INTERNATIONAL ECONOMIC REVIEW
Vol. 38, No. 4, November 1997

AN AXIOMATIZATION OF THE WALRAS CORRESPONDENCE IN INFINITE DIMENSIONAL SPACES*

BY MUKUL MAJUMDAR AND BEZALEL PELEG

Cornell University, U.S.A.
The Hebrew University of Jerusalem, Israel, and Cornell University, U.S.A.

This paper presents a generalization of earlier results on the axiomatization of the Walras correspondence to generalized (pure exchange) economies, where the commodity space is the positive cone in an ordered locally convex topological vector space. Our main result characterizes the Walras correspondence completely over an 'acceptable' class of economies in terms of consistency, converse consistency, and weak versions of Pareto optimality and nonemptiness. Important examples of economies that are 'acceptable' are given in detail.

1. INTRODUCTION

During the last twenty years consistency has become the unifying property in the axiomatizations of all major solutions in game theory (see Aumann 1989, pp. 43–44, for a comprehensive list of references up to 1987, the survey papers by Thomson 1990, and Driessen 1991, and the 1996 paper by Peleg and Tijs). Consistency has also been applied successfully to some practical problems like taxation and apportionment (Thomson 1990). The role of consistency is explained in Aumann (1989, pp. 43–44) in the following way: "Consistency implies that it is not too important how the player set is chosen. One can confine attention to a 'small world,' as the outcome for the denizens of this world will be the same as if we had looked at them in a 'big world.'"

The consistency properties of the core have been extensively investigated (see Peleg 1992, and Tadenuma 1992). Recently van den Nouweland et al. (1996) ([$N-P-T$]) characterized the Walras correspondence by its consistency properties. The reader is referred to [$N-P-T$] for a review of the various axiomatizations of the Walras correspondence in finite-dimensional economies. Our aim is to generalize the results of [$N-P-T$] to infinite-dimensional economies. In the next paragraph we motivate our goal.

It is now commonly accepted that private ownership economies with an infinite dimensional commodity space represent in some cases real-life economic problems (see Mas-Colell and Zame 1991, pp. 1836–1837). The problems of extending the fundamental results on existence and optimality of Walrasian equilibria to infinite-dimensional commodity spaces have given rise to a substantial literature in mathematical economics in the last twenty five years (for recent surveys of the literature

* Manuscript received February 1995.

see Aliprantis et al., 1989, and Mas-Colell and Zame 1991). It is well known that there are several mathematical difficulties that one must overcome in going from finite to infinite dimension. In our case, the main tool in $[N-P-T]$, the second fundamental theorem of welfare economics, is unavailable in infinite-dimensional economies. Therefore, we had to rely on special techniques (see the proofs of Lemmas 5.2 and 6.3). Thus, in our view, our paper is an interesting contribution to the overall effort of extending general equilibrium theory to infinite-dimensional commodity spaces.

$[N-P-T]$ contains an axiomatic characterization of the Walras correspondence for generalized economies with smooth preferences. We generalize their result to some classes of generalized (pure exchange) economies with an infinite dimensional commodity space. We now compare our results with those of $[N-P-T]$. The commodity space in our model is the cone L^+ of the nonnegative vectors in an ordered locally convex topological vector space L. In $[N-P-T]\, L = R^l$ and $L^+ = R^l_+$. We assume that the utility functions are continuous, quasi-concave, strictly increasing, smooth, and interior. $[N-P-T]$ assume only weak monotonicity of the utility functions. Also, the assumptions of interiority are different: $[N-P-T]$ assume that the indifference curves (of interior bundles) do not meet the boundary of R^l_+. This assumption can be generalized to infinite dimensional commodity spaces in the following way: The closure of any indifference curve which contains a strictly positive point does not contain any points that are not strictly positive. We chose an alternative condition: The upper-level sets of the utility functions cannot be *strictly* supported at points that are not strictly positive (see the precise definition in Section 3). In both models the initial endowments are strictly positive. The main difference between the two models is the following. In the $[N-P-T]$ model the second fundamental theorem of welfare economics holds, whereas in our framework it may not be true. Therefore, we have to assume a very weak form of that theorem, which we call admissibility. A class of economies that satisfies all our former conditions and additional two mild assumptions, is called acceptable. The $[N-P-T]$ theorem holds for acceptable classes of economies, that is, the Walras correspondence on an acceptable class of economies is completely characterized by consistency, converse consistency, and weak versions of Pareto-optimality and nonemptiness.

Our main results on existence of acceptable classes of economies are presented in Sections 5 and 6. In Section 5 we prove a general existence theorem for economies with countably many commodities. The family of separable utilities is investigated in Section 6. We find that, under the standard assumptions of finance theory, classes of economies with separable utilities are acceptable. Our results in Sections 5 and 6 rely heavily on Mas-Colell and Zame (1991), Araujo and Monteiro (1989), and Bewley (1972).

2. THE MODEL

The commodity space L is a Hausdorff locally convex topological vector space over R. We assume that $\geq$ is a partial order (i.e., it is reflexive and transitive) on L

with the following properties:

(2.1) $L^+ = \{x \in L | x \geq 0\}$ is a closed convex cone;

(2.2) $L^+ \cap (-L^+) = \{0\}$ and $L^+ \neq \{0\}$.

