Repeated Games with Incomplete Information on Both Sides

Mor Amitai *

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Abstract

We analyze the set of equilibria of two-person repeated games with incomplete information on both sides. We show that each equilibrium generates a martingale with certain properties. Moreover, for games, satisfying a certain condition that we call "tightness", it is shown that the converse also holds: each such martingale generates an equilibrium.

*Institute of Mathematics and Center for Rationality and Interactive Decision Theory, The Hebrew University, Jerusalem.
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1 Introduction

A game with incomplete information is a game in which different players have different information on the structure of the game. Independent incomplete information (see, for example, Myerson 1991 section 2.8) is a situation in which there exists a common prior distribution over the possible set of games, known to all the players, and in addition, each player might have additional information which is independent of the information of the other player. Repeated games with incomplete information were much studied in the last thirty years (see the book of Aumann and Maschler 1995). Aumann, Maschler, and Stearns (Aumann and Maschler 1968 and Aumann, Maschler, and Stearns 1968) analyzed the two player zero sum case (see also Blackwell 1956) and had the first results on the non-zero sum case. In particular, they showed that the set of equilibria might be empty (in the non-zero sum incomplete information on both sides). However, when it is not empty, it might be a very complex set. Hart (Hart 1985) introduced the concept of bi-martingales and used it to characterize the set of equilibria in the one-sided information case (see also Aumann and Hart 1986). Lately Simon, Spież, and Toruńczyk (1995) proved that this set is not empty. In this work we study games with incomplete information on both sides and show that in this case the bi-martingales used in the characterization in the one-sided information case can be replaced by an appropriate class of "admissible martingales". The characterization has a lot in common with the characterization of the equilibria in the general two player incomplete information cheap talk games (Amitai 1996) and in particular the admissible martingales were first introduced there.

However, unlike both the one sided information case (Hart 1985) and the general Cheap-Talk case (Amitai 1996), admissible martingales are not sufficient, in general, to generate equilibria. We therefore introduce the concept of tightness. In this case admissible martingales and equilibria are equivalent.

In section 2 we define the model and discuss some properties of repeated games with incomplete information on both sides and in section 3 we define the notion of admissible martingales. In sections 4 and 5 we state and prove the main result and in section 6 we give an example demonstrating the difficulties in generalizing the result.

2 The Model

As every repeated game with incomplete information is equivalent to a repeated game with independent incomplete information (see Myerson 1991 page 73, Aumann and Maschler 1995 section 4.2, and for a detailed proof, Amitai 1996) we can restrict ourselves to games with independent incomplete information. We define a class of repeated games with incomplete information on both sides. The repeated game is played after the players have received their private information and is defined by the following:

1. Two players: player 1 and player 2.
2. A finite set of actions I for player 1, and a finite set of actions J for player 2.
3. Two finite sets, K and L, such that to each pair $(k \in K, l \in L)$ there corresponds a pair of $I \times J$ matrices $(A^{k,l}\ , B^{k,l})$. $A^{k,l} = (A^{k,l}(i,j))_{i\in I, j\in J}$, $B^{k,l} = (B^{k,l}(i,j))_{i\in I, j\in J}$.
4. Two probability vectors: $p \in \Delta(K)$, $p = (p(k))_{k\in K}$ and $q \in \Delta(L)$, $q = (q(l))_{l\in L}$.

5. Let n be a natural number or $n = \infty$. We define the game $\Gamma_n(p,q)$.

6. The game has two phases:

   The Information Phase : Nature chooses $\mathbf{k} \in K$ according to p and $\mathbf{l} \in L$ according to q. The choices are made independently, i.e, $Prob(\mathbf{k} = k \text{ and } \mathbf{l} = l) = p(k)q(l)$. $\mathbf{k}$ is told to player 1 and $\mathbf{l}$ is told to player 2.

   The Action Phase : This phase is divided into periods t=1,2,3...,n . For each t, player 1 chooses an action $i_t \in I$ and player 2 chooses an action $j_t \in J$. The choices are made simultaneously. The payoff to player 1 in period t is $a_t := A^{\mathbf{k},\mathbf{l}}(i_t, j_t)$ and the payoff to player 2 in period t is $b_t := B^{\mathbf{k},\mathbf{l}}(i_t, j_t)$.

7. The players have perfect recall.

8. 1,2,3,4,5,6,7 are common knowledge to both players.

9. For a finite n the payoff of the game is defined by the sum of the payoffs in the n periods of the game. For the infinite repeated game it will be defined later, together with the definition of equilibrium (definition 2.3).

The players have perfect recall, hence i_t and j_t are functions of the history of length $t-1$, namely, $h_{t-1} := ((i_1,j_1),(i_2,j_2),...,(i_{t-1},j_{t-1}))$. let $h_\infty := ((i_1,j_1),(i_2,j_2),...,(i_t,j_t),...)$ be the infinite sequence defined by the actions of the players in the game. Let $H_t = (I \times J)^t$ be the set of histories of length t. Define $H_0 = \{\phi\}$. Let $H_\infty = \prod_{t=1}^{\infty}(I \times J)$ be the set of infinite histories. On H_∞ , we define for every t, a finite field $\mathcal{H}_t$ as follows: h^1_∞ , $h^2_\infty \in H_\infty$ are in the same atom of $\mathcal{H}_t$ if and only if $h^1_\infty(u) = h^2_\infty(u)$ for every $1 \le u \le t$ (recall that $h_\infty(u)$ is the pair of actions chosen by the players at period u, according to the infinite history h_∞). Let $\mathcal{H}_\infty$ be the σ-field generated by $\{\mathcal{H}_t\}_{t=0}^{\infty}$. Our basic probability space is $(\Omega, \mathcal{A}) = (K \times L \times H_\infty , 2^K \otimes 2^L \otimes \mathcal{H}_\infty)$. A point in Ω is a triple (k, l, h_∞), where (k,l) is a possible state of nature and $h_\infty \in H_\infty$ is an history of the game. When defining sequences of random variables, we will use the following notation: $a_t, b_t, c_t, ...$ will usually be random variables measurable with respect to $(H_t, \mathcal{H}_t)$, and $a_{h_t}, b_{h_t}, c_{h_t}, ...$ will denote $a_t(h_t), b_t(h_t), c_t(h_t), ...$. For $x \in \Delta(I)$ and $y \in \Delta(J)$ we will write $A^{k,l}(x,y)$ instead of $\sum_{i\in I, j\in J} x(i)y(j)A^{k,l}(i,j)$ and $B^{k,l}(x,y)$ instead of $\sum_{i\in I,j\in J} x(i)y(j)B^{k,l}(i,j)$. Since $\Gamma_\infty(p,q)$ is a game with perfect recall, we can restrict ourselves to behavior strategies (see Aumann 1964). To shorten the writing, whenever we write 'strategy' we will mean a behavior strategy. Let $\mathbb{N}$ denotes the set of natural numbers $\{1,2,3,...\}$.

