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Individual rights and collective responsibility: the rights–egalitarian solution

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Abstract

The problem of distributing a given amount of a divisible good among a set of agents which may have individual entitlements is considered here. A solution to this problem, called the Rights–Egalitarian Solution is proposed. This allocation rule divides equally among the agents the difference between the aggregate entitlements and the amount of the good available. A relevant feature of the analysis developed is that no sign restriction is established on the parameters of the model (that is, the aggregate entitlement may exceed or fall short of the amount of the good, agents' rights may be positive or negative, the allocation may involve a redistribution of agents' holdings, etc.). This paper provides several characterizations of this rule and analyzes its game theoretical support. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Real life allocation problems often admit the same mathematical description and yet their resolution is different. Cooperative game theory exhibits a similar feature: it provides several solutions that can be applied to the same game. Which solution is *better* usually depends not only on the mathematical description of the game but also on the type of problem at hand. This conveys the idea that one should be very careful when deciding what solution concept should be applied. This paper refers to the distribution of an estate among a group of agents, when these have claims but also are held collectively

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responsible for the discrepancies between rights and worth. In order to motivate the discussion, let start by presenting a simple numerical example of a two-person problem.

Consider a problem of distributing an estate of \$100 between two persons 1 and 2 who claim \$30 and \$120, respectively. Obviously the claims cannot be met, so the question is how to share the loss in some kind of fair way. There are three different rules that immediately come to mind (even though one can think of more sophisticated rules):

1. *The proportional rule*: The share is in proportion to the claims. This yields an allocation vector of (20, 80).
2. *The Talmud rule* (see Aumann and Maschler, 1985): Agent 1 admits that he has no right to \$70 out of the total whereas agent 2 claims the whole estate. They therefore argue about the remainder, \$30, which it is only fair to share equally. This yields the allocation vector (15, 85).
3. *The divorce rule*:¹ This rule gives each agent her or his claim and asks both to share the deficit equally. This yields yet another allocation vector of (5, 95).

Such a simple mathematical problem, yet common sense already suggests three different allocation vectors! Which one should be recommended? Or, more to the point, under what circumstances should we consider any of them?

Perhaps one can draw conclusions from observing the ways people apply these rules in real life. The proportional rule is the one most often used. Think for example, about a company owned by shareholders. The Talmud rule is also used for two-person situations, for instance in problems dealing with bankruptcy.² The divorce rule, which is the topic of our study, is also used: many divorce cases are settled by wife and husband taking home everything each brought to the marriage, and the rest—surplus or debts—being shared equally.

There are essential differences in the above applications. In the first *the agents own the estate and claims represent the parts in the estate that each person owns—not necessarily an absolute amount of money that any agent claims*. If the estate is larger than the total amount of claims, they all benefit from it. And (usually) if the estate suffers losses they are not responsible for the losses. The proportional solution, obviously, considers only the ratio of the claims, since indeed the ratios are all that matters.

The second case is different: *the claimants claim only the money. They will not enjoy more profits if the estate is larger than the total amount of claims and are not responsible for the debts if the estate is too small*. Hence, in the Talmud solution, the attitude is *to ignore everything that is not available*. One asks \$120?—too bad. Only

¹The name is derived from the application to be discussed subsequently.

²This rule can be given several interpretations other than the one above, but the only one for three or more agents that is consistent with the two-person case which is the Talmud rule (see Aumann and Maschler, 1985), is never used—not even in Rabbinical courts.

\$100 is available. Let us therefore truncate her claim to whatever is left. Together they ask too much? Nobody cares. They will only get part of their claim and that's it.

In the third case *the agents own the estate. Moreover, they are all responsible for losses and enjoy the profits.* The claims represent real entities that must be satisfied before any further step is taken. Consequently, the attitude in the divorce solution is: *The claims are absolute—each is going to get her or his claim in full. The responsibilities are also absolute: they both should share the loss, if there is one.*

From the above it follows that what is a sensible solution depends not only on the mathematical structure embodied in the numbers corresponding to the estate and the claims, but also on the nature of the problem (namely, the nature of the 'property rights' associated with estate and claims). In particular,

- If the parties understand that their claims are good only to the extent that the resources are available, then the proportional and the Talmud rule (and any other which is customary in a case of bankruptcy) should be considered.
- If claims are absolute, e.g., they can be enforced in court and, on the other hand, responsibilities are also absolute, the divorce solution should be considered.

Bearing in mind these considerations let us continue with the numerical example by slightly modifying the numbers and see what these rules suggest. Consider again a problem of distributing an estate between two persons 1 and 2 with claims \$30 and \$120 respectively, but suppose that now the worth of the estate is \$0. Both the proportional and the Talmud rule solve the problem by giving \$0 to each agent, whereas the divorce solution stipulates that agent 1 should pay \$45 to agent 2.

Which solution is admissible depends again on the nature of the problem and the assignment of property rights. Suppose, for instance, that the allocation problem corresponds to the case of two partners that jointly invest \$150 in a plantation and assume that the crop is totally lost due to a natural disaster. Had the partners created an Ltd. firm, the assignment of property rights is clear: each one risks what she invests. Hence (0,0) is the only sensible solution to this problem. If however, they had created a society with unlimited liability, the divorce rule becomes a meaningful solution because it amounts to a fair share of the losses.

This illustrates that in some situations it is not sensible to restrict the solutions to be nonnegative (i.e, to preserve the *status quo*).

Consider still a different variant of the original example. The problem is that of distributing an estate of \$100 between two people with claims of $-\$30$ and \$120. The novelty here is that one of the agents has a *negative entitlement* (think of the case of a divorce where $-\$30$ means that agent 1 enters the marriage bringing a debt). Note that there is a surplus of \$10 given by the difference between the estate and the aggregate claims. The proportional rule proposes an allocation of $(-100/3, 400/3)$ whereas the divorce rule yields an allocation of $(-25, 125)$. The proportional rule is hardly acceptable in this context: it requires a contribution from agent 1 greater than her debt, in spite of the existing surplus! Indeed the example can be modified to even greater absurdity, that the agent with debts gets paid and the agent with claims has still to contribute [the reader is invited to try with a vector of claims $(-130, 120)$]. The divorce

rule gives each agent one half of the net worth of the estate, so that both agents get a benefit from the solution.³

The main purpose of this paper is to extend the divorce rule to more general allocation problems; namely, problems that share the chief characteristics of divorce settlements even though they may involve a number $n \geq 2$ of agents. Therefore, the ensuing discussion refers to a family of distribution problems whose key features are:

(i) *With respect to the budget and the rights.* We assume that *the budget is absolute*, in the sense that it must be fully distributed and *the rights are absolute too*, in the sense that they must be satisfied and the group is fully responsible for that.