$x \in L$ is *positive* (denoted $x > 0$), if $x \geq 0$ and $x \neq 0$. Denote by L^* the dual space of L. $f \in L^*$ is *nonnegative* (denoted $f \geq 0$), if $f(x) \geq 0$ for all $x \geq 0$. $(L^*)^+ = \{f \in L^* | f \geq 0\}$. $f \in L^*$ *is positive* (denoted $f > 0$), if $f \geq 0$ and there exists $x \in L^+$ such that $f(x) > 0$. Assumptions (2.1) and (2.2) guarantee the existence of $f > 0$ in $(L^*)^+$.

A *generalized (pure exchange) economy* (GE) is a list $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$ where $N = N(E)$ is a nonempty finite set of *traders*; w^i is the *initial endowment* of $i \in N$; u^i: $L^+ \to R$ is the *utility function* of $i \in N$; and θ is the *net trade vector* of E (with the outside world), which satisfies $\Sigma_{i \in N} w^i + \theta \geq 0$. The vector θ is essential for the definition of consistency (see Section 3). We shall denote by $|N(E)|$ the number of members of $N(E)$. Throughout the paper we will assume that:

(2.3) u^i is continuous, quasi-concave, and strictly increasing (i.e., if $x, y \in L^+$ and $x - y > 0$, then $u^i(x) > u^i(y)$), for all $i \in N$.

An *allocation* for a GE E is a list $(x^i)_{i \in N(E)}$ such that $x^i \in L^+$ for all $i \in N$, and $\Sigma_{i \in N} x^i = \Sigma_{i \in N} w^i + \theta$. The set of all allocations for E is denoted by $A(E)$. $\underset{\sim}{x} \in A(E)$ is *Pareto optimal* (PO) if there is no $\underset{\sim}{y} \in A(E)$ such $u^i(y^i) > u^i(x^i)$ for all $i \in N$. We denote by PO(E) the set of all Pareto-optimal allocations of E.

In order to define the Walras equilibrium we first have to define prices. Clearly, if p: $L^+ \to R$ is a system of prices, then p must be linear, continuous, and positive, that is, p must be a positive member of $(L^*)^+$. However, in some cases $(L^*)^+$ is too big and contains linear functionals that cannot be interpreted as price systems in the usual way (see Bewley 1972, and Mas-Colell and Zame 1991). More precisely, there exist functionals that do not assign prices to individual commodities in a way that determines the value of commodity bundles containing infinitely many commodities. So the choice of a price cone depends on the structure of L^* (e.g., if $L = l_\infty$ then l_1^+ is the usual choice). Hence we define prices in the following way. Let π be a closed subspace of L^*, such that (i) $\pi^+ = \pi \cap (L^*)^+$ contains a positive vector, and (ii) if $x \in L$ and $f(x) = 0$ for all $f \in \pi$, then $x = 0$. A *price vector* is a positive member of π^+ (see Aliprantis et al., 1989). If $x \in L^+$ and $p \in \pi^+$, then we denote $p(x) = p \cdot x$.

Now we can define competitive equilibrium. Let E be a GE. $\underset{\sim}{x} \in A(E)$ is a *Walras* allocation if there exists $p \in \pi^+, p > 0$, such that for every $i \in N$ and $x \in L^+$: (i) $p \cdot x^i \leq p \cdot w^i$, and (ii) $[p \cdot x \leq p \cdot w^i] \Rightarrow [u^i(x^i) \geq u^i(x)]$. The pair $(\underset{\sim}{x}, p)$ is a competitive (i.e., Walras) equilibrium. The set of all Walras allocations of E is denoted by $W(E)$.

REMARK 2.1. Let E be a GE and let $(\underset{\sim}{x}, p)$ be a Walras equilibrium for E. Then $p \cdot x^i = p \cdot w^i$ for all $i \in N$.

The remark follows from (2.1), (2.2), and (2.3). The following corollaries of Remark 2.1 are straightforward.

COROLLARY 2.2. *If $(\underset{\sim}{x}, p)$ is a Walras equilibrium of a GE E, then $p \cdot \theta = 0$. Thus, competitive equilibria are budget-balanced.*

COROLLARY 2.3. *Every Walras allocation is Pareto-optimal, that is, the first fundamental theorem of welfare economics is true in our model.*

Let $x \in L^+$. x is *strictly* positive (denoted $x \gg 0$) if $p \cdot x > 0$ for every price vector p. Now we add the following condition:

(2.4) A GE$\langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$

has strictly positive endowments if $w^i \gg 0$ for every $i \in N$.

Denote by ξ the class of all GE's which satisfy (2.3) and (2.4). Let $E \in \xi$, $\underset{\sim}{x} \in A(E)$, and $S \subset N(E)$, $S \neq \emptyset$. The *reduced* economy with respect to (w.r.t.) $\underset{\sim}{x}$ and S is

$$E^{S,\underset{\sim}{x}} = \langle S; (w^i)_{i \in S}; (u^i)_{i \in S}; \theta + \sum_{i \in N \setminus S} (w^i - x^i) \rangle$$

If we denote $\hat{\theta} = \theta + \sum_{i \in N \setminus S}(w^i - x^i) = \sum_{i \in S}(x^i - w^i)$, then we obtain $\hat{\theta} + \sum_{i \in S} w^i = \sum_{i \in S} x^i \geq 0$. Hence $E^{S,\underset{\sim}{x}} \in \xi$. Thus, ξ is closed (as in Peleg and Tijs 1996). Let $\xi_0 \subset \xi$. A *solution* on ξ_0 is a function φ that assigns for every $E \in \xi_0$ a subset $\varphi(E)$ of $A(E)$. For example, $W(\cdot)$ is a solution on ξ.