Definition 2.1: A *strategy* σ of player 1 in $\Gamma_\infty(p,q)$ is a function $\sigma : K \times \bigcup_{t\in\mathbb{N}} H_{t-1} \to \Delta(I)$. A *strategy* τ of player 2 in $\Gamma_\infty(p,q)$ is a function $\tau : L \times \bigcup_{t\in\mathbb{N}} H_{t-1} \to \Delta(J)$. For $h_t \in H_t$ let σ_{h_t} and τ_{h_t} be the strategies of playing according to σ and τ (respectively) given h_t, i.e, [1]

$$\sigma_{h_t}(k, h_r) := \sigma(k, (h_t, h_r))$$

$$\tau_{h_t}(l, h_r) := \tau(l, (h_t, h_r))$$

[1] $(h_t, h_r) \in H_{t+r}$ is the strategy h_t followed by the strategy h_r.

Definition 2.2: A strategy σ of player 1 in $\Gamma_\infty(p,q)$ is called *non−revealing* if $\sigma(k,h_t)=\sigma(k',h_t)$ for all $k,k'\in K$ and h_t. A strategy τ of player 2 in $\Gamma_\infty(p,q)$ is called *non − revealing* if $\tau(l,h_t)=\tau(l',h_t)$ for all $l,l'\in L$ and h_t.

Let Σ^i be the set of strategies of player i for $i=1,2$ and let Σ^i_{nr} be the set of non-revealing strategies of player i for $i=1,2$. Denote:

$$Z:=\max_{k\in K,l\in L,i\in I,j\in J}\{|A^{k,l}(i,j)|,|B^{k,l}(i,j)|\}+1.$$

That is, Z is a strict upper bound of the absolute value of the possible payoffs.

Every 4-tuple $(\sigma,\tau,k,l)\in\Sigma^1\times\Sigma^2\times K\times L$ defines a *probability* measure $\pi_{\sigma,\tau,k,l}$ on $(H_\infty,\mathcal{H}_\infty)$, i.e, for an history $h_t:=((i_1,j_1),(i_2,j_2),...,(i_t,j_t))$, we denote by $\pi_{\sigma,\tau,k,l}(h_t)$ the probability that the first action played by player 1 is i_1, the first action played by player 2 is j_1, the second action played by player 1 is i_2, and so on, given that $\mathbf{k}=k$, $\mathbf{l}=l$, player 1 plays according to σ and player 2 plays according to τ. We derive from $\pi_{\sigma,\tau,k,l}$ another probability measure on $(K\times L\times H_\infty, 2^K\otimes 2^L\otimes\mathcal{H}_\infty)$ by:

$$P_{\sigma,\tau,p,q}(k,l,h_t):=p(k)q(l)\pi_{\sigma,\tau,k,l}(h_t)$$

Note that $P_{\sigma,\tau,p,q}(\mathbf{k}=k,\mathbf{l}=l)=\sum_{h_t\in H_t}P_{\sigma,\tau,p,q}(k,l,h_t)=p(k)q(l)$. Denote by $E_{\sigma,\tau,k,l}$ the expectation with respect to $\pi_{\sigma,\tau,k,l}$ and by $E_{\sigma,\tau,p,q}$ the expectation with respect to $P_{\sigma,\tau,p,q}$. Similarly we derive $|K|$ probability measures on $(L\times H_\infty,2^L\otimes\mathcal{H}_\infty)$ and $|L|$ probability measures on $(K\times H_\infty, 2^K\otimes\mathcal{H}_\infty)$ by:

$$P^{k\cdot}_{\sigma,\tau,q}(l,h_t):=q(l)\pi_{\sigma,\tau,k,l}(h_t)$$

and

$$P^{\cdot l}_{\sigma,\tau,p}(k,h_t):=p(k)\pi_{\sigma,\tau,k,l}(h_t)$$

Let $E^{k\cdot}_{\sigma,\tau,q}$ and $E^{\cdot l}_{\sigma,\tau,p}$ be the expectations with respect to $P^{k\cdot}_{\sigma,\tau,q}$ and $P^{\cdot l}_{\sigma,\tau,p}$ respectively. We will denote $P_{\sigma,\tau,p,q}$, $E_{\sigma,\tau,p,q}$, $P^{k\cdot}_{\sigma,\tau,q}$, $E^{k\cdot}_{\sigma,\tau,q}$, $P^{\cdot l}_{\sigma,\tau,p}$, and $E^{\cdot l}_{\sigma,\tau,p}$ by P, E, $P^{k\cdot}$, $E^{k\cdot}$, $P^{\cdot l}$, and $E^{\cdot l}$ respectively. Denote by $\mathbf{a}_T$ and $\mathbf{b}_T$ the average random payoffs to player 1 and player 2, respectively, up to period T, i.e,

$$\mathbf{a}_T:=\frac{1}{T}\sum_{t=1}^{T}a_t=\frac{1}{T}\sum_{t=1}^{T}A^{\mathbf{k},\mathbf{l}}(i_t,j_t)$$

and

$$\mathbf{b}_T:=\frac{1}{T}\sum_{t=1}^{T}b_t=\frac{1}{T}\sum_{t=1}^{T}B^{\mathbf{k},\mathbf{l}}(i_t,j_t)$$

Definition 2.3: $a\in\mathbb{R}^K$ and $b\in\mathbb{R}^L$ are *equilibrium payoffs* in $\Gamma_\infty(p,q)$ if there exist $\sigma\in\Sigma^1$ and $\tau\in\Sigma^2$ such that:

E1 : $a^k=\lim_{T\to\infty}E^{k\cdot}(\mathbf{a}_T)$ for all $k\in K$.

