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A difficulty with Nash's program: A proof of a special case

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Abstract

Let g be a cooperative game and let N be the set of players of g. According to Nash's Program N can find a noncooperative game G such that some Nash equilibrium of G may serve as a solution to g. We show that the implementation of Nash's Program might face some difficulties. In this paper we restrict ourselves to finite games. However, we proved in a previous unpublished paper that the same difficulties also appear when infinite games are allowed. © 1997 Elsevier Science S.A.

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1. Introduction

Nash's Program consists of the following claim: "...that analysis of any cooperative game G should be based on a formal bargaining model $B(G)$, involving bargaining moves and countermoves by the various players and resulting in an agreement about the outcome of the game. Formally, this bargaining model $B(G)$ would always be a noncooperative game in extensive form (or possibly in normal form), and the solution of the cooperative game G would be defined in terms of the equilibrium points of this noncooperative bargaining game $B(G)$" (see Harsanyi and Selten, 1988, p. 21).

It is well known that the set of noncooperative games that resolve a cooperative game in the foregoing sense, may contain more than one game (Harsanyi and Selten, 1988, p. 22). Therefore, Nash's Program involves, at least implicitly, a choice problem. By suggesting an explicit model for the formulation of Nash's Program we show that the choice problem might present some difficulties.

2. Preliminaries

Let $N=\{1,...,n\}$, $n \geq 2$ be a set of players and let V be a fixed alphabet. We assume that $V \supset R$

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where R is the set of real numbers. A (finite) extensive game is a sextuple (K, P, U, C, p, h) where K is a (finite) game tree, P is the player partition, U is the information partition, C is the choice partition, p is the probability assignment, and h is the payoff function (see Selten, 1975, Section 2). Let ξ be the set of all finite extensive games which are fully described by letters from V. An *extensive game form* is a list (K, P, U, C, p, ρ, A) where K is a game tree, P is the player partition, U is the information partition, C is the choice partition, p is the probability assignment, A is a finite set (of *outcomes*) and $\rho: \partial K \rightarrow A$ is a surjective function from ∂K, the set of endpoints of K, onto A. (Game forms will also be described by letters from V.) We recall that the probability of each chance move is positive (according to standard assumptions). Hence, if $f=(K, P, U, C, p, \rho, A)$ is an extensive game form, then every endpoint of K can be reached with positive probability (by a suitable choice of strategies). Therefore, every outcome can be obtained with positive probability.

Let $\sim$ be an equivalence relation on ξ (i.e., $\sim$ is a reflexive, symmetric, and transitive binary relation on ξ). If G_1 and G_2 are in ξ and $G_1 \sim G_2$, then we shall say that G_1 is equivalent to G_2. For $G_0 \in \xi$ we denote $\hat{G}_0 = \{G \in \xi | G \sim G_0\}$. Furthermore, $\hat{\xi} = \{\hat{G} | G \in \xi\}$.

Definition 2.1: An equivalence relation $\sim$ on ξ is *regular* if it satisfies the following condition: If $G \in \xi$ and G_0 is a proper subgame of G, then G is not equivalent to G_0. The notations of this section will be used in the sequel.

3. Obedience and choice

Let D be a nonempty family of cooperative games. We assume that the set of players of each $g \in D$ is N. (D may be, e.g., a family of coalitional games.). Let $\sim$ be a regular equivalence relation on ξ.

Definition 3.1: A selection rule for D is a correspondence $\Gamma: D \rightarrow \hat{\xi}$ that satisfies nonemptiness (i.e., $\Gamma(g) \neq \emptyset$ for each $g \in D$).

Let Γ be a selection rule for D. If $g \in D$ and $\Gamma(g)$ is finite, then we denote by $|\Gamma(g)|$ the number of members of $\Gamma(g)$. Also, we denote $|\Gamma(g)| = \infty$ if $\Gamma(g)$ is infinite.

Definition 3.2: If $g \in D$ and $|\Gamma(g)| < \infty$ then a *choice procedure* for $\Gamma(g)$ is an extensive game form $(K, P, U, C, p, \rho, \Gamma(g))$.

Intuitively, a choice procedure is a (generalized) voting rule that may be used by N in order to choose a member of $\Gamma(g)$. Let $g \in D$, $\Gamma(g) = \{\hat{G}_1, ..., \hat{G}_k\}$, $2 \le k \le \infty$, and $H_i \in \hat{G}_i, i = 1, ..., k$. If $f=(K, P, U, C, p, \rho, \Gamma(g))$ is a choice procedure for $\Gamma(g)$, then we denote by $G^* = G^*(f; H_1, ..., H_k)$ the following two-stage game:

(i) Play f in order to choose $\hat{G}_i \in \Gamma(g)$. The outcome of the choice is common knowledge.

(ii) If $\hat{G}_i \in \Gamma(g)$ is chosen, then play H_i. (The games $H_1, ..., H_k$ are also common knowledge.)

Clearly, $G^*(f; H_1, ..., H_k)$ is a well defined member of ξ. The following result is a consequence of the foregoing definitions.

Lemma 3.3: Let $g \in D$, $\Gamma(g) = \{\hat{G}_1, ..., \hat{G}_k\}$, $2 \le k < \infty$, and $H_i \in \hat{G}_i$, $i = 1, ..., k$. If f is a choice procedure for $\Gamma(g)$ and $G^* = G^*(f; H_1, ..., H_k)$, then $\hat{G}^* \notin \Gamma(g)$.