3. SOME PROPERTIES OF THE WALRAS CORRESPONDENCE

In this section we prove that the Walras correspondence satisfies consistency and converse consistency under relatively mild conditions. Let $\xi_0 \subset \xi$ and let φ be a solution on ξ_0. ξ_0 is *φ-closed* if for every $E \in \xi_0$, $S \subset N(E)$, $S \neq \emptyset$, and $\underset{\sim}{x} \in \varphi(E)$, $E^{S,\underset{\sim}{x}}$ is in ξ_0. φ is *consistent* (CONS) on ξ_0 if: (i) ξ_0 is φ-closed, and (ii) for every $E \in \xi_0$, $S \subset N(E)$, $S \neq \emptyset$, and $\underset{\sim}{x} \in \varphi(E)$, $\underset{\sim}{x}^S \in \varphi(E^{S,\underset{\sim}{x}})$ (here $\underset{\sim}{x}^S$ is the restriction of $\underset{\sim}{x}$ to S).

LEMMA 3.1. *If $\xi_0 \subset \xi$ and it is $W(\cdot)$-closed, then $W(\cdot)$ is consistent on ξ_0.*

PROOF. Let $E \in \xi_0$, $S \subset N(E)$, $S \neq \emptyset$, and $\underset{\sim}{x} \in W(E)$. Then $E^{S,\underset{\sim}{x}} \in \xi_0$ because ξ_0 is $W(\cdot)$-closed. Also, there is a price vector p such that $(\underset{\sim}{x}, p)$ is a competitive equilibrium of E. As the reader may easily verify, $(\underset{\sim}{x}^S, p)$ is a Walras equilibrium for $E^{S,\underset{\sim}{x}}$. Thus $\underset{\sim}{x}^S \in W(E^{S,\underset{\sim}{x}})$. Q.E.D.

We now shall define converse consistency. Let $\xi_0 \subset \xi$ and let φ be a solution on ξ_0. φ is conversely consistent (COCONS) if for every $E \in \xi_0$ with at least two agents ($|N(E)| \geq 2$), and for every $\underset{\sim}{x} \in \text{PO}(E)$, the following condition is satisfied. If for every $S \subset N(E)$, $S \neq \emptyset$, $N(E)$, $E^{S,\underset{\sim}{x}} \in \xi_0$ and $\underset{\sim}{x}^S \in \varphi(E^{S,\underset{\sim}{x}})$, then $\underset{\sim}{x} \in \varphi(E)$. If the utility functions are smooth, then $W(\cdot)$ satisfies COCONS. Therefore we need the following notations and definitions.

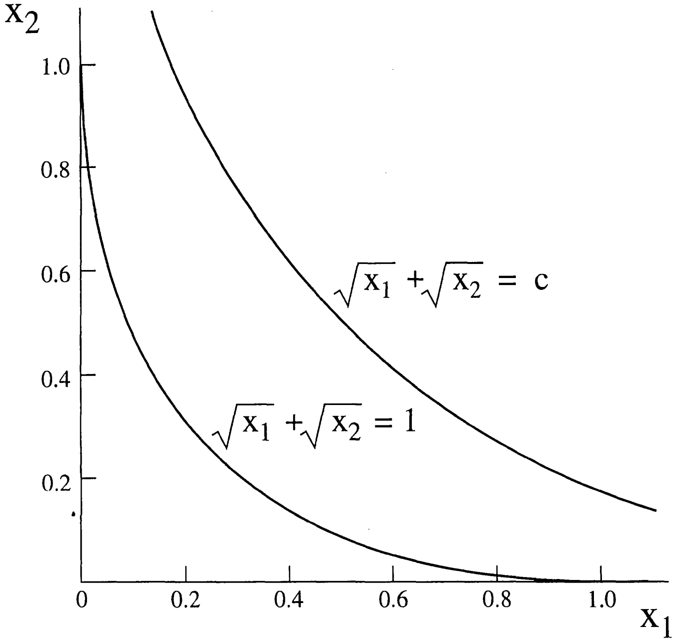

FIGURE 1

AN EXAMPLE OF AN INTERIOR FUNCTION IN R^2. POINTS ON THE AXES, EXCEPT 0, ARE INSEPARABLE FROM THEIR UPPER LEVEL SET. EXAMPLE: $(1,0)$ CANNOT BE STRICTLY SEPARATED FROM THE SET $\{x \in R^2_+ \mid \sqrt{x_1} + \sqrt{x_2} > 1\}$

Let $A, B \subset L^+$ such that $A \cap B = \varnothing$. A price vector p *strictly* separates A and B if $p \cdot a < p \cdot b$ for every $a \in A$ and $b \in B$. Now let $E \in \xi$, $i \in N(E)$, and $x \in L^+$. We denote $G^i(x) = \{y \in L^+ | u^i(y) > u^i(x)\}$. u^i is *interior* if for every $x \in L^+ \backslash \{0\}$ that is *not* strictly positive, the sets $\{x\}$ and $G^i(x)$ are *not* strictly separated (i.e., there is no price vector that strictly separates $\{x\}$ and $G^i(x)$). u^i is *smooth* if for every $x \gg 0$ there exists *at most one* price vector p that strictly separates $\{x\}$ and $G^i(x)$ and satisfies $p \cdot w = 1$ (here and in the sequel $w = \sum_{i \in N(E)} w^i$).

EXAMPLE 3.2. Let $L = l_\infty$ with the $\sigma(l_\infty, l_1)$-topology and the usual order and let $\pi = l_1$. For $x = \langle x(1), x(2), \dots \rangle \in l_\infty^+$ let $u(x) = \sum_{t=1}^{\infty} 2^{-t} \sqrt{x(t)}$. Then u satisfies (2.3) and is both smooth and interior.