E2 : $b^l=\lim_{T\to\infty}E^{\cdot l}(\mathbf{b}_T)$ for all $l\in L$.

E3 : $a^k\geq\limsup_{T\to\infty}E^{k\cdot}_{\sigma',\tau}(\mathbf{a}_T)$ for all $k\in K$ and $\sigma'\in\Sigma^1$.

E4 : $b^l \geq \limsup_{T\to\infty} E^{\cdot l}_{\sigma,\tau'}(\mathbf{b}_T)$ for all $l \in L$ and $\tau' \in \Sigma^2$.

Note that if $a \in \mathbb{R}^K$ and $b \in \mathbb{R}^L$ are equilibrium payoffs in $\Gamma_\infty(p,q)$ then $a \in (-Z,Z)^K$ and $b \in (-Z,Z)^L$.

Definition 2.4: $a \in \mathbb{R}^K$ and $b \in \mathbb{R}^L$ are *non revealing* equilibrium vector payoffs in $\Gamma_\infty(p,q)$ if there exist $\sigma \in \Sigma^1_{nr}$ and $\tau \in \Sigma^2_{nr}$ satisfying E1,E2,E3,E4. Note that $\sigma \in \Sigma^1_{nr}$ and $\tau \in \Sigma^2_{nr}$ imply (using induction and Bayes' rule):

E5 : $P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h_t) = p(k)$ for all h_t such that $P_{\sigma,\tau,p,q}(h_t) > 0$.

E6 : $P_{\sigma,\tau,p,q}(\mathbf{l} = l \mid h_t) = q(l)$ for all h_t such that $P_{\sigma,\tau,p,q}(h_t) > 0$.

Let

$NR := \{(a,b,p,q) \in \mathbb{R}^K \times \mathbb{R}^L \times \Delta(K) \times \Delta(L)$ s.t. (a,b) is a non revealing equilibrium in $\Gamma_\infty(p,q)\}$

Definition 2.5: Let $(a,b,p,q) \in \mathbb{R}^K \times \mathbb{R}^L \times \Delta(K) \times \Delta(L)$. $(a,b,p,q) \in NR^+$ if and only if there exist non-revealing strategies $\sigma \in \Sigma^1_{nr}$ and $\tau \in \Sigma^2_{nr}$ such that:

E1' : $a^k = \lim_{T\to\infty} E^{k\cdot}(\mathbf{a}_T)$ for all $k \in K$ such that $p(k) > 0$.

E2' : $b^l = \lim_{T\to\infty} E^{\cdot l}(\mathbf{b}_T)$ for all $l \in L$ such that $q(l) > 0$.

E3 : $a^k \geq \limsup_{T\to\infty} E^{k\cdot}_{\sigma',\tau}(\mathbf{a}_T)$ for all $k \in K$ and $\sigma' \in \Sigma^1$.

E4 : $b^l \geq \limsup_{T\to\infty} E^{\cdot l}_{\sigma,\tau'}(\mathbf{b}_T)$ for all $l \in L$ and $\tau' \in \Sigma^2$.

We need some notations. Let $a_{q,i,j}$ and $b_{p,i,j}$ be the expected vector payoffs (of player 1 and player 2 respectively) when the players play the actions i and j (respectively) constantly, i.e,

$$a^k_{q,i,j} := \sum_{l\in L} q(l) A^{k,l}(i,j) \quad \forall k \in K$$

$$b^l_{p,i,j} := \sum_{k\in K} p(k) B^{k,l}(i,j) \quad \forall l \in L$$

Let

$$F_{p,q} := \bigcup_{(i,j)\in I\times J} \{(a_{q,i,j}, b_{p,i,j})\}$$

$conv(F_{p,q})$ is[2] the set of (jointly) feasible payoffs in the two sided incomplete information one shot game with payoff matrices $(A^{k,l}, B^{k,l})$ and probability vectors p and q. For $x, y \in \mathbb{R}^M$ let $x \leq y$ denotes $x^m \leq y^m$ for all $m \in M$. We will use $x << y$ to denote $x^m < y^m$ for all $m \in M$. Let

$$F := \bigcup_{p,q} \{(a,b,p,q) \in [-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L) \text{ s.t. } (a,b) \in conv(F_{p,q})\}$$

[2] $conv(X)$ denotes the convex hull of the set X.

$$F^+ := \{(a,b,p,q) \in [-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L) \text{ s.t. there are } c \leq a \text{ and } d \leq b \text{ s.t.}$$

$$(c,d,p,q) \in F,$$
$$p(k) > 0 \text{ implies } a^k = c^k, \text{ and}$$
$$q(l) > 0 \text{ implies } b^l = d^l\}$$

We now introduce the concept of Banach limit (see Dunford and Schwartz 1958, page 73). We will use Banach limits in section 4. Let l_∞ be the space of all real bounded sequences. From the Hahn-Banach theorem (see, for example, Dunford and Schwartz 1958) follows the existence of a Banach limit which is a real operator $\mathcal{L} : l_\infty \to \mathbb{R}$ with the following properties (and many more):