(ii) *With respect to the allocation rule.* The allocation rule admits negative values, in accordance with the absolute character of the rights (i.e., the operation of the rule may involve a *redistribution* of agents' holdings). The interest of allowing for negative values in the solution function derives from the fact that some allocation problems are inherently redistribution problems (e.g., divorce settlements, the bankruptcy of a society with unlimited liability).

(iii) *With respect to the domain of problems.* The analysis will be applied to a wider than usual domain of problems. In the general case no restriction will be imposed on the values that the parameters of the model take. In particular, the estate can be either positive or negative, the claims may well have negative components, and the estate may exceed or fall short of the aggregate claims. A negative budget simply means a cost to be shared. A negative claim corresponds to a debt. An estate greater (resp. smaller) than the aggregate claims represents a problem of distributing a surplus (resp. sharing a deficit).

Related problems have been analyzed in the literature from different viewpoints. O'Neill (1982) started the literature on the model of rights arbitration. As for the cost/surplus-sharing approach, see for instance Moulin (1988, Ch. 4–6). In dealing with the analysis of bankruptcy-like situations, see Maschler (1982), Aumann and Maschler (1985), Curiel et al. (1988), Dagan and Volij (1993) or Dagan (1995). The novelty of the approach here resides in the three features explained above.

The work is organized as follows. Section 2 generalizes the divorce rule to n -person problems. The solution concept that arises from this extension will be called the *rights-egalitarian* rule. The characterization of this rule is taken up in Section 3, by means of conventional axioms. We provide several characterizations because each additional one extends the scope of cases to which our solution can be applied. A smaller family of problems is considered in Section 4: that in which both the budget and the entitlements are positive and the budget falls short of the claims. This case corresponds to a variant of the standard bankruptcy problem, in which redistribution is permitted (think of the allotment of fishing quotas or the distribution of the Government's budget among the different ministries). The interest of treating this restricted family of problems is twofold. On the one hand, to check whether the properties that characterize the rights-egalitarian solution preserve their bite in this restricted setting.

³The Talmud rule was not defined for this case, but the nucleolus of the game, $[v(\{1\})=100, v(\{2\})=-30, v(\{1,2\})=100]$, yields also an interesting solution; namely $(-15, 115)$, where 120 is truncated and the 'loss' is shared.

This is important because, in many cases, the bigger the domain of problems the smaller the set of allocation rules that satisfy certain properties. On the other, to provide an environment that enables the comparison of the performance of this rule with respect to the proportional solution (which, as we saw, does not make sense if negative and positive claims exist). Finally, Section 5 refers to the game-theoretic support of the rule.

2. The rights–egalitarian rule

The purpose of this section is to extend the divorce rule, discussed above for 2-person situations, to the more general case of n agents. One can achieve this in one of two ways: One way is to apply the concept of consistency and find out what solution is generated. Another way consists of directly applying the key ideas of this solution to the n -person case. We shall use both procedures, after presenting the general domain of problems.

Consider a problem involving the distribution of a given amount of *money* among a number of *agents*, each of which is characterized by a monetary *entitlement*. The money being distributed will be called the *budget*. It represents the worth jointly owned by the agents. Its origin can stem from one of many circumstances (e.g., administrative decisions, an enterprise to be liquidated, inheritance, etc.). The vector of entitlements represents the agents' individual rights (e.g., needs, claims, benefits or private loans, shares, inheritance wills and others). By agents we mean people or more general instances, such as expenditure categories, departments or institutions. Money will refer to actual money or to any unit of account of rights and worth pertinent to the problem under consideration (e.g., square miles, calories, gallons, etc.). An allocation problem will thus be described by a triple $[N, E, \mathbf{c}]$, where N represents the set of agents, $E \in \mathbb{R}$ the budget, and $\mathbf{c} \in \mathbb{R}^N$, the vector of entitlements.

Consider then a set of potential agents, \mathcal{N} , and let N be any finite subset of \mathcal{N} .

Definition 1. An allocation problem is a triple $[N, E, \mathbf{c}]$, such that N is a set of agents, $N \subseteq \mathcal{N}$, N finite, $|N| \geq 2$, $\mathbf{c} \in \mathbb{R}^N$ describes a vector of entitlements and $E \in \mathbb{R}$, a given budget.

Let us call Ω the family of all allocation problems. For any $\omega = [N, E, \mathbf{c}] \in \Omega$, call $C(\omega) = \sum_{i \in N} c_i$, and let $H(\omega)$ stand for the hyperplane

$$H(\omega) = \left\{ z \in \mathbb{R}^N \mid \sum_{i \in N} z_i = E \right\}.$$

Definition 2. An **allocation rule** is a function $F: \Omega \rightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$, such that for any $\omega = [N, E, \mathbf{c}] \in \Omega$, $F(\omega) \in H(\omega)$.

Thus, an allocation rule is a mechanism such that: (a) provides us with a unique solution for any problem in Ω ; and (b) exhausts the budget.

One of the distinguished features that we want to capture, as indicated in Section 1, is

that all the agents are, above all, responsible that claims should be paid in full. This is translated into the following axiom:⁴

Axiom 1 (RESPONSIBILITY). For any $\omega = [N, E, \mathbf{c}]$ in Ω , and each $i \in N$, $F([N, E, \mathbf{c}]) = (0, \dots, 0, c_1, 0, \dots, 0) + F([N, E - c_i, (c_{-i}, 0)])$.

Axiom 1 says that every agent achieves the same outcome either if we apply the allocation rule directly or if first she is given her claim and then proceed to distribute the rest among the agents with agent i having no more claims.

Another relevant value judgment that was implicit in the discussion of Section 1 was anonymity. The next axiom is a weak form of anonymity:

Axiom 2 (SYMMETRY). For any $\omega = [N, E, \mathbf{c}] \in \Omega$, if $c_i = c_j$ for all $i, j \in N$, then $F_i(\omega) = F_j(\omega)$, for all $i, j \in N$.

Symmetry is a very mild condition which says that if all agents have identical entitlements, then the rule should divide the budget equally among them.