Denote by ξ_s the class of all GE's in ξ with smooth and interior utility functions. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$ be a two-person (i.e., $|N(E)| = 2$) member of ξ_s. E is *admissible* if it satisfies the following condition. If $\underset{\sim}{x} \in PO(E)$ and for every $i \in N(E) x^i \in W(E^{\{i\}, \underset{\sim}{x}})$, then $\underset{\sim}{x} \in W(E)$. A class $\xi_0 \subset \xi_s$ is *admissible* if every two-person GE $E \in \xi_0$ is admissible. Now we can formulate the following result:

THEOREM 3.3. *If ξ_0 is an admissible class of GE's, then $W(\cdot)$ satisfies COCONS on ξ_0.*

Theorem 3.3 follows from Lemma 3.4.

LEMMA 3.4. *Let $E \in \xi_s$ satisfy $|N(E)| \geq 3$, let $\underset{\sim}{x} \in A(E)$, and let B be a set of (unordered) pairs of agents. If the graph $(N(E), B)$ is connected and for each $S \in B \underset{\sim}{x}^S \in W(E^{S, \underset{\sim}{x}})$, then $\underset{\sim}{x} \in W(E)$.*

PROOF OF LEMMA 3.4. Let $i \in N(E)$. Because $(N(E), B)$ is connected there is $S \in B$ such that $i \in S$. Because $\underset{\sim}{x}^S \in W(E^{S, \underset{\sim}{x}})$, there exists $p^S \in \pi^+$, $p^S > 0$, such that $(\underset{\sim}{x}^S, p^S)$ is a competitive equilibrium and $p^S \cdot w = 1$. Because u^i is interior we must have $x^i \gg 0$. ($x^i = 0$ is impossible because $w^i \gg 0$ and u^i is strictly increasing.) Using the smoothness of the utility functions we conclude that if $S, T \in B$, $S \cap T \neq \emptyset$, and $p^S \cdot w = p^T \cdot w = 1$, then $p^S = p^T$. Hence, by the connectedness of $(N(E), B)$, $p^S = p^T = p$ for all $S, T \in B$. Therefore, $(\underset{\sim}{x}, p)$ is a Walras equilibrium. Q.E.D.

PROOF OF THEOREM 3.3. Let $E \in \xi_0$ satisfy $|N(E)| \geq 2$ and let $\underset{\sim}{x} \in PO(E)$. Assume further that for each $S \subset N(E)$, $S \neq \emptyset$, $N(E)$, $E^{S, \underset{\sim}{x}} \in \xi_0$ and $\underset{\sim}{x}^S \in W(E^{S, \underset{\sim}{x}})$. If $|N(E)| = 2$ then E must be admissible. Hence, by definition $\underset{\sim}{x} \in W(E)$. If $|N(E)| \geq 3$ then $\underset{\sim}{x} \in W(E)$ by Lemma 3.4. Q.E.D.

4. AN AXIOMATIZATION OF THE WALRAS CORRESPONDENCE

A class $\xi_a \subset \xi$ is *acceptable* if: (i) ξ_a is admissible; (ii) $W(E) \neq \emptyset$ for every one-person economy E of ξ_a; and (iii) ξ_a is $W(\cdot)$-closed. We shall characterize the Walras solution on acceptable classes. First, we need two additional definitions. Let $\xi_0 \subset \xi$. A solution φ on ξ_0 satisfies PO(2) if for every two-person GE $E \in \xi_0$, $\varphi(E) \subset \text{PO}(E)$. φ satisfies NEM(1) (nonemptiness for one-person economies), if for every $E \in \xi_0$ with $|N(E)| = 1$, $\varphi(E) \neq \emptyset$.

Now we can formulate the following theorem.

THEOREM 4.1. *Let ξ_a be an acceptable class of economies. If a solution φ on ξ_a satisfies* NEM(1), PO(2), CONS, *and* COCONS, *then $\varphi(E) = W(E)$ for all $E \in \xi_a$.*

PROOF. Let φ be a solution on ξ_a that satisfies our four axioms and let $E \in \xi_a$. We shall prove by induction on $|N(E)|$ that $\varphi(E) = W(E)$. If $|N(E)| = 1$ then $|A(E)| = 1$. Therefore, by NEM(1), $\varphi(E) = W(E)$ ($W(\cdot)$ satisfies NEM(1) because ξ_a is acceptable). Now let $|N(E)| = 2$. If $\underset{\sim}{x} \in W(E)$ then $\underset{\sim}{x}$ is PO and $x^i \in W(E^{\{i\}, \underset{\sim}{x}}) = \varphi(E^{\{i\}, \underset{\sim}{x}})$ for every $i \in N(E)$ ($W(\cdot)$ is consistent on ξ_a because ξ_a is $W(\cdot)$-closed). By COCONS of φ, $\underset{\sim}{x} \in \varphi(E)$. Thus, $\varphi(E) \supset W(E)$. Similarly, if $\underset{\sim}{x} \in \varphi(E)$, then $\underset{\sim}{x}$ is PO (by PO(2) of φ), and $x^i \in \varphi(E^{\{i\}, \underset{\sim}{x}}) = W(E^{\{i\}, \underset{\sim}{x}})$ for all $i \in N(E)$. Therefore, $\underset{\sim}{x} \in W(E)$ by COCONS of $W(\cdot)$. Hence $W(E) \supset \varphi(E)$. Thus, $\varphi(E) = W(E)$ if $|N(E)| = 2$.