1. $\mathcal{L}(\alpha x + \beta y) = \alpha\mathcal{L}(x) + \beta\mathcal{L}(y)$ for all $x, y \in l_\infty$ and $\alpha, \beta \in \mathbb{R}$.
2. $\mathcal{L}(\{x_{n+1}\}_{n=1}^\infty) = \mathcal{L}(\{x_n\}_{n=1}^\infty)$ for all $\{x_n\}_{n=1}^\infty \in l_\infty$.
3. $\liminf_{n\to\infty} x_n \leq \mathcal{L}(\{x_n\}_{n=1}^\infty) \leq \limsup_{n\to\infty} x_n$ for all $\{x_n\}_{n=1}^\infty \in l_\infty$. (in particular, the existence of $\lim_{n\to\infty} x_n$ implies $\lim_{n\to\infty} x_n = \mathcal{L}(\{x_n\}_{n=1}^\infty)$)

We will write $\mathcal{L}[x_n]$ instead of $\mathcal{L}(\{x_n\})$. Let $\Delta_1^1(p,q)$ be the one shot zero-sum game, with payoff matrix $\sum_{k\in K}\sum_{l\in L} p(k)q(l)A^{k,l}$. Let $\Delta_1^2(p,q)$ be the one shot zero-sum game, with payoff matrix $\sum_{k\in K}\sum_{l\in L} p(k)q(l)B^{k,l}$. Denote by $U^i(p,q)$ the value of the game $\Delta_1^i(p,q)$ for $i = 1, 2$. For a two dimensional function u, Denote by $vex_1 u$ the convexification of u with respect to the first variable, and by $vex_2 u$ the convexification of u with respect to the second variable. Similarly, denote by $cav_1 u$ and $cav_2 u$ the concavification of u with respect the first and the second variables.

Definition 2.6: Let $p \in \Delta(K)$ and $q \in \Delta(L)$.

$$W_q^1 := \{a \in \mathbb{R}^K \text{ s.t. } \exists\tau \in \Sigma^2 \; s.t. \; \forall k \in K \text{ and } \forall\sigma \in \Sigma^1 \; \limsup_{T\to\infty} E_{\sigma,\tau,q}^{k\cdot}(a_T) \leq a^k\}$$

$$\underline{W}_q^1 := \{a \in \mathbb{R}^K \text{ s.t. } \exists\tau \in \Sigma^2 \; s.t. \; \forall\mathcal{L}(\text{Banach limit })\forall k \in K \text{ and } \forall\sigma \in \Sigma^1 \; \limsup_{T\to\infty} E_{\sigma,\tau,q}^{k\cdot}(a_T) \leq a^k\}$$

$$W_p^2 := \{b \in \mathbb{R}^L \text{ s.t. } \exists\sigma \in \Sigma^1 \; s.t. \; \forall l \in L \text{ and } \forall\tau \in \Sigma^2 \; \limsup_{T\to\infty} E_{\sigma,\tau,p}^{\cdot l}(b_T) \leq b^l\}$$

$$\underline{W}_p^2 := \{b \in \mathbb{R}^L \text{ s.t. } \exists\sigma \in \Sigma^1 \; s.t. \; \forall\mathcal{L}(\text{Banach limit })\forall l \in L \text{ and } \forall\tau \in \Sigma^2 \; \limsup_{T\to\infty} E_{\sigma,\tau,p}^{\cdot l}(b_T) \leq b^l\}$$

That is, $a \in W_q^1$ if and only if player 2 can guarantee that player 1 will not get more than a^k for all $k \in K$ simultaneously, and $b \in W_p^2$ if and only if player 1 can guarantee that player 2 will not get more than b^l for all $l \in L$ simultaneously. From the study of the zero-sum case if follows that

$$W_q^1 = \underline{W}_q^1 = \{a \in \mathbb{R}^K \; s.t. \; \sum_{k\in K} p(k)a^k \geq (vex_2 cav_1 U^1)(p,q) \;\; \forall p \in \Delta(K)\}$$

and

$$W_p^2 = \underline{W}_p^2 = \{b \in \mathbb{R}^L \; s.t. \; \sum_{l\in L} q(l)b^l \geq (vex_1 cav_2 U^2)(p,q) \;\; \forall q \in \Delta(L)\}$$

(see Mertens, Sorin, and Zamir (1994) page 341).

W_q^1 is convex and upper semi continuous with respect to q. W_p^2 is convex and upper semi continuous with respect to p. Denote:

$$IR := \{(a,b,p,q) \in [-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L) \text{ s.t. } a \in W_q^1 \text{ and } b \in W_p^2\}$$

That is, $(a,b,p,q) \in IR$ if and only if (a,b) are *individually rational* payoffs in $\Gamma_\infty(p,q)$.

Lemma 2.7: *IR is a convex set.*

Proof: Denote $IR_1 := \{(a,q) \text{ s.t. } a \in W_q^1\}$ and $IR_2 := \{(b,p) \text{ s.t. } b \in W_p^2\}$. $IR = IR_1 \times IR_2$, hence it is sufficient to prove that IR_1 and IR_2 are convex. We will prove that IR_1 is convex, the proof for IR_2 is similar. For $p \in \Delta(K)$ let

$$IR_1^p := \{(a,q) \in \mathbb{R}^K \times \Delta(L) \ s.t. \sum_{k\in K} p(k)a^k \geq (vex_2 cav_1 U^1)(p,q)\ \}$$

$IR_1 = \bigcap_{p\in\Delta(K)} IR_1^p$, hence it is enough to show that IR_1^p is convex for every $p \in \Delta(K)$. $(vex_2 cav_1 U^1)(p,q)$ is convex as a function of q, hence IR_1^p is convex. ▮

Lemma 2.8:

1. $NR = F \cap IR$.

2. $NR^+ = F^+ \cap IR$.