It is easy to see that *there is one and only one rule F on Ω that satisfies responsibility and symmetry*. This rule may be regarded as conceding agents all their claims, and then dividing equally whatever is left—be it positive or negative. We call this rule the *rights–egalitarian rule* to express the fact that rights are honored (axiom 1) and further, the remainder is divided equally (axiom 2). To put it formally:

Definition 3. The **rights–egalitarian allocation rule** F^{RE} , is defined by

$$F_i^{\text{RE}}([N, E, \mathbf{c}]) := c_i + \frac{1}{n} \left(E - \sum_{i \in N} c_i \right), \text{ for any } [N, E, \mathbf{c}] \in \Omega.$$

This rule divides equally the net worth $E - C(\omega)$ among the n agents. When $E > C(\omega)$, the resulting allocation coincides with the equal-gains solution from the rights point \mathbf{c} . If $E < C(\omega)$, then our solution corresponds to the equal-loss (or claims–egalitarian) solution from the claims point \mathbf{c} . For 2-person problems the ‘rights egalitarian’ rule is obviously the divorce rule. Therefore, both concepts will be identified from now on.

Remark. The rights–egalitarian rule can be viewed as a combination of the equal-award/equal-loss principles. Both principles are common in the literature dealing with the division of a surplus (Moulin, 1987), the bankruptcy problem (Young, 1987; Dagan, 1995), the axiomatic bargaining theory (Kalai, 1977; Chun, 1988a; Chun and Peters, 1991; Herrero and Marco, 1993), and the bargaining with claims problem (Bossert, 1993; Herrero, 1994). Nevertheless our solution function differs from the standard treatment of these problems because it applies over a larger domain of problems (and

⁴To the best of our knowledge, this is a new property. We call c_{-i} the $(n-1)$ vector resulting from \mathbf{c} by deleting its i th coordinate.

also, as we shall see later, because it uses a different reference outcome as the proper upper or lower bound for admissible solutions).

The next property refers to the effect of a change in the number of agents. The following notation will be used: If $\omega = [N, E, \mathbf{c}] \in \Omega$, and $S \subset N$, $S \neq N$, we shall call $\mathbf{c}_s = (c_i)_{i \in s}$. If F is an allocation rule, call $\omega_s = [S, \sum_{i \in s} F_i(\omega), \mathbf{c}_s]$ the *reduced problem relative to F* for the subset of agents S .

Axiom 3 (CONSISTENCY). *Let F be an allocation rule. F is consistent if for any $\omega = [N, E, \mathbf{c}] \in \Omega$, any nonempty subset of agents $S \subset N$, and any $i \in S$, $F_i(\omega_s) = F_i(\omega)$.*

Consistency has to do with the possibility of renegotiation among a group of agents, whenever they face the total amount assigned to them by the solution. When F is consistent, if some agents leave, bringing with them their allotted shares, they cannot change their outcomes by using again the rule over the reduced problem. Consistency is considered an important stability feature of solution concepts (cf. Young, 1987; Thomson and Lensberg, 1989). Trivially, F^{RE} satisfies consistency.

A weaker requirement is that of asking for this property to hold only for two-person problems. Formally:

Axiom 4 (BI-CONSISTENCY). *For any $\omega = [N, E, \mathbf{c}] \in \Omega$, and for any pair $\{i, j\}$ of distinct individuals in N , $F_k(\omega_{\{i,j\}}) = F_k(\omega)$, $k = i, j$.*

The following result is obtained:

Proposition 1. *F^{RE} is the only bi-consistent rule on Ω that coincides with the divorce rule.*

Proof. Obviously, F^{RE} is bi-consistent and even consistent, and coincides with the two person rights–egalitarian (or divorce) rule.

Consider now a problem $\omega = [N, E, \mathbf{c}] \in \Omega$, with $|N| > 2$, and let F be a bi-consistent allocation rule which coincides with the divorce rule. Suppose that $F(\omega) = x$. Now, for two agents $\{i, j\} \subset N$, consider the problem $\omega_{ij} = [\{i, j\}, x_i + x_j; (c_i, c_j)]$. By bi-consistency, $F_i(\omega_{ij}) = x_i$, $F_j(\omega_{ij}) = x_j$. Moreover, since F coincides with F^{RE} for two person problems, there exists some $\lambda \in \mathbb{R}$, such that $x_i = c_i + \lambda$, $x_j = c_j + \lambda$. Take now a third agent, k , and the problem $\omega_{jk} = [\{j, k\}, x_j + x_k; (c_j, c_k)]$. Again, by bi-consistency, for some $\mu \in \mathbb{R}$, $x_j = c_j + \mu$, $x_k = c_k + \mu$. Consequently, $\lambda = \mu$. As previous construction can be done for any $i, j, k \in N$, it follows that $F(\omega) = F^{\text{RE}}(\omega)$. ■

3. Characterizations

Several properties will now be considered. They will serve the purpose of characterizing the rights–egalitarian rule and thus show its main properties (something that is important when we come to consider the application of this rule to different problems). The first of these properties is the following:

Axiom 5 (COMPATIBILITY). For any $\omega = [N, E, \mathbf{c}] \in \Omega$, $C(\omega) = E$ implies $F(\omega) = \mathbf{c}$.

Compatibility is an obvious restriction on the allocation rule. It establishes that if the claims are feasible the rule should give each agent her claim. We regard this as a fundamental property of social justice.

The next property is related to the possibility of solving these problems sequentially. To motivate its relevance think of the following situation. A company is being dissolved, and its worth has to be distributed among its creditors. The company's worth corresponds to the market value of a number of items, such as real estate, machinery, financial assets, etc. Each of these items can be sold independently of the others and hence at different points in time. One would like the final distribution of the revenues to be independent of the order in which these items are sold.

More formally: Let $\omega = [N, E, \mathbf{c}] \in \Omega$ and let E_1, E_2 be such that $E_1 + E_2 = E$. We consider now the possibility of solving the distribution problem ω in two steps. First we solve the problem $\omega_1 = [N, E_1, \mathbf{c}]$ in which we take E_1 instead of E . Let $\mathbf{x}_1 = F(\omega_1)$. Then we consider the remaining problem given by $\omega_2 = [N, E_2, \mathbf{c} - \mathbf{x}_1]$, whose solution is \mathbf{x}_2 . We require the outcome to be independent of such a sequential process, that is, $F(\omega) = F(\omega_1) + F(\omega_2)$. This is a property which prevents the manipulation of the outcome by framing conveniently the sequential process. Following Young (1988), we shall call this property *composition*. Formally:

Axiom 6 (COMPOSITION). For any $\omega = [N, E, \mathbf{c}] \in \Omega$, and any $E_1, E_2 \in \mathbb{R}$ such that $E_1 + E_2 = E$, it follows that $F(\omega) = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 = F([N, E_1, \mathbf{c}])$ and $\mathbf{x}_2 = F([N, E_2, \mathbf{c} - \mathbf{x}_1])$.