To complete the proof let $k \geq 2$ and assume that $\varphi(E) = W(E)$ if $E \in \xi_a$ and $|N(E)| \leq k$. Now let $E \in \xi_a$ satisfy $|N(E)| = k + 1$. If $\underset{\sim}{x} \in W(E)$, then $\underset{\sim}{x}$ is PO and $\underset{\sim}{x}^S \in W(E^{S, \underset{\sim}{x}}) = \varphi(E^{S, \underset{\sim}{x}})$ for all $S \subset N(E), S \neq \emptyset, N(E)$. Hence, by COCONS of φ, $\underset{\sim}{x} \in \varphi(E)$. If $\underset{\sim}{x} \in \varphi(E)$, then for all $S \subset N(E), S \neq \emptyset, N(E)$, $\underset{\sim}{x}^S \in \varphi(E^{S, \underset{\sim}{x}}) = W(E^{\{S\}, \underset{\sim}{x}})$. Hence, by Lemma 3.4, $\underset{\sim}{x} \in W(E)$. Q.E.D.

5. EXISTENCE OF ACCEPTABLE CLASSES OF ECONOMIES I: COUNTABLY MANY COMMODITIES

In this section we find some acceptable classes of economies with countably many commodities.

5.1. *Finite-dimensional Commodity Spaces.* Let l be the number of commodities. Using the notations of Sections 2 and 3 we choose $L = \pi = R^l$. Let u: $R^l_+ \to R$. u is *acceptable* if it is continuous, quasi-concave, strictly monotonic, and for each $j = 1, \dots, l(\partial u / \partial x_j)$ has the following two properties:

$$\frac{\partial u}{\partial x_j} \text{ is a continuous function from } R^l_+ \text{ to } [0, \infty]. \tag{5.1}$$

$$\text{If } \bar{x} \in R^l_+ \text{ then } \frac{\partial u(\bar{x})}{\partial x_j} = \infty \text{ iff } \bar{x}_j = 0. \tag{5.2}$$

For the sake of completeness we prove the following simple result.

LEMMA 5.1. *If u is acceptable then u is interior.*

PROOF. Let $\bar{x} \in R^l_+, \bar{x} \neq 0$, satisfy $\bar{x}_j = 0$. Assume, on the contrary, that there exists $p > 0$ that strictly separates $\{\bar{x}\}$ and $G(\bar{x}) = \{y \in R^l_+ \mid u(y) > u(\bar{x})\}$. Because u is strictly monotonic we must have $p \gg 0$. Because $\bar{x} \neq 0$ there is $1 \leq h \leq l$ such that $\bar{x}_h > 0$. Choose $0 < \varepsilon < \bar{x}_h$ and $\delta > 0$ such that $p_h \varepsilon = p_j \delta$. Because $p \cdot \bar{x} = p \cdot z$ where $z = \bar{x} - \varepsilon e^h + \delta e^j$ ($e^t \equiv$ the t-th unit vector of R^l), we must have $u(z) \leq u(\bar{x})$. Hence

$$0 \geq u(z) - u(\bar{x}) = (-\varepsilon)\frac{\partial u}{\partial x_h}(y) + \delta \frac{\partial u}{\partial x_j}(y)$$

for some $y \in [\bar{x}, z]$. Substituting in $\delta = (p_h / p_j)\varepsilon$ we obtain $p_j(\partial u / \partial x_h)(y) \geq p_h(\partial u(y) / \partial x_j)$. Because ε is arbitrary $p_j(\partial u(\bar{x}) / \partial x_h) \geq p_h(\partial u(\bar{x}) / \partial x_j)$, which contradicts our assumption $\partial u(\bar{x}) / \partial x_j = \infty$. Q.E.D.

Denote by U_a the class of all acceptable utility functions. Then each $u \in U_a$ (see (5.1)) is smooth and interior. We now define a class ξ_a of GE's by the following rule. $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$ is in ξ_a if: (i) $w^i \gg 0$ for all $i \in N$; (ii) $u^i \in U_a$ for every $i \in N$; and (iii) if $|N| = 1$ then $W(E) \neq \emptyset$. Then ξ_a is admissible by the choice of U_a. Also, ξ_a is $W(\cdot)$-closed. Hence, ξ_a is acceptable.

5.2. *Infinite-dimensional Commodity Spaces.* Let s be the vector space of all real sequences with the usual order, and let s^* be the space of all finite sequences. We assume that the commodity space L, its topology τ, and the space π of prices satisfy the following conditions.

(5.3) L and π are (linear) subspaces of s and both contain s^*.

(5.4) If $x \in L$ and $f \in \pi$ then $\sum_{t=1}^{\infty} |f(t)x(t)| < \infty$ and $f(x) = \sum_{t=1}^{\infty} f(t)x(t)$.

(5.5) $\sigma(L, \pi) \subset \tau$ and (L, τ) is a locally convex topological vector space ($\sigma(L, \pi)$ is the weak topology of the pair (L, π)).

(5.6) π is a closed subspace of $(L, \tau)^*$.

As the reader may easily verify

$$L^+ = \{x \in L | x = \langle x(1), x(2), \dots \rangle \text{ and } x(t) \geq 0 \text{ for all } t\}$$

satisfies (2.1) and (2.2). The foregoing conditions guarantee that price vectors are represented by (nonnegative) sequences in the class of GE's that we shall define.