Proof: We will prove the second part. The proof of the first part is similar and a little simpler. Assume that $(a,b) \in NR^+$ with non revealing strategies σ and τ satisfying the conditions of definition 2.5. Let $\overline{a}_T := \frac{1}{T}\sum_{t=1}^T a_{q,i_t,j_t}$ and $\overline{b}_T := \frac{1}{T}\sum_{t=1}^T b_{p,i_t,j_t}$. For every $T \in \mathbb{N}$ we have $(\overline{a}_T, \overline{b}_T) \in conv(F_{p,q})$ (because $(a_{q,i_t,j_t}, b_{p,i_t,j_t}) \in F_{p,q}$ for all $t \in \mathbb{N}$), hence $E_{\sigma,\tau,p,q}(\overline{a}_T, \overline{b}_T) \in conv(F_{p,q})$ and also $\limsup_{T\to\infty} E_{\sigma,\tau,p,q}(\overline{a}_T, \overline{b}_T) \in conv(F_{p,q})$, because $F_{p,q}$ is a finite set and hence close. $E(\overline{a}_T^k) = E(\sum_{l\in L} q(l)\frac{1}{T}\sum_{t=1}^T A^{k,l}(i_t,j_t)) = E(\sum_{l\in L} q(l)\frac{1}{T}\sum_{t=1}^T A^{k,l}(i_t,j_t) \mid \mathbf{k} = k)$, because the strategies, and hence the histories, are independent of k. Similarly $E(\overline{b}_T^l) = E((\sum_{k\in K} p(k)\frac{1}{T}\sum_{t=1}^T B^{k,l}(i_t,j_t) \mid \mathbf{l} = l)$. Now E1',E2',E3, and E4 of definition 2.5 imply that $(a,b) \in F^+$. E3 and E4 also imply that (a,b) is individually rational and $(a,b,p,q) \in IR$. Now we assume that $(a,b,p,q) \in F^+ \cap IR$ and show that $(a,b,p,q) \in NR^+$. $(a,b,p,q) \in F^+$, hence there exist $c \leq a$ and $d \leq b$ such that $(c,d) \in conv(F_{p,q})$, $p(k) > 0$ implies $c^k = a^k$ and $q(l) > 0$ implies $d^l = b^l$. Hence $(c,d) = \sum_{i\in I,j\in J}\lambda_{i,j}(a_{q,i,j}, b_{p,i,j})$ (where $\sum_{i\in I,j\in J}\lambda_{i,j} = 1$ and $\lambda_{i,j} \geq 0$). For all $n \in N$ define [3] $\mu_n : I \times J \to \mathbb{R}$ by $\mu_n(i,j) := \lfloor n\lambda_{i,j}\rfloor$. Denote $s_n := \sum_{i\in I,j\in J}\mu_n(i,j)$. Note that $n - |I| \times |J| < s_n \leq n$. Define a sequence C_n of s_n joint actions of the two players. For every $(i,j) \in I \times J$, C_n contains $\mu_n(i,j)$ copies of (i,j). When we say that the two players play C_n we mean that they play a sequence of s_n joint actions according to C_n. We define σ and τ together. The strategies σ and τ are to play C_1, then twice C_2, then six times C_3, ... then $n!$ times C_n and so on. The limit of the average payoffs exists and equals (c,d) because s_n and $(n-1)!$ are negligible with respect to $n!$. A player can not defect without being detected. If player 1 defects, player 2 will switch to a strategy guaranteeing that player 1 will not get more than a^k for all $k \in K$ (such a

[3] $\lfloor x \rfloor$ is the greatest integer not exceeding x.

strategy exists because $a \in W_q^1$). If player 2 defects player 1 will switch to a strategy guaranteeing that player 2 will not get more than b^l for all $l \in L$. σ and τ satisfy conditions E1', E2', E3, and E4 of definition 2.5. ∎

Let $\Gamma = (K, L, I, J, \{A^{k,l}\}_{k\in K, l\in L}, \{B^{k,l}\}_{k\in K, l\in L})$. That is, Γ is the structure of the game $\Gamma_n(p,q)$ without n, p, and q. In the next definition we define tight games. A tight game is a game Γ in which for every individually rational pair of vector payoffs there exists a feasible pair of vector payoffs, not exceeding the individually rational payoffs.

Definition 2.9: A game Γ is called *tight* if for every $(a, b, p, q) \in IR$ there exists a mixed joint action, $w_{a,b}^{p,q} \in \Delta(I \times J)$, yielding a vector payoff not exceeding a for player 1 , and a vector payoff not exceeding b for player 2, i.e,

1. $\sum_{l\in L} q(l) A^{k,l}(w_{a,b}^{p,q}) \le a^k$ for all $k \in K$.
2. $\sum_{k\in K} p(k) B^{k,l}(w_{a,b}^{p,q}) \le b^l$ for all $l \in L$.

3 Admissible Splits

In this section we introduce three definitions based upon the concept of admissible splits (for the motivation, examples and geometrical properties see Amitai 1996).

Let $x = (a, b, p, q) \in [-Z, Z]^K \times [-Z, Z]^L \times \Delta(K) \times \Delta(L)$ and let n and m be positive integers. Let $S = (\{x_{u,v}\}_{1\le u\le m, 1\le v\le n}, \mu, \lambda) \in ([-Z, Z]^K \times [-Z, Z]^L \times \Delta(K) \times \Delta(L))^{m\cdot n} \times \Delta([m]) \times \Delta([n])$, where $x_{u,v} = (a_{u,v}, b_{u,v}, p_{u,v}, q_{u,v})$.

Definition 3.1:
S is called an $m \times n-$ *admissible split* of x if

1. $x = \sum_{u=1}^{m} \sum_{v=1}^{n} \mu(u)\lambda(v) x_{u,v}$
2. (a) $a = \sum_{v=1}^{n} \lambda(v) a_{u,v}$ for all u such that $1 \le u \le m$.
   (b) $b = \sum_{u=1}^{m} \mu(u) b_{u,v}$ for all v such that $1 \le v \le n$.
   (c) $p_{u,v} = p_{u,v'}$ for all $1 \le u \le m$ and $1 \le v, v' \le n$.
   (d) $q_{u,v} = q_{u',v}$ for all $1 \le u, u' \le m$ and $1 \le v \le n$.

S is called an *exact* $m \times n$-admissible split if it is an $m \times n$-admissible split and in addition:

3. $\mu(u) > 0$ and $\lambda(v) > 0$ for all $1 \le u \le m$ and $1 \le v \le n$ (The split is into exactly $m \cdot m$ points).