The following characterization of F^{RE} is obtained:

Proposition 2. F^{RE} is the unique allocation rule in Ω satisfying compatibility, symmetry and composition.

Proof. Obviously, F^{RE} satisfies symmetry and compatibility. To see that it also satisfies composition, take an arbitrary problem $\omega = [N, E, \mathbf{c}] \in \Omega$, and let $\omega_1 = [N, E_1, \mathbf{c}]$, $\omega_2 = [N, E_2, \mathbf{c} - F^{\text{RE}}(\omega_1)]$, with $E = E_1 + E_2$. By definition, $F_i^{\text{RE}}(\omega) = c_i + [E - C(\omega)]/n$. Now observe that $F_i^{\text{RE}}(\omega_1) = c_i + [E_1 - C(\omega)]/n = x_{1i}$, and $F_i(\omega_2) = c_i - x_{1i} + [E_2 - C(\omega) + E_1]/n = E_2/n$. Hence, $F^{\text{RE}}(\omega) = F^{\text{RE}}(\omega_1) + F^{\text{RE}}(\omega_2)$.

Let us consider now an allocation rule F fulfilling all the requirements, and let $\omega = [N, E, \mathbf{c}] \in \Omega$. If we take $\omega_1 = [N, C(\omega), \mathbf{c}]$, it follows by compatibility that $F(\omega_1) = \mathbf{c}$. Let now $\omega_2 = [N, E - C(\omega), \mathbf{0}]$. As ω_2 is a symmetric problem, symmetry implies that $F(\omega_2) = [E - C(\omega)]/n$. Now, by composition, $F_i(\omega) = c_i + [E - C(\omega)]/n = F(\omega)$. ■

Suppose now that we face two allocation problems with the same agents, the same budget and vectors of claims given by \mathbf{c}, \mathbf{c}' . Consider then a third problem with the same agents, the same budget and a vector of claims that is a convex combination of \mathbf{c} and \mathbf{c}' (i.e. $\mathbf{c}'' = \lambda\mathbf{c} + (1 - \lambda)\mathbf{c}'$ for some $\lambda \in [0, 1]$). We would like the solution to this third

problem be given by the convex combination of the solutions (that is, λ times the solution to the first problem plus $(1 - \lambda)$ times the solution of the second one). Formally:

Axiom 7 (CLAIMS LINEARITY). Let $\omega = [N, E, \mathbf{c}]$, $\omega' = [N, E, \mathbf{c}'] \in \Omega$, and $\lambda \in [0, 1]$. Then, $F([N, E, \lambda \mathbf{c} + (1 - \lambda)\mathbf{c}']) = \lambda F(\omega) + (1 - \lambda)F(\omega')$.

Claims linearity may also be seen as dealing with the case in which the agents are uncertain about the rights point (it may be either \mathbf{c} or \mathbf{c}'). This happens when the claims correspond to the value of assets with uncertain market value (e.g., the case of a partnership in which agents contributions consisted of real assets). The rule makes it appealing (for risk-neutral agents) to sign a contingent contract rather than wait until all uncertainties are resolved. Related properties appear in Chun and Thomson (1992) and Herrero (1998) in the bargaining with claims case.

This brings us to another characterization of F^{RE} :

Proposition 3. F^{RE} is the unique allocation rule in Ω satisfying compatibility, symmetry and claims linearity.

Proof. Obviously, F^{RE} satisfies these properties. To prove the converse part, notice that compatibility implies that if $\omega = [N, E, \mathbf{c}] \in \Omega$ is such that $C(\omega) = E$, then $F(\omega) = \mathbf{c} = F^{RE}(\omega)$. We shall therefore consider the case $C(\omega) \neq E$.

For any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, denote by $L[\mathbf{x}, \mathbf{y}]$ the straight line containing \mathbf{x} and \mathbf{y} . Let $\omega = [N, E, \mathbf{c}]$ and take a point $\mathbf{d} \in \mathbb{R}^n$ such that $d_i = d_j$ for all $i, j \in N$, and d_i is strictly in between E/n and $C(\omega)/n$. Let now $\mathbf{b} \in \mathbb{R}^n$ be the point given by $\mathbf{b} = L[\mathbf{c}, \mathbf{d}] \cap H(\omega)$. By construction, there exists $\lambda \in (0, 1)$ such that $\mathbf{d} = \lambda \mathbf{b} + (1 - \lambda)\mathbf{c}$. Symmetry implies that $F_i([N, E, \mathbf{d}]) = E/n$, for any $i \in N$. Now consider the problem $\omega' = [N, E, \mathbf{b}]$. Since $\mathbf{b} \in H(\omega')$, compatibility implies that $F(\omega') = \mathbf{b}$. Now, by letting $\mathbf{1} = (1, 1, \dots, 1)$ and applying claims linearity it follows that $\mathbf{1} \cdot E/n = \lambda \mathbf{1} \cdot \mathbf{b} + (1 - \lambda)F(\omega)$, so that $L[\mathbf{c}, F(\omega)]$ and $L[\mathbf{d}, \mathbf{0}]$ are parallel lines. Therefore, $F(\omega) = F^{RE}(\omega)$. ■

Remark. Note that the same result is obtained if we use the weaker property of claims concavity [i.e. $F([N, E, \lambda \mathbf{c} + (1 - \lambda)\mathbf{c}']) \geq \lambda F(\omega) + (1 - \lambda)F(\omega')$] rather than claims linearity.

In order to introduce the next axiom define, for each $\omega = [N, E, \mathbf{c}]$ and every agent i ,

$$r_i(\omega) = E - \sum_{j \in N \setminus \{i\}} c_j = E - C(\omega) + c_i.$$

The number $r_i(\omega)$ tells us what the agent gets if all other agents obtain their entitlements. Let $\mathbf{r}(\omega)$ denote the n -vector whose components are $r_i(\omega)$, $i \in N$. We call $\mathbf{r}(\omega)$ the *reference vector*, because it can be thought of as an endogenous pseudo status quo (resp. an endogenous pseudo ideal point), which establishes a lower bound (resp. an upper bound) for the values that any sensible allocation rule can take on [depending upon the relative situation of \mathbf{c} and $H(\omega)$].