We now define a class of utility functions on L^+. u: $L^+ \to R$ is *acceptable* if it is τ-continuous, quasi-concave, strictly increasing, and for each $t = 1, 2, \dots$, the following conditions are satisfied.

(5.7) $$\frac{\partial u}{\partial x(t)} \text{ is a } \tau\text{-continuous function from } L^+ \text{ to } [0, \infty].$$

(5.8) $$\text{If } \bar{x} \in L^+, \text{ then } \frac{\partial u(\bar{x})}{\partial x(t)} = \infty \text{ iff } \bar{x}(t) = 0.$$

Denote by U_a the class of all acceptable utility functions. If $u \in U_a$ then u is interior (see the Proof of Lemma 5.1). Also, every $u \in U_a$ is smooth.

Now we define the following class ξ_a of generalized economies. $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$ is in ξ_a if

(5.9) $$w^i \gg 0 \text{ for all } i \in N \text{ (i.e., } w^i(t) > 0 \text{ for } t = 1, 2, \dots,).$$

(5.10) $$u^i \in U_a \text{ for every } i \in N.$$

(5.11) $$\text{If } |N| = 1 \text{ then } W(E) \neq \varnothing.$$

We shall prove that ξ_a is acceptable. Indeed, as the reader may easily verify, ξ_a is $W(\cdot)$-closed. Thus, it is sufficient to prove that ξ_a is admissible. This is done in the following Lemma which is proved by using the partial equilibrium technique.

LEMMA 5.2. *Let* $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle \in \xi_a$, *and let* $\underset{\sim}{x} \in A(E)$ *satisfy the following two conditions*: (i) $\underset{\sim}{x} \in \mathrm{PO}(E)$, *and* (ii) *for all* $i \in N$ $x^i \in W(E^{\{i\}, \underset{\sim}{x}})$. *Then* $\underset{\sim}{x} \in W(E)$.

PROOF. For each $i \in N$ there exists $p^i \in \pi, p^i > 0$, such that (x^i, p^i) is a Walras equilibrium of $E^{\{i\}, \underset{\sim}{x}}$. Because u^i is interior, $x^i \gg 0$ for each $i \in N$. Also, the strict monotonicity of u^i implies that $p^i \gg 0$ for all $i \in N$. Let $j \in N$. We shall prove that for each $i \in N$ there exists $\alpha^i > 0$, such that $p^i = \alpha^i p^j$. Indeed, let $t > 1$ be a natural number. Consider the following two-commodity GE $\hat{E} = \langle N; (\hat{w}^i)_{i \in N}; (\hat{u}^i)_{i \in N}; \hat{\theta} \rangle$ where

(5.12) $\hat{w}^i = \langle w^i(1), w^i(t) \rangle$ for all $i \in N$;

(5.13) $\hat{u}^i$: $R^1_+ \times R^t_+ \to R$ is given by

$$\hat{u}^i(y^i(1), y^i(t)) = u^i(y^i(1), x^i(2), \dots, x^i(t-1), y^i(t), x^i(t+1), \dots);$$

(5.14) $\hat{\theta} = \langle \theta(1), \theta(t) \rangle$

Clearly, $\hat{\underset{\sim}{x}} = (\langle x^i(1), x^i(t) \rangle)_{i \in N}$ is in PO($\hat{E}$). Hence, by the second fundamental theorem of welfare economics there exist $\hat{p}(1) > 0$ and $\hat{p}(t) > 0$, such that $\langle \hat{p}(1), \hat{p}(t) \rangle$ supports the allocation $\hat{\underset{\sim}{x}}$ in $\hat{E}$. Therefore by the smoothness of $\hat{u}^i$, $i \in N$,

$$\frac{p^i(t)}{p^i(1)} = \frac{\hat{p}(t)}{\hat{p}(1)} = \frac{p^j(t)}{p^j(1)}$$

Thus, $p^i(t) = (p^i(1)/p^j(1))p^j(t)$ and our claim is proved. Now we conclude that (x^i, p^j) is a Walras equilibrium of $E^{\{i\}, \underset{\sim}{x}}$ for all $i \in N$. Therefore, $(\underset{\sim}{x}, p^j)$ is a Walras equilibrium for E. Q.E.D.

Now we present two examples of acceptable classes of GE's.

EXAMPLE 5.3. Let $L = l_\infty$, $\pi = l_1$, and τ the Mackey topology of (l_∞, l_1) (Bewley 1972). If $x \in l_\infty$ and there exists $r > 0$, such that $x(t) \geq r$ for $t = 1, 2,$ then we denote $x \gg 0$. Let ξ_a be the class of generalized economies $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$, such that $w^i \gg 0$ for all $i \in N$, and (5.10) and (5.11) are satisfied. As the reader may easily verify, ξ_a is acceptable. We also remark that if $E \in \xi_a$ is a closed GE (i.e., $\theta(E) = 0$), then $W(E) \neq \varnothing$ (Bewley 1972).

EXAMPLE 5.4. Let $L = l_p$, $1 \leq p < \infty$, $\pi = l_q$, where $1/p + 1/q = 1$, and τ the norm topology of l_p. Let ξ_a be the class of all GE's $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$ that satisfy (5.9)–(5.11). By the foregoing analysis ξ_a is acceptable.