Remarks :

1. From 1. and 2(c) it follows that $p = \sum_{u=1}^{m} \mu(u) p_{u,v}$ for all $1 \le v \le n$.
2. From 1. and 2(d) it follows that $q = \sum_{v=1}^{n} \lambda(v) q_{u,v}$ for all $1 \le u \le m$.

Definition 3.2:
Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two finite fields ($\mathcal{F}_2$ is thus a refinement of $\mathcal{F}_1$). Let X^1 and X^2 be $[-Z, Z]^K \times$

$[-Z,Z]^L \times \Delta(K) \times \Delta(L)$-valued random variables, measurable with respect to $\mathcal{F}^1$ and $\mathcal{F}^2$ respectively. X^2 is called an *(exact)* $m \times n$ *- admissible split* of X^1 if for every atom f^1 of $\mathcal{F}^1$, such that $P(f^1) > 0$, there exists an (exact) $m \times n$-admissible split $S = (\{x^2_{u,v}\}_{1 \le u \le m\ ,\ 1 \le v \le n},\ \mu,\ \lambda)$ of $x_{f^1} := E(X^1 \mid f^1)$, such that f^1 is partitioned into disjoint $\mathcal{F}^2$-measurable sets $\{f^2_{u,v}\}_{1 \le u \le m\ ,\ 1 \le v \le n}$ (thus $\bigcup_{1 \le u \le m\ ,\ 1 \le v \le n} f^2_{u,v} = f^1$ and $f^2_{u,v} \bigcap f^2_{u',v'} = \phi$ if $u \ne u'$ or $v \ne v'$) satisfying:

1. $P(f^2_{u,v} \mid f^1) = \mu(u)\lambda(v)$.
2. $X^2 = x^2_{u,v}$ on $f^2_{u,v}$ (i.e, $x^2_{u,v} = E(X^2 \mid f^2_{u,v})$) whenever $P(f^2_{u,v}) > 0$.

Let $\mathbb{N}_0$ be the set of non-negative integers, i.e, $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$.

Definition 3.3:
Let $x = (c, d, w, s) \in [-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L)$. Let $C \subset [-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L)$. An *(exact)* $m \times n$*-admissible martingale* starting at x and converging to C is a sequence $\{X_t\}_{t \in \mathbb{N}_0} = \{(c_t, d_t, w_t, s_t)\}_{t \in \mathbb{N}_0}$ of $[-Z,Z]^K \times [-Z,Z]^L \times \Delta(K) \times \Delta(L)$-valued random variables satisfying:

3.3.1 $X_0 =$ x a.s. (almost surely).

3.3.2 There exists a nondecreasing sequence $\{\mathcal{F}_t\}_{t \in \mathbb{N}_0}$ of finite fields ($\mathcal{F}_0 = \{\phi, \Omega\}$) with respect to which $\{X_t\}_{t \in \mathbb{N}_0}$ is a *martingale*, i.e: X_t is measurable with respect to $\mathcal{F}_t$ and $X_t = E(X_{t+1} \mid \mathcal{F}_t)$ a.s.

3.3.3 X_{t+1} is an *(exact)* $m \times n$ *- admissible split* of X_t for every $t \in \mathbb{N}_0$.

3.3.4 Every a.s. limit X_∞ of $\{X_t\}_{t \in \mathbb{N}_0}$ satisfies $X_\infty \in C$ a.s.

4 Main Result

We can now state and prove the main result.

Theorem 4.1:
Let $0 << p \in \Delta(K)$ and $0 << q \in \Delta(L)$ and let $\Gamma_\infty(p,q)$ be an infinite repeated game of independent incomplete information on both sides. If $(a,b) \in \mathbb{R}^K \times \mathbb{R}^L$ are equilibrium payoffs in $\Gamma_\infty(p,q)$, then there exists an $|I| \times |J|$ - admissible martingale starting at (a,b,p,q) and converging to NR^+. If Γ is tight then also the converse holds, i.e, if there exists an $|I| \times |J|$ - admissible martingale starting at (a,b,p,q) and converging to NR^+ then (a,b) are equilibrium payoffs in $\Gamma_\infty(p,q)$.

Proof: In order to prove the theorem we will need a few definitions and lemmas. In this section we will build an admissible martingale when an equilibrium is given, and in the next section we will prove the second part of the theorem.

Fix σ and τ, equilibrium strategies in $\Gamma_\infty(p,q)$ with equilibrium payoffs (a,b). For every history h_t we want to define several random variables: $p_{h_t}(k)$ and $q_{h_t}(l)$, the a posteriori probabilities of the events $(\mathbf{k} = k)$ and $(\mathbf{l} = l)$ given the history h_t. $\mu_{h_t}(i)$, the probability of the action i being played by player 1 after the history h_t. $\lambda_{h_t}(j)$, the probability of the action j being played by player 2 after h_t, and a_{h_t} and b_{h_t}, the expected vector payoffs given h_t. These variables are not defined for every history and we will have to extend the definitions. We will define p_{h_t} and μ_{h_t}, together, by induction. Similarly we will define q_{h_t} and λ_{h_t}. We will use q_{h_t} to define a_{h_t} and p_{h_t} to define b_{h_t}. Formally: Let $p_{h_0} := p$ and define (using induction on t) for all $i \in I$, $j \in J$ and $k \in K$:

$$\mu_{h_t}(i) := \sum_{k \in K} p_{h_t}(k)\sigma(k, h_t)(i)$$

and

$$p_{(h_t,(i,j))}(k) := \begin{cases} \frac{p_{h_t}(k)\sigma(k,h_t)(i)}{\mu_{h_t}(i)} & \mu_{h_t}(i) \neq 0 \\ p_{h_t}(k) & \text{otherwise} \end{cases}$$

