$

as a subset the set of all Schur concave permutations. (See for example Marshall and Olkin (1979), Chapter 3.)

Remark 2.1. Note that if the maximum of

$$EX_t \in \sum_{\pi_1}$$

satisfy (2.5).

Remark 2.2. For any given $EX_t \in \sum_{\pi_1}$, necessarily

$$EX_t \in \sum_{\pi_1}$$

corresponding to the desired

$$EX_t \in \sum_{\pi_1}$$

for all

$$EX_t \in \sum_{\pi_1}$$

and

$$EX_t \in \sum_{\pi_1}$$

through

$$EX_t \in \sum_{\pi_1}$$

The randomization part need not necessarily be

$$EX_t \in \sum_{\pi_1}$$

a.s.
where

and

\( Z_t \) do not suffice. Rather, the randomization needed to obtain an optimal value and rule depends to consider non-randomized rules. Furthermore, randomization depending only to the product function.

in detail in Assaf and Samuel-Cahn (1998). The approach and proof in that paper is particular solution to our maximization problem is easily obtained. Let

\( i \) to stop at time \( h \) is symmetric and increasing but condition (2.7) fails. Nevertheless the

\( \gamma \)-values satisfying

\( S_t \) are the values given in (2.3). Note that the 'stopping regions' are half spaces (i.e. linear),

\( (x_1,...,x_d) \). Further results and aspects of this example are studied

\( 2 \) holds, since here (2.7) is equivalent to the statement

\( X \)'s are exchangeable, and consider

\( t \) is monotone increasing and (2.7) holds, since here (2.7) is equivalent to the statement

Certainly if \( x \) be a random variable, let \( \phi \) be the random variable satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying

\( \gamma \)-values satisfying
The variance at time $t$ is given in the following result.

**Proposition 4.1.**

**Remark 4.1.**

**Example 4.1.**

The corresponding limit for $h$ is given in the following result.

**Remark 4.2.**

**Example 4.2.**

The optimal rule for the former would clearly be to stop when

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

where the intercept.

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

where

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

is just

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]

Asymptotic comparison of $h$ and $\delta$ for

\[
\arg\max_{j} \sum_{i=1}^{n} x_{it} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{j}{2} \cdot \left( \max_{i} x_{it} - \frac{1}{n} \sum_{i=1}^{n} x_{it} \right)
\]
is attained at some point(s) on the edge {sufficiently large.

Note that (4.5) implies that the conditional probability that $h^*$ occurs for all $j$ and $k$ is 0 a.s., the value in (4.1) is

Proof of Proposition 4.1. Then $(\sum_{1}^{m} x_{1}^{n})_{\text{occurs}}$ be such that

$$h^* + h(n)$$

is a triangle in the plane, and the maximum value of any monotone increasing

1, $\cdots$, $X$.

... ... ...

$$\cdots$$

$$D = \cdots$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$A_{m}$ denote the event \(\{x_{1} < \delta \} \cap \cdots \cap \{x_{m} < \delta \}\). For the sake of simplicity, let us denote by $A_{m}$ the event \(\{x_{1} < \delta \} \cap \cdots \cap \{x_{m} < \delta \}\). To show equality, it suffices to show that $h$ satisfies $0 \leq \ell \leq 1$ for all $x \in \mathbb{R}$.
method as the proofs in Section 2. For the particular case where one has
\[ a \]
where randomization may be required on \( X \) and \( h \).

For monotone sequence of \( Z_i \) random variables, with finite expectations, the definitions of the set \( \mathcal{H} \) rather surprising when considering the abundance of rules in \( T_n \).

Infinite horizon structure the optimal rule for the \( Z_i \) sequence of constants. This is a rule optimal for the sequence \( T_n \) is a rule optimal for the sequence \( X \) can be generalized to the arbitrarily dependent case, i.e. where \( X \) are only slightly more involved. The parallel to Theorem 2.4 now is that there exists a vector \( c \) that if \( h \) is such that
\[ a \] and \( X \) is exchangeable, \( d \) is again defined through
\[ a \] then the rule
\[ a \] is optimal.

Remarks. 
1. The constructions in this section can be generalized to \( n \) dependent random variables. The 'optimal choice set' problem is defined as follows.

2. The parallel to Corollary 2.1 we have
\[ a \].

3. Theorem 2.4 now is that there exists a vector \( c \) and with the previous exchangeability assumptions one deals with an infinite sequence \( X \).