6. EXISTENCE OF ACCEPTABLE CLASSES OF ECONOMIES II: SEPARABLE UTILITIES

Let (Ω, Σ, μ) be a probability space and $1 \leq p < \infty$. The commodity space in this section is $L = L_p(\Omega, \Sigma, \mu)$. The price cone is $L_q^+(\Omega, \Sigma, \mu) \setminus \{0\}$, where $1/p + 1/q = 1$. The class U_a of admissible utilities is determined by the following conditions. u: $L_p^+ \to R$ is in U_a if:

(6.1) u is norm-continuous. There is a function v: $R_+ \times \Omega \to R_+$ with the following properties.

(6.2) $v(\cdot, \sigma)$ is continuous, strictly increasing, and concave for every $\sigma \in \Omega$.

(6.3) $v(0, \sigma) = 0$ for all $\sigma \in \Omega$.

(6.4) $\frac{d}{dr} v(\cdot, \sigma)$ is a continuous function from R_+ into $[0, \infty]$ for every $\sigma \in \Omega$.

(6.5) $\frac{d}{dr} v(0, \sigma) = \infty$ for all $\sigma \in \Omega$.

(6.6) $v(r, \cdot)$ is measurable for every $r \in R_+$.

(6.7) $u(f) = \int_{\Omega} v(f(\sigma), \sigma)\, d\mu(\sigma)$ for all $f \in L_p^+$.

We now prove that each member of U_a is interior and smooth.

LEMMA 6.1. *If $u \in U_a$ then u is interior.*

PROOF. Let $f \in L_p^+$ satisfy $0 < \mu(\{\sigma | f(\sigma) > 0\}) < 1$. Assume, on the contrary, that there exists $g \in L_q$, $g > 0$, such that if $x \in L_p^+$ and $u(x) > u(f)$, then $g \cdot x > g \cdot f$. Because u is strictly increasing, $g \gg 0$. Let $A = \{\sigma | f(\sigma) > 0\}$ and $B = \Omega \backslash A$. Now

$$A = \bigcup_{n=1}^{\infty} \left\{ \sigma | f(\sigma) \geq \frac{1}{n} \right\}.$$

Hence, we can choose $K_1 > 0$ and $A_1 \subset A$, such that $f(\sigma) \geq 2K_1$ if $\sigma \in A_1$ and $\mu(A_1) > 0$. Repeating a similar argument twice, we can find $A_2 \subset A_1$, $K_2 > 0$ and $K_3 > 0$; with that $g(x) \geq K_2$ and $(d/dr)v(K_1, \sigma) \leq K_3$ for all $\sigma \in A_2$, and $\mu(A_2) > 0$. Now, if $0 < \varepsilon < K_1$, the following inequalities are true:

$$\int_{A_2} \varepsilon g(\sigma)\, d\mu(\sigma) \geq \varepsilon K_2\, \mu(A_2); \tag{6.8}$$

$$\int_{A_2} [v(f(\sigma), \sigma) - v(f(\sigma) - \varepsilon, \sigma)]\, d\mu(\sigma) \leq \varepsilon K_3\, \mu(A_2). \tag{6.9}$$

Similarly, we can choose $B_1 \subset B$, such that $\mu(B_1) > 0$ and $g(\sigma) \leq K_4$ for all $\sigma \in B_1$. Using the assumption that $(d/dr)v(0, \sigma) = \infty$ for all σ, we obtain that for every $K_5 > 0$.

$$B_1 = \bigcup_{l=1}^{\infty} \bigcap_{h=l}^{\infty} \left\{ \sigma \in B_1 | \frac{d}{dr} v\left(\frac{1}{h}, \sigma\right) \geq K_5 \right\}$$

Choose $K_5 > K_4 K_3 / K_2$ and l_0, such that

$$\bigcap_{h=l_0}^{\infty} \left\{ \sigma \in B_1 | \frac{d}{dr} v\left(\frac{1}{h}, \sigma\right) \geq K_5 \right\} = B_2$$

is of positive probability. Clearly, if $0 < \delta < 1/l_0$, then

$$\int_{B_2} v(\delta, \sigma)\, d\mu(\sigma) \geq \delta K_5\, \mu(B_2). \tag{6.10}$$

Now we choose $0 < \varepsilon < K_1$ and $0 < \delta < 1/l_0$, such that

$$(6.11) \qquad \varepsilon \int_{A_2} g(\sigma)\, d\mu(\sigma) = \delta \int_{B_2} g(\sigma)\, d\mu(\sigma)$$

(6.8) and $B_2 \subset B_1$ imply

$$(6.12) \qquad \varepsilon K_2 \mu(A_2) \le \varepsilon \int_{A_2} g(\sigma)\, d\mu(\sigma) = \delta \int_{B_2} g(\sigma)\, d\mu(\sigma) \le \delta K_4 \mu(B_2)$$

Furthermore, $u(f) \ge u(f - \varepsilon\chi_{A_2} + \delta\chi_{B_2})$ (here χ_C is the indicator of C for all $C \in \Sigma$). Thus, by (6.9) and (6.10)

$$(6.13) \qquad \begin{aligned} \delta K_5 \mu(B_2) &\le \int_{B_2} v(\delta, \sigma)\, d\mu(\sigma) \\ &\le \int_{A_2} (v(f(\sigma), \sigma) - v(f(\sigma) - \varepsilon, \sigma)]\, d\mu(\sigma) \\ &\le \varepsilon K_3 \mu(A_2) \end{aligned}$$

By (6.13) and (6.12)

$$(6.14) \qquad \frac{K_5}{K_3}\frac{\mu(B_2)}{\mu(A_2)} \le \frac{\varepsilon}{\delta} \le \frac{K_4}{K_2}\frac{\mu(B_2)}{\mu(A_2)}$$

Now, (6.14) contradicts the choice of K_5. Q.E.D.