Proof: Let $f=(K, P, U, C, p, \rho, \Gamma(g))$. Then every member of $\Gamma(g)$ may be reached (in f) with positive probability. Let $\hat{G} \in \Gamma(g)$. Then $G \sim H_i$ for some $1 \le i \le k$. Hence, G is equivalent to a proper subgame of G^*. Thus, $G^* \notin \hat{G}$ because $\sim$ is regular.

The following definitions provide an intuitive reformulation of Lemma 3.3.

Definition 3.4: Let $g \in D$ and denote $\Gamma^*(g) = \cup\{\hat{G} | \hat{G} \in \Gamma(g)\}$. *N obeys Γ at g* if the interaction between the members of N, as far as the solution of g is concerned, is restricted to a play of a game $G \in \Gamma^*(g)$.

Definition 3.5: Let $g \in D$, $\Gamma(g) = \{\hat{G}_1,...,\hat{G}_k\}$ and $2 \leq k < \infty$. We say that *N chooses from Γ(g) at g* if there exists a choice procedure f for $\Gamma(g)$ and extensive games $H_1,...,H_k$, where $H_i \sim G_i$, $i = 1,...,k$, such that the members of N play the game $G^*(f; H_1,..., H_k)$ in order to solve g.

The following theorem is a corollary of Lemma 3.3.

Theorem 3.6: *Let $g \in D$ and $2 \leq |\Gamma(g)| < \infty$. If N obeys Γ at g, then N cannot choose from Γ at g.*

Theorem 3.6 has been generalized in Peleg (1996) in two directions: (i) The set $\Gamma(g)$ was allowed to be infinite. (ii) The class ξ of finite extensive games was replaced by a class of infinite extensive games.

4. A difficulty with Nash's program

We now recall that two extensive games $G = (K, P, U, C, p, h)$ and $G_* = (K_*, P_*, U_*, C_*, p_*, h_*)$ are isomorphic if there exists a permutation π of N and a bijection $\varphi: K \to K_*$ such that the pair (π, φ) respects the structure of both G and G_*. (For a precise definition see Definition 3.5 of Peleg et al., 1996.) If G is isomorphic to G_*, then we shall write $G \sim_i G_*$. Also, we denote by $\hat{\xi}_i$ the set of all equivalence classes with respect to $\sim_i$. Obviously, $\sim_i$ is a regular equivalence relation.

Our second step is to present an interpretation of Nash's program. Let g be a cooperative game. An extensive game $G = (K, P, U, C, p, h)$ *resolves* g if there exists a Nash equilibrium (*NE*) σ of G such that:

(i) The game G specifies a bargaining procedure that may lead to a solution of g.

(ii) $h(\sigma)$ is an agreeable payoff distribution for g.

Now denote $\Gamma_N(g) = \{\hat{G} \in \hat{\xi}_i | G \text{ resolves } g\}$

Our (naive) interpretation of Nash's program is equivalent to the following two conditions:

(4.1): $\Gamma_N(g) \neq \emptyset$ iff g can be resolved by finite extensive games. (If $\Gamma_N(g) = \emptyset$ then g is resolved by infinite noncooperative extensive games (see Peleg, 1996, Section 4)).

(4.2): If $\Gamma_N(g) \neq \emptyset$ then the interaction between the members of N, as far as the solution of g is concerned, is restricted to a play of an *NE* of some $G \in \Gamma_N^*(g)$, where $\Gamma_N^*(g) = \cup\{\hat{G} | \hat{G} \in \Gamma_N(g)\}$.

The precise formulation of Nash's program is given by:

Nash's hypothesis. Let g be a cooperative game. Then there exists a unique subset $\Gamma_N(g) \subset \xi_i$ with the following properties:

(i) If $\Gamma_N(g) \neq \emptyset$ then the interaction between the members of N, as far as the solution of g is concerned, is restricted to a play of an *NE* of some $G \in \Gamma_N^*(g)$.

(ii) If $\Gamma_N(g) = \emptyset$ then there exists unique set $\tilde{\Gamma}_N(g)$ of (suitably defined) infinite extensive games such that the interaction between the members of N is restricted to a play of an *NE* of some $G \in \tilde{\Gamma}_N(g)$.

Remark 4.1: Assume that there exists a cooperative game g such that $2 \leq |\Gamma_N(g)| < \infty$. By Definition 3.4 N obeys Γ_N at g. Hence, by Theorem 3.6, N cannot choose from Γ_N at g. Thus, the question who chooses the (noncooperative extensive) game from $\Gamma_N(g)$ naturally arises. Unfortunately, as far as we can see, there is no simple answer to this question. If, for example, Nature makes the choice, then the lottery used by her must be unknown to the players. [A lottery that is known to the players consists of a choice procedure for $\Gamma_N(g)$ (see Definition 3.2).]

Remark 4.2: Peleg (1996) contains applications to economics and the game theory of Remark 4.1. In particular, it offers a solution for the foregoing difficulty.

Remark 4.3: Actually, in Remark 4.1 we may further assume that the players of g have conflicting preferences over $\Gamma_N(g)$. This is possible because there are at least two players. (The preferences of the players may be, for example, based on some selection theory that associates an equilibrium payoff vector with each extensive game.) The foregoing observation justifies our use of Definition 3.2, that is, we must use (in some cases) a choice procedure in order to choose a member of $\Gamma_N(g)$.

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