The next axiom, new in the literature, called *reflection*, refers to two different problems having identical set of agents and identical budget, and such that the rights point in one problem is the reference vector of the other (one problem is the mirror image of the other, hence the name). That is, $\omega = [N, E, \mathbf{c}]$, $\omega' = [N, E, \mathbf{r}(\omega)]$. Reflection requires that both problems have identical solutions. Notice that since \mathbf{c} and $\mathbf{r}(\omega)$ are separated by $H(\omega) = H(\omega')$, ω and ω' are problems of different type, namely in one of them we have to allocate losses (with respect to the claims point), and in the other we have to allocate gains. The intuitive idea behind the principle of reflection is: Suppose $[N, E, \mathbf{c}]$ is an allocation problem in which, say, $C(\omega) > E$. An agent can view it in two ways:

(i) *Aggressive*: The firm is bankrupt. I and the others endeavor to salvage whatever we can get, presenting our c_i as evidence of our rights.

(ii) *Understanding*: I am ready to grant, provided the others do the same, that c_i cannot be achieved since the firm is already bankrupt. But I (and similarly the others too) deserve at least $r_i(\omega)$, which is feasible, so instead of being involved in paying debts to each other, I am willing to replace my unrealistic claim by this reference value, as long as everyone else does the same.

The principle of reflection requires that it does not matter whether the agents' rights are the original claims or the rights are the *undisputed amounts*, the solution function will yield each of them the same outcome.

Axiom 8 (REFLECTION). Let $\omega = [N, E, \mathbf{c}]$, and let $\omega' = [N, E, \mathbf{r}(\omega)]$. Then $F(\omega) = F(\omega')$.

Now we obtain another characterization of F^{RE} :

Proposition 4. An allocation rule F satisfies compatibility, claims linearity and reflection if and only if $F = F^{\text{RE}}$.

Proof. F^{RE} satisfies compatibility and claims linearity. It is easy to check that it also satisfies reflection: Take $\omega = [N, E, \mathbf{c}]$, and $\omega' = [N, E, \mathbf{r}(\omega)]$. Then,

$$\begin{aligned} F_i^{\text{RE}}(\omega) &= c_i + \frac{E - C(\omega)}{n} = c_i + E - C(\omega) + \frac{E - nE + nC(\omega) - C(\omega)}{n} \\ &= r_i(\omega) + \frac{E - R(\omega)}{n} = F_i^{\text{RE}}(\omega'). \end{aligned}$$

On the other hand, take the point $\mathbf{z} \in \mathbb{R}^n$, defined by $z_i = c_i + (E - C(\omega))/n$. It is immediate to observe that $\mathbf{z} = \lambda \mathbf{c} + (1 - \lambda)\mathbf{r}(\omega)$, where $\lambda = (n - 1)/n$. Thus, consider also the problem $\omega'' = [N, E, \mathbf{z}]$. Since $C(\omega'') = E$, compatibility implies that $F(\omega'') = \mathbf{z}$. Furthermore, by reflection, $F(\omega) = F(\omega')$. Now, by claims linearity, $\mathbf{z} = \lambda F(\omega) + (1 - \lambda)F(\omega') = F(\omega)$, and therefore, $F(\omega) = \mathbf{z} = F^{\text{RE}}(\omega)$. ■

One can also consider the sensitivity of the solution to changes in the population when the budget remains fixed. Let $\omega = [N, E, \mathbf{c}]$ be a distributive problem in Ω , and let

$\mathbf{x}=F(\omega)$ be the solution provided by an allocation rule F . Suppose that new agents enter the picture, while the budget remains unaltered. How should the solution be affected? A well established fairness principle says that all agents initially present have to be affected “equally”. That is the content of the next axiom:

Axiom 9 (UNIFORM POPULATION MONOTONICITY). For any two problems $\omega=[N, E, \mathbf{c}]$, and $\omega'=[N \cup M, E, (\mathbf{c}, \mathbf{c}')] \in \Omega$, $F_i(\omega) - F_i(\omega') = F_j(\omega) - F_j(\omega')$, for all $i, j \in N$.

According to this property, all agents initially present are *equally* affected by the incorporation of new claimants. (Cf. with the usual population monotonicity property in Thomson and Lensberg (1989) or Chun and Thomson (1992).)

The following result is obtained:

Proposition 5. An allocation rule F satisfies compatibility and uniform population monotonicity if and only if $F = F^{\text{RE}}$.

Proof. First notice that F^{RE} satisfies uniform population monotonicity: Take $\omega=[N, E, \mathbf{c}]$ and $\omega'=[N \cup M, E, (\mathbf{c}, \mathbf{c}')]$. For any $i \in N$, $F_i(\omega) = c_i + (E - C(\omega)/n)$ and $F_i^{\text{RE}}(\omega') = c_i + (E - C(\omega')/n + m)$.

Consequently, $F_i^{\text{RE}}(\omega) - F_i^{\text{RE}}(\omega') = (C'n + m[E - C(\omega)]/n(n + m))$, where $C' = \sum_{j \in M} c_j$.

Now let F be a rule which satisfies compatibility and uniform population monotonicity. Let $\omega=[N, E, \mathbf{c}]$ be a problem. If $C(\omega) = E$, then by compatibility, $F(\omega) = \mathbf{c} = F^{\text{RE}}(\omega)$. Thus, assume that $C(\omega) \neq E$. Take then an additional agent, k , such that $c_k = E - C(\omega)$, and the problem $\omega'=[N \cup \{k\}, E, (\mathbf{c}, c_k)]$. Since $C(\omega') = E$, again by compatibility, $F(\omega') = (\mathbf{c}, c_k)$. Furthermore, by uniform population monotonicity, for any two agents $i, j \in N$, $F_i(\omega) - F_i(\omega') = F_j(\omega) - F_j(\omega')$, i.e., $F_i(\omega) - c_i = F_j(\omega) - c_j$, and, consequently, $F_i(\omega) = c_i + \lambda$. Furthermore, $\sum_{i \in N} F_i(\omega) = E$, and therefore, $F(\omega) = F^{\text{RE}}(\omega)$. ■

In order to see that all provided characterization results are tight, consider the following rules:

Rule D: Let \mathcal{N} be the set of potential agents, that we assume completely ordered. Call agent 1 the first agent in the previously mentioned ordering. Now, define the rule D as follows: Let $\omega=[N, E, \mathbf{c}] \in \Omega$. If $1 \in N$, $D_1(\omega) = c_1$, and $D_j(\omega) = F_j^{\text{RE}}([N \setminus \{1\}, E - c_1, \mathbf{c}_{-1}])$ if $j \neq 1$. If $1 \notin N$, then $D(\omega) = F^{\text{RE}}(\omega)$.