LEMMA 6.2. *If $u \in U_a$, then u is smooth.*

PROOF. Let $u \in U_a$ and let $x \in L_p, x \gg 0$. If $G(x) = \{y \in L_p^+ | u(y) > u(x)\}$ and $g \in L_q^+$ strictly separates $\{x\}$ and $G(x)$, then g supports $\{y \in L_p^+ | u(y) \ge u(x)\}$ at x. Therefore, as found by Mas-Colell and Zame (1991, p. 1875), there exists $t > 0$ such that $g(\sigma) = t(d/dr)v(x(\sigma), \sigma)$ for almost all $\sigma \in \Omega$. Q.E.D.

Now we define a class ξ_a of GE's by the following rules. $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle$ is in ξ_a if:

$$(6.15) \qquad w^i \in L_p(\Omega, \Sigma, \mu) \text{ and } w^i \gg 0 \text{ for all } i \in N;$$

$$(6.16) \qquad u^i \in U_a \text{ for all } i \in N;$$

$$(6.17) \qquad \text{If } |N| = 1, \text{ then } W(E) \neq \emptyset.$$

Clearly, ξ_a is $W(\cdot)$-closed. The following lemma implies that ξ_a is acceptable.

LEMMA 6.3. *If $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; \theta \rangle \in \xi_a$, $\underset{\sim}{x} = (x^i)_{i \in N} \in PO(E)$, and $x^i \in W(E^{\{i\}, \underset{\sim}{x}})$ for every $i \in N$, then $\underset{\sim}{x} \in W(E)$.*

PROOF. For each $i \in N$ there exists $g^i \in L_q$, $g^i > 0$, such that (x^i, g^i) is a competitive equilibrium of $E^{\{i\}, \underset{\sim}{x}}$. Because u^i, $i \in N$, is interior, $x^i \gg 0$ for all $i \in N$. Now consider the GE $\hat{E} = \langle N; (x^i)_{i \in N}; (u^i)_{i \in N}; 0 \rangle$. Because g^i supports $\{y \in L_p^+ | u^i(y) \geq u^i(x)\}$ at x^i for every $i \in N$, $\hat{E}$ satisfies the assumptions of Theorem 11.1 reported in Mas-Colell and Zame (1991). Hence, $\hat{E}$ has a quasi-equilibrium $((y^i)_{i \in N}, g)$. Thus, for every $i \in N$

$$[u^i(z) \geq u^i(y^i)] \Rightarrow [g \cdot z \geq g \cdot x^i] \quad \text{for all } z \in L_p^+.$$

$x^i \gg 0$ implies that $g \cdot x^i > 0$ for all $i \in N$. Because the functions u^i, $i \in N$, are strictly monotonic, $g \gg 0$ and $((y^i)_{i \in N}, g)$ is a Walras equilibrium of $\hat{E}$. Now, from $u^i(y^i) \geq u^i(x^i)$, $i \in N$, and $\underset{\sim}{x} \in \mathrm{PO}(E)$, it follows that $u^i(x^i) = u^i(y^i)$ for all $i \in N$. Hence $(\underset{\sim}{x}, g)$ is a Walras equilibrium of $\hat{E}$. The assumption that u^i is smooth, $i \in N$, implies that for each $i \in N$ there exists $t^i > 0$, $t^i \in R$, such that $g = t^i g^i$. Finally, $g^i \cdot x^i = g^i \cdot w^i$ for each $i \in N$ implies that $(\underset{\sim}{x}, g)$ is a competitive equilibrium of E. Q.E.D.

REFERENCES

ALIPRANTIS, C.D., D.J. BROWN, AND O. BURKINSHAW, *Existence and Optimality of Competitive Equilibria* (Berlin: Springer-Verlag, 1989).

ARAUJO, A. AND P.K. MONTEIRO, "Equilibrium without Uniform Conditions," *Journal of Economic Theory* 48 (1989), 416–427.

AUMANN, R.J., "Game Theory," in J. Eatwell, M. Milgate, and P. Newman, eds., *The New Palgrave: Game Theory* (London: Macmillan, 1989).

BEWLEY, T., "Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory* 4 (1972), 514–540.

DRIESSEN, T.S.H., "A Survey of Consistency Properties in Cooperative Game Theory," *SIAM Review* 33 (1991), 43–59.

MAS-COLELL, A. AND W.R. ZAME, "Equilibrium Theory in Infinite Dimensional Spaces," in W. Hildenbrand and H. Sonnenschein, eds., *Handbook of Mathematical Economics*, Vol. IV (Amsterdam: North-Holland, 1991, pp. 1835–1898).

PELEG, B., "Axiomatizations of the Core," in R.J. Aumann and S. Hart, eds., *Handbook of Game Theory*, Vol. I (Amsterdam: North-Holland, 1992, pp. 398–412).

——— AND S. TIJS, "The Consistency Principle for Games in Strategic Form," *International Journal of Game Theory* 25 (1996), 13–34.

TADENUMA, K., "Reduced Games, Consistency, and the Core," *International Journal of Game Theory* 20 (1992), 325–334.

THOMSON, W., "The Consistency Principle," in T. Ichiishi, A. Neyman, and Y. Tauman, eds., *Game Theory and Applications* (San Diego: Academic Press, 1990, pp. 187–215).

VAN DEN NOUWELAND, A, B. PELEG AND S. TIJS, "Axiomatic Characterizations of the Walras Correspondence for Generalized Economies," *Journal of Mathematical Economics*, 25 (1996), 355–372.