Let $q_{h_0} := q$ and define $\forall i \in I$, $\forall j \in J$ and $\forall l \in L$:

$$\lambda_{h_t}(j) := \sum_{l \in L} q_{h_t}(l)\tau(l, h_t)(j)$$

and

$$q_{(h_t,(i,j))}(l) := \begin{cases} \frac{q_{h_t}(l)\sigma(l,h_t)(j)}{\lambda_{h_t}(j)} & \lambda_{h_t}(j) \neq 0 \\ q_{h_t}(l) & \text{otherwise} \end{cases}$$

Note that $p_{(h_t,(i,j))}$ is independent of j and τ and $q_{(h_t,(i,j))}$ is independent of i and σ. $p_{h_t}(k) = 0$ implies $p_{(h_t,(i,j))}(k) = 0$ and $q_{h_t}(l) = 0$ implies $q_{(h_t,(i,j))}(l) = 0$. Note that $p_{h_t}(k) = P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h_t)$, $q_{h_t}(l) = P_{\sigma,\tau,p,q}(\mathbf{l} = l \mid h_t)$, $\mu_{h_t}(i) = P_{\sigma,\tau,p,q}(i_{t+1} = i \mid h_t)$ and $\lambda_{h_t}(j) = P_{\sigma,\tau,p,q}(j_{t+1} = j \mid h_t)$ whenever the right side of the equations exists, i.e, whenever $P(h_t) > 0$. σ and τ are equilibrium strategies hence $\lim_{T\to\infty} E^{k\cdot}(a_T)$ exists, but this does not imply the existence of $\lim_{T\to\infty} E^{k\cdot}(a_T \mid h_s)$ even when $P(h_s) > 0$. We therefore use Banach limits. Fix a Banach limit $\mathcal{L}$ and define for all $k \in K$ and $l \in L$

$$a^k_{h_t} := \mathcal{L}[E^{k,\cdot}_{\sigma_{h_t},\tau_{h_t},q_{h_t}}(a_T)]$$

$$b^l_{h_t} := \mathcal{L}[E^{\cdot,l}_{\sigma_{h_t},\tau_{h_t},p_{h_t}}(b_T)]$$

Note that a_{h_t} is independent of p_{h_t} and b_{h_t} is independent of q_{h_t}. We will denote by p_t, q_t, μ_t, λ_t, a_t and b_t the random variables whose values given h_t are p_{h_t}, q_{h_t}, μ_{h_t}, λ_{h_t}, a_{h_t} and b_{h_t} respectively. Define, for every $h_t \in H_t$, $k \in K$ and $l \in L$

$$X^k_{h_t} := \sup_{\sigma'} \mathcal{L}[E^{k,\cdot}_{\sigma',\tau_{h_t},q_{h_t}}(a_T)]$$

$$Y^l_{h_t} := \sup_{\tau'} \mathcal{L}[E^{\cdot,l}_{\sigma_{h_t},\tau',p_{h_t}}(b_T)]$$

Clearly $a^k_{h_t} \leq X^k_{h_t} < Z$ and $b^l_{h_t} \leq Y^l_{h_t} < Z$. For all $t \geq 1$ and $h_t \in H_t$ denote by $h_t^{-1} \in H_{t-1}$ the history such that $h_t = (h_t^{-1}, (i,j))$. Let $h_0^{-1} := \phi$ and $P(h_0^{-1}) := 1$.

The following lemma is the main part of the first part of the proof, and in it the admissible martingale is being built.

Lemma 4.2:

1. *For all $h_t \in H_t$ and $k \in K$ there exists $c^k_{h_t} \in \mathbb{R}$ such that:*

   (a) *$X^k_{h_t} \leq c^k_{h_t} < Z$ for all $k \in K$ and h_t. If $P^{k\cdot}(h_t) > 0$ then $c^k_{h_t} = X^k_{h_t} = a^k_{h_t}$.*

(b) $c^k_{h_0} = a^k$ *for all* $k \in K$.

(c) $c^k_{h_t} = \sum_{j\in J} \lambda_{h_t}(j) c^k_{(h_t,(i,j))}$ *for all* $i \in I$ *and* h_t.

2. *For all* $h_t \in H_t$ *and* $l \in L$ *there exists* $d^l_{h_t} \in \mathbb{R}$ *such that:*

(a) $Y^l_{h_t} \le d^l_{h_t} < Z$ *for all* $l \in L$ *and* h_t. *If* $P^{\cdot l}(h_t) > 0$ *then* $d^l_{h_t} = Y^l_{h_t} = b^l_{h_t}$.

(b) $d^l_{h_0} = b^l$ *for all* $l \in L$.

(c) $d^l_{h_t} = \sum_{i\in I} \mu_{h_t}(i) d^l_{(h_t,(i,j))}$ *for all* $j \in J$ *and* h_t .

Proof: We will prove the first part of the lemma (the other part is similar), using induction on t. For $t = 0$, $h_t = \phi$. Define $c^k_{h_0} := a^k$. Condition (a) is satisfied for h_0 because σ and τ are equilibrium strategies. Fix $k \in K$, $t \in \mathbb{N}$, $h_t \in H_t$ and $i \in I$. We assume that $c^k_{h_t}$ is defined correctly and define $c^k_{(h_t,(i,j))}$ for all $j \in J$.
case 1: $P^{k\cdot}(h_t)\sigma(k,h_t)(i) = 0$. In this case $P^{k\cdot}(h_t,(i,j)) = 0$ for all $j \in J$, hence we have to prove only condition (c) (for h_t) and the first part of (a) (for $(h_t,(i,j))$). Define:

$$U_i := \sum_{j\in J} \lambda_{h_t}(j) X^k_{h_t,(i,j)}$$

Using the induction hypothesis we have

$$U_i \le \max_{i'\in I} U_{i'} = X^k_{h_t} \le c^k_{h_t} < Z = \sum_{j\in J} \lambda_{h_t}(j) Z \tag{1}$$

and $X^k_{h_t,(i,j)} < Z$ for all $j \in J$. Therefore we can choose for all $j \in J$ (simultaneously), $c^k_{h_t,(i,j)}$ such that $X^k_{h_t,(i,j)} \le c^k_{h_t,(i,j)} < Z$, and such that (c) is satisfied :

$$c^k_{h_t,(i,j)} := X^k_{h_t,(i,j)} + (Z - X^k_{h_t,(i,j)}) \frac{c^k_{h_t} - U_i}{Z - U_i}$$