Rule G: For $\omega=[N, E, \mathbf{c}] \in \Omega$, $E_i(\omega) = E/n$ for all $i \in N$.

Rule P: Let $\omega=[N, E, \mathbf{c}] \in \Omega$. If

1. $C(\omega) = E$; then $P(\omega) = \mathbf{c}$.
2. $C(\omega) \neq E$, then consider the following possibilities:
 - (a) $E \geq 0$, $\exists i \in N$ such that $c_i > 0$. If $c_j \leq 0$, $P_j(\omega) = 0$; if $c_k > 0$, then $P_k(\omega) = \lambda c_k$, where λ is chosen in such a way that $\sum_{i \in N} P_i(\omega) = E$.

- (b) $E \geq 0, \mathbf{c} < 0$. Then, $P_i(\omega) = E/n$ for all $i \in N$.
- (c) $E < 0$. Then, $P(\omega) = -P([N, -E, -\mathbf{c}])$.

Rule R: Consider again the set of potential agents totally ordered. Let $\omega = [N, E, \mathbf{c}] \in \Omega$. If $|N| \geq 3$, then $R(\omega) = F^{RE}(\omega)$; for $|N| = 2$, let $N = \{i, j\}$, with $i < j$ (in the mentioned ordering). Then, if $C(\omega) \geq E$, $R(\omega) = (c_i, E - c_i)$; if $C(\omega) < E$, $R(\omega) = (E - c_j, c_j)$.

Rule S: Let $\omega = [N, E, \mathbf{c}] \in \Omega$. If $|N| = 2$, $S(\omega) = F^{RE}(\omega)$; if $|N| \geq 3$, $S(\omega) = E(\omega)$.

The behavior of these allocation rules with respect to the axioms in Propositions 1 to 4 is summarized in Table 1. This Table serves the purpose of separating the axioms in the propositions.

Table 1

	F^{RE}	D	G	P	R	S
Symmetry	+	-	+	+	-	+
Responsibility	+	+	-	-	-	-
Compatibility	+	+	-	+	+	-
Composition	+	+	+	-	+	+
Claims linearity	+	+	+	-	-	+
Reflection	+	-	+	-	+	+
2-consistency	+	+	+	+	-	-
U.pop.mon	+	-	+	-	-	-

4. The restricted model

The rights–egalitarian rule will now be considered within a smaller domain of problems: those in which both claims and estate are nonnegative, and the estate falls short of the claims. As it was already mentioned, the interest of this analysis is to see the extent to which the characterizations developed in Sections 2 and 3 apply to bankruptcy-like situations, and to compare the rights egalitarian rule with the proportional rule (cf. Chun, 1988b).

The restricted class of problems under consideration, called Ω^+ , is formally defined as follows:

$$\Omega^+ = \{\omega = [N, E, \mathbf{c}] \in \Omega \mid \mathbf{c} \geq 0, E \geq 0 \text{ and } C(\omega) \geq E\}.$$

An allocation rule is now a function $F: \Omega^+ \rightarrow \cup_{N \in \mathcal{N}} \mathbb{R}^N$, such that for any $\omega = [N, E, \mathbf{c}] \in \Omega^+$, $F(\omega) \in H(\omega)$.

Observe that this situation corresponds to bankruptcy-like problems in which redistribution is permitted (i.e. $F_i(\omega) < 0$ is admissible). A special class of distribution problems is that of pure redistribution problems, namely those with $E = 0$.

Among the axioms used in Sections 2 and 3, some are meaningless in this restricted framework. This is the case for the axioms of *responsibility* and *reflection*. The difficulty derives from the fact that the allocation problems associated with the application of these

properties may well not be in Ω^+ . The translation of the remaining axioms to this context is straightforward, and will not be repeated here.

Concerning the characterization results, it is immediate to check that Propositions 1 and 3 also apply to this framework, that is:

Proposition 1'. F^{RE} is the only bi-consistent allocation rule in Ω^+ that coincides with the divorce rule.

Proposition 3'. F^{RE} is the unique allocation rule in Ω^+ satisfying compatibility, symmetry and claims linearity.

Proposition 2, however, does not hold in this restricted domain. The proportional solution also satisfies compatibility, symmetry and composition in Ω^+ . Proposition 5 does not hold either. Consider the following rule: For a problem $[N, E, \mathbf{c}]$, call $d_i = \min\{c_i, E\}$, $i \in N$. Now, define $F([N, E, \mathbf{c}]) = F^{\text{RE}}([N, E, \mathbf{d}])$. F also satisfies compatibility and uniform population monotonicity in Ω^+ .

Consider now the following axiom:

Axiom 10 (MINIMUM RESPONSIBILITY). For any $\omega = [N, E, \mathbf{c}]$ in Ω^+ , and each $i \in N$, $F([N, E, \mathbf{c}]) = (0, \dots, 0, r_i(\omega), 0, \dots, 0) + F([N, E - r_i(\omega), (\mathbf{c}_{-i}, c_i - r_i)])$.

Minimum responsibility shares the spirit of the responsibility axiom. Since guaranteeing the claims may put us outside the domain, let us guarantee only the amounts given by the reference point $r_i(\omega)$. By applying this procedure sequentially we end up by assigning to every agent her guaranteed minimal amount of money. Then the rule F is applied to the resulting problem in which the available budget is now $E - R(\omega) = (n - 1)(C(\omega) - E) \geq 0$, and the claims are given by $\mathbf{c} - \mathbf{r}(\omega) \geq 0$. Note that $[N, E - R(\omega), \mathbf{c} - \mathbf{r}(\omega)]$ is a symmetric problem. Consequently,

Proposition 2'. An allocation rule F on Ω^+ satisfies symmetry and minimum responsibility if and only if $F = F^{\text{RE}}$.

Remark. Observe that this weaker notion of responsibility is actually sufficient to get the characterization result in Proposition 2. It is however a less intuitive property in the general context of Sections 2 and 3.

Let us finally consider two additional axioms, in order to compare the performance of the rights-egalitarian rule and the proportional rule in Ω^+ :

Axiom 11 (BUDGET LINEARITY). For any two problems $\omega = [N, E, \mathbf{c}]$, $\omega' = [N, E', \mathbf{c}]$ in Ω^+ , and any $\lambda \in [0, 1]$, $F(\omega'') = \lambda F(\omega) + (1 - \lambda)F(\omega')$, where $\omega'' = [N, \lambda E + (1 - \lambda)E', \mathbf{c}]$.