4. All results can be extended to the case of infinite horizon, where
\[ a \].

5. The main difficulty in finding an optimal rule is that of finding \( c \) such that if we define
\[ a \]
then the rule
\[ a \] is optimal for \( X \) is always a half-space, i.e. of the form
\[ a \].

6. For details, see Glickman (1999).
3.2. There are 4 equally likely outcomes for the return to the prophet. Since he observes randomized decision rules \( g \) corresponding to the return to the prophet, the supremum in (5.1) is attained at some boundary point \( x \).}

The interpretation is that for a given \( h \), the hyperplanes for the statistician and the prophet, respectively. A notable exception is the hyperplane to \( P_n \) which states that for any independent non-negative random variables the best-known prophet inequality is the inequality

\[
E(Y_n) \geq \sum_{i=1}^n E(Y_i) / \sum_{i=1}^n 1
\]

Optimal multivariate stopping rules

In the usual univariate setting, for a given sequence \( Y_1, Y_2, \ldots, Y_n \) of independent non-negative random variables, the equivalent of

\[
E(Y_n) \geq \sum_{i=1}^n E(Y_i) / \sum_{i=1}^n 1
\]

Valls and Kertz (1992) for an excellent review. The best-known prophet inequality is the inequality

\[
E(Y_n) \geq \sum_{i=1}^n E(Y_i) / \sum_{i=1}^n 1
\]

Valls and Kertz (1992) for an excellent review. The best-known prophet inequality is the inequality

\[
E(Y_n) \geq \sum_{i=1}^n E(Y_i) / \sum_{i=1}^n 1
\]

Valls and Kertz (1992) for an excellent review. The best-known prophet inequality is the inequality

\[
E(Y_n) \geq \sum_{i=1}^n E(Y_i) / \sum_{i=1}^n 1
\]
equals 1 and 0 with probabilities $p_t$.

Example 5.2. Of the present section, cannot be improved upon; cf. Hill and Kertz (1981). Now let random variables are used in showing that the constant '2' mentioned in the first paragraph (5.2) cannot hold for all non-negative independent strategies) for the statistician and prophet will be optimal for all increasing functions $h$. Theorem 5.1.

Let $X_i$ be independent and identically distributed random variables, $i = 1, \ldots, n$, and suppose $h$ be exchangeable, $i = 1, \ldots, n$. Then

$$
\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right) = \frac{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right)}{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right)}
$$

Thus the optimal $h$-values are given by

$$
h_k = \frac{\text{EX}_n - \text{E} \left( f(X_k) \right)}{\text{EX}_n}
$$

for $k = 1, \ldots, n$. Note that for this example the $h$-values are achieved when

$$
\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right) = \frac{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right)}{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right)}
$$

It follows that

$$
\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right) = \frac{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right)}{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( f(X_i) \right)}{\text{EX}_n} \right)}
$$

Thus, for the special case of $f(x) = x^2$, the optimal $h$-values are given by

$$
h_k = \frac{\text{EX}_n - \text{E} \left( x^2 \right)}{\text{EX}_n}
$$

for $k = 1, \ldots, n$. Note that for this example the $h$-values are achieved when

$$
\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( x^2 \right)}{\text{EX}_n} \right) = \frac{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( x^2 \right)}{\text{EX}_n} \right)}{\text{maxi} \left( \frac{\text{EX}_n - \text{E} \left( x^2 \right)}{\text{EX}_n} \right)}
$$
Though the right hand side is not expressed in terms of probabilities. Assume for example that $X_i$ all statistician and prophet, based on the sequence of the exchangeable non-negative $X_i$. Thus, since all terms in each of the summands are equal, implies $EX_{d,h} = EX_{d,h}$. Let $\hat{\sigma}$ be the maximum value. For example, for $\tilde{X}$ and $\hat{\sigma}$, which, since all terms in each of the summands are equal, implies $\sum_{i=1}^{n} X_i$. Let $\hat{\sigma}$ be the maximum value of $X_i$. Assume for example that $X_i$ is exchangeable, then:

\[ \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i \]

Now by the univariate prophet inequality we have $EX_{d,h} = EX_{d,h}$. Let $\hat{\sigma}$ be the maximum value of $X_i$. Assume for example that $X_i$ is exchangeable, then:

\[ \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i \]

The authors are grateful to Dr. Uwe Saint-Mont for his thorough reading and valuable comments on an earlier version of the paper.

References