Budget linearity can be interpreted in terms of allocation problems with uncertain budget (similarly to claims linearity). The next axiom is self explanatory:

Axiom 12 (STATUS QUO PRESERVATION). For any $\omega \in \Omega^+$, $F(\omega) \geq 0$.

The next result follows:

Proposition 6. *The proportional solution is the unique allocation rule in Ω^+ satisfying compatibility, budget linearity and status quo preservation.*

Proof. The proportional solution clearly satisfies all the requirements. Let now F be an allocation rule satisfying the properties. For $\omega = [N, E, \mathbf{c}] \in \Omega^+$, let $\omega' = [N, C(\omega), \mathbf{c}]$, $C(\omega) \neq 0$, and $\omega'' = [N, 0, \mathbf{c}]$. It is clear that $E = \lambda C(\omega) + (1 - \lambda)0$ for some $\lambda \in [0, 1]$, namely, $\lambda = E/C(\omega)$. Furthermore, compatibility implies that $F(\omega') = \mathbf{c}$, and status quo preservation implies that $F(\omega'') = \mathbf{0}$. Hence, budget linearity implies $F(\omega) = \lambda \mathbf{c}$, so F is the proportional solution. ■

This result suggests that a way of comparing the performance of the rights–egalitarian and the proportional solutions on Ω^+ is to say that *both satisfy compatibility, budget linearity and symmetry but the first one satisfies claims linearity whereas the second one satisfies status quo preservation.*

Notice that examples in Table 1 also serve the purpose of proving that the characterization results in propositions 2' and 3' are tight. As for proposition 6, F^{ER} satisfies budget linearity and compatibility and fails to satisfy status quo preservation, whereas G satisfies budget linearity and status quo preservation, but fails to satisfy compatibility.

5. Game theoretical support

In this section we will examine to what extent the rights egalitarian solution is supported by solutions of game theory and by game theoretical analysis. This study is interesting by itself and might also prove useful if one wants to extend the rights egalitarian solution to more complicated cases. To avoid trivialities we shall assume now that the cardinality of N is at least 3. Our first task then will be to convert every allocation problem $\omega = [N, E, \mathbf{c}] \in \Omega$ into a TU game on N . Two coalition functions come to mind:

1. The *direct* coalition function:

$$v(S) = \begin{cases} \sum_{i \in S} c_i, & S \neq N, \\ E, & S = N. \end{cases}$$

2. The *complementary* coalition function:

$$z(S) = \begin{cases} E - \sum_{i \in S} c_i, & S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

The second line in each of these definitions reflects the understanding that eventually E will be distributed among the players.

The first line represents different views about the rights: in the direct case, the members of a coalition S , $S \neq N$, are aggressive: It is as if they say: “You owe us⁵ $c(S)$ and we shall sue you if we are not fully compensated. Every further settlement should start from this premise”. The members of N are also aggressive: They demand $c(N)$, but they are also responsible to pay the debt of the group, if there is one, and are entitled to the excess, if there is one. Altogether the balance is E .

In the complementary case, the players are more passive: each nonempty coalition S bases its demands on the recognition that the other players are entitled to $c(S^c)$, where S^c denotes the complementary coalition to S . Negotiation should start from recognizing this fact.

The next results show that the rights–egalitarian solution to $[N, E, \mathbf{c}]$ is equal to the Shapley value and the prenucleolus of both (N, v) and (N, z) . Moreover, it coincides with the tau value, the nucleolus, the prekernel and the kernel of the direct game $(N; v)$ if $c(N) \leq E$, and with the same solutions of the complementary game $(N; z)$, if $c(N) \geq E$.

The following propositions facilitate the proofs of the aforementioned results.

Proposition 7. *The games $(N; v)$ and $(N; z)$ are dual games.*

Proof. Indeed, for each S , $S \subseteq N$, $v(S) = E - z(S^c)$. ■

Proposition 8. *If $c(N) \leq E$, then $(N; v)$ is a convex game and $(N; z)$ is a concave game. If $c(N) \geq E$, then $(N; v)$ is a concave game and $(N; z)$ is a convex game.*

Proof. By Proposition 7 and de-Morgan rules, it is sufficient to consider the games $(N; v)$. Clearly, $v(S) + v(T) = v(S \cup T) + v(S \cap T)$, whenever $S \cup T \neq N$. This relation also holds if $S = N$ or $T = N$. In the remaining case, $S \cup T = N$, $S \neq N$, $T \neq N$; therefore,

$$\begin{aligned} v(S) + v(T) &= c(S) + c(T) = c(S \cup T) + c(S \cap T) \\ &= c(N) + c(S \cap T) \begin{cases} \leq v(S \cup T) + v(S \cap T), & \text{if } c(N) \leq E, \\ \geq v(S \cup T) + v(S \cap T), & \text{if } c(N) \geq E. \end{cases} \end{aligned}$$

■

We shall show now that both the Shapley value and the prenucleolus of these games coincide with the ‘rights egalitarian’ solution.

Proposition 9. *The rights–egalitarian solution to $[N, E, \mathbf{c}]$ is equal to the Shapley value for both $(N; v)$ and $(N; z)$.*

Proof. By Proposition 7, it is sufficient to consider $(N; v)$, because a game and its dual have the same Shapley value. Denote by $(N; v^N)$ the carrier game on player i (that is, $v^i(S) = 1$ if $i \in S$ and $v^i(S) = 0$, otherwise). Then,

⁵We denote $\sum_{i \in S} c_i = c(S)$, if $S \neq \emptyset$, and $= 0$, otherwise. Note that $C(\omega) = c(N)$.

$$(N; v) = c_1(N; v^1) + c_2(N; v^2) + \cdots + c_n(N; v^n) + [E - c(N)](N; v^N),$$

where $(N; v^N)$ is the carrier game on N ; namely, the unanimity game. The sum of the Shapley values of the games that appear on the right hand side is the rights–egalitarian solution of the original problem. \blacksquare

Proposition 10. *The rights–egalitarian solution x to $[N, E, c]$ is equal to the prenucleolus of both $(N; v)$ and $(N; z)$.*

Proof. The excess of a coalition S , $\emptyset \neq S \neq N$, at the rights–egalitarian solution \mathbf{x} , for the game $(N; v)$ is:

$$e^v(S, \mathbf{x}) = v(S) - \mathbf{x}(S) = c(S) - \left[c(S) + \frac{E - c(N)}{n} s \right] = \frac{s}{n} [c(N) - E].$$

Thus the excess is monotonic in the size of the coalition. Hence, the collections of the highest, second highest, . . . , coalitions are the collections of single-person, 2-person, . . . , coalitions if $c(N) \leq E$, or in the reverse order if $c(N) \geq E$, not counting the coalitions \emptyset and N . These collections are balanced collections, hence it follows from Kohlberg's (1971) theorem, as formulated in the case of the prenucleolus by Sobolev (1975), that \mathbf{x} is the prenucleolus as well as the anti-prenucleolus⁷ of $(N; v)$. By proposition 7, \mathbf{x} is the prenucleolus point of the game $(N; z)$, because $e^v(S, \mathbf{x}) \leq (T, \mathbf{x})$ implies that $e^z(T^c, \mathbf{x}) \leq e^z(S^c, \mathbf{x})$. \blacksquare

Proposition 11. *If $c(N) \leq E$, then the rights egalitarian solution coincides with the nucleolus and the kernel of the game $(N; v)$. If $c(N) \geq E$, then the rights egalitarian solution coincides with the nucleolus and the kernel of the game $(N; z)$.*

Proof. We know by Proposition 8 that $(N; v)$ is a convex game when $c(N) \leq E$, and that $(N; z)$ is a convex game when $c(N) \geq E$. Proposition 10 shows that the rights egalitarian rule is equal to the prenucleolus of both games. Hence we can apply the result in Maschler and Peleg (1966) that shows that for convex games the prenucleolus coincides with the nucleolus and the kernel. That and Proposition 8 give us the desired results for both claims. \blacksquare

We see that some solution concepts such as the Shapley value and the prenucleolus coincide with the rights–egalitarian solutions for both $(N; v)$ and $(N; z)$ (Propositions 9 and 10) whereas other solution concepts such as the nucleolus and the kernel coincide with the rights–egalitarian solution only when the game $(N; v)$ or $(N; z)$ is convex (Propositions 8 and 11). The next Proposition provides a general condition for a solution

⁶The relationship between the Shapley value and the rights–egalitarian solution was extended in Maschler (1982) to the case in which also more than one person coalitions have claims.

⁷Namely, it is the outcome, among the set of preimputations that lexicographically maximizes the vector of excesses arranged in a nondecreasing order.

concept to coincide with the rights–egalitarian solution for either $(N; v)$ or $(N; z)$, whichever is convex. To this end we introduce the following terminology:

Let D^N denote the set of all convex games with player set N . A solution on D^N is a map ψ assigning to each convex game $(N; u) \in D^N$, an element $\psi(N; u) \in \mathbb{R}^N$. A solution on D^N satisfies:

Efficiency (EFF). If for all $(N; u) \in D^N$: $\sum_{i \in N} \psi_i(N; u) = u(N)$.

Covariance (COV). If for all $(N; u) \in D^N$, all $\lambda > 0$, and all additive games $(N; b)$ such that $b(\{i\}) = b_i$, $\mathbf{b} = (b_i)_{i \in N}$: $\psi[(N; b + \lambda u)] = \mathbf{b} + \lambda \psi(N; u)$.

Strong Symmetry (SSYM). If for all $(N; u) \in D^N$ and for all $i, j \in N$ with $u(S \cup \{i\}) - u(S) = u(S \cup \{j\}) - u(S)$ for all $S \in 2^N$, $S \neq \emptyset$, $S \neq N$, with $i, j \notin S$: $\psi_i(N; u) = \psi_j(N; u)$.

Remark. If a solution ψ satisfies these conditions, then the second one also holds when $\lambda = 0$.

We have then the following result:

Proposition 12.⁸ Let ψ be a solution on D^N that satisfies efficiency, covariance and strong symmetry. If the game $(N; v)$ is convex, then $\psi(N; v) = F^{\text{RE}}([N, E, \mathbf{c}])$. Similarly, if the game $(N; z)$ is convex, then $\psi(N; z) = F^{\text{RE}}([N, E, \mathbf{c}])$.

Proof. Let $[N, E, \mathbf{c}]$ be an allocation problem. Define the additive game $(N; a)$ by $a(S) = \sum_{i \in S} c_i$, for $S \neq \emptyset$, $a(\emptyset) = 0$. Further, let $(N; u_N)$ be the unanimity game for the grand coalition, namely,

$$u_N(S) = \begin{cases} 1, & \text{if } S = N, \\ 0, & \text{otherwise.} \end{cases}$$

Let us call $(N; u^*)$ the dual game of $(N; u_N)$, namely

$$u^*(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Now, we can write games $(N; v)$ and $(N; z)$ as follows:

$$v(S) = a(S) + (E - c(N))u_N(S), \text{ for all } S,$$

$$z(S) = a(S) + (E - c(N))u^*(S), \text{ for all } S.$$

Let now $(N; v) \in D^N$. Then, by covariance, $\psi_i(N; v) = c_i + (E - C(N))\psi_i(N; u_N)$. Moreover strong symmetry and efficiency imply that $\psi(N; u_N) = 1/n$. Consequently,

⁸We are grateful to Stef Tijs who provided us with this proposition and the subsequent corollary.

$$\psi_i(N; v) = c_i + \frac{1}{n} (E - c(N)) = F^{\text{RE}}([N, E, c]).$$

The proof for $(N; z)$ is similar. ■

This proposition provides an alternative proof of some of the previous results, using known properties of the solutions concerned and as a bonus adds results concerning other solutions that happen to satisfy the conditions of the proposition. We summarize some of them as:

Corollary. *If $c(N) \leq E$, the rights egalitarian solution to $[N, E, c]$ coincides with the Shapley value ψ , the Tau-value \mathbb{T} , the prenucleolus \mathbb{PN} , the nucleolus \mathbb{N} , the prekernel \mathbb{PK} and the kernel \mathbb{K} of the corresponding direct game $(N; v)$.*

If $c(N) \geq E$, the rights egalitarian solution to $[N, E, c]$ coincides with the Shapley value ψ , the Tau-value \mathbb{T} , the prenucleolus \mathbb{PN} , the nucleolus \mathbb{N} , the prekernel \mathbb{PK} and the kernel \mathbb{K} of the corresponding complementary game $(N; z)$.

Proof. It follows from Proposition 8 and the fact the these solutions satisfy the conditions of Proposition 12. For the case of the kernel, prekernel and nucleolus, see Maschler and Peleg (1966); furthermore, notice that the nucleolus is a kernel point, and the prenucleolus is a prekernel point. For the Tau value, see Tijs (1981) and Tijs and Otten (1993). ■

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