The inequality $X^k_{h_t,(i,j)} \le c^k_{h_t,(i,j)}$ follows from the inequalities: $Z > X^k_{h_t,(i,j)}$, $c^k_{h_t} \ge U_i$, and $Z > U_i$. The inequality $c^k_{h_t,(i,j)} < Z$ follows from the inequalities $Z > X^k_{h_t,(i,j)}$ and $Z > c^k_{h_t}$ because the later yields $\frac{c^k_{h_t} - U_i}{Z - U_i} < 1$. Thus, we proved that $X^k_{h_t,(i,j)} \le c^k_{h_t,(i,j)} < Z$. To complete this part of the proof we have to show that (c) is satisfied:

$$\sum_{j\in J} \lambda_{h_t}(j) c^k_{h_t,(i,j)} = \sum_{j\in J} \lambda_{h_t}(j) X^k_{h_t,(i,j)} + \frac{c^k_{h_t} - U_i}{Z - U_i} \left(\sum_{j\in J} \lambda_{h_t}(j) Z - \sum_{j\in J} \lambda_{h_t}(j) X^k_{h_t,(i,j)}\right)$$

$$= U_i + \frac{c^k_{h_t} - U_i}{Z - U_i}(Z - U_i) = c^k_{h_t}$$

and (c) is satisfied.
case 2: $P^{k\cdot}(h_t)\sigma(k,h_t)(i) > 0$. For all $j \in J$ define $c^k_{(h_t,(i,j))} := X^k_{(h_t,(i,j))} \ge a^k_{(h_t,(i,j))}$. Fix $j \in J$ and denote $(h_t,(i,j))$ by h_s. We will show that if $P^{k\cdot}(h_s) > 0$ then $c^k_{h_s} = a^k_{h_s}$. The idea is that if $a^k_{h_s} < X^k_{h_s}$ and $P^{k\cdot}(h_s) > 0$ then type k of player 1 can achieve more than a^k by playing σ and

switching to a strategy guaranteeing him almost $X^k_{h_s}$ after h_s, a contradiction. We will choose an arbitrary strategy σ' and show that player 1 can gain no more than $a^k_{h_s}$ playing σ' after h_s (otherwise he can get more than a^k by playing σ and switching to σ' after h_s). Therefore $c^k_{h_s} = a^k_{h_s}$ if $P^{k\cdot}(h_s) > 0$. Formally, let σ' be a strategy of player 1. Define σ'' as follows:

$$\sigma''(k', h_r) := \begin{cases} \sigma'(k', h_x) & \text{for } h_r = (h_s, h_x) \\ \sigma(k', h_r) & \text{otherwise} \end{cases}$$

σ'' is the strategy of playing σ and switching to σ' if h_s has occurred. Denote $E^{k\cdot}$ by E_k and $P^{k\cdot}$ by P_k. Denote $E^{k\cdot}_{\sigma'',\tau,q}$ by E''_k and $P^{k\cdot}_{\sigma'',\tau,q}$ by P''_k. Denote the set of strategies different from h_s (i.e, $H_s \backslash \{h_s\}$) by "*not* h_s". σ and τ are equilibrium strategies, therefore

$$a^k = \lim_{T\to\infty} E_k(a_T) = \mathcal{L}[E_k(a_T)] = P_k(h_s)\mathcal{L}[E_k(a_T \mid h_s)] + (1 - P_k(h_s))\mathcal{L}[E_k(a_T \mid \textit{not } h_s)]$$

$$\geq \limsup_{T\to\infty} E''_k(a_T) \geq \mathcal{L}[E''_k(a_T)] = P''_k(h_s)\mathcal{L}[E''_k(a_T \mid h_s)] + (1 - P''_k(h_s))\mathcal{L}[E''_k(a_T \mid \textit{not } h_s)]$$

$P_k(h_s) = P''_k(h_s)$ and $E_k(a_T \mid \textit{not } h_s) = E''_k(a_T \mid \textit{not } h_s)$ and assuming that $P_k(h_s) > 0$ we get

$$a^k_{h_s} = \mathcal{L}[E_k(a_T \mid h_s)] \geq \mathcal{L}[E''_k(a_T \mid h_s)] = \mathcal{L}[E^{k;\cdot}_{\sigma',\tau_{h_s},q_{h_s}}(a_T)]$$

This is true for all σ', hence $P^{k\cdot}(h_s) > 0$ implies $a^k_{h_s} \geq X^k_{h_s}$ and therefore $c^k_{h_s} = X^k_{h_s} = a^k_{h_s}$. We have proved (a) for $(h_t, (i,j))$ for all $j \in J$. We will use (a) to prove (c). From the induction hypothesis $P^{k\cdot}(h_t) > 0$ implies $c^k_{h_t} = a^k_{h_t} = X^k_{h_t}$.

$$c^k_{h_t} = a^k_{h_t} = \sum_{i' \text{ s.t. } \sigma(k,h_t)(i')>0} \sigma(k, h_t)(i') \sum_{j\in J} \lambda_{h_t}(j) a^k_{(h_t,(i',j))}$$

$$= \sum_{i' \text{ s.t. } \sigma(k,h_t)(i')>0} \sigma(k, h_t)(i') \sum_{j\in J} \lambda_{h_t}(j) c^k_{(h_t,(i',j))} \quad (2)$$

because $P^{k\cdot}(h_t) > 0$, $\sigma(k, h_t)(i') > 0$ and $\lambda_{h_t}(j) > 0$ imply $P^{k\cdot}(h_t, (i', j)) > 0$, which implies $a^k_{(h_t,(i',j))} = c^k_{(h_t,(i',j))}$.

$$c^k_{h_t} = X^k_{h_t} = \max_{i'\in I} \sum_{j\in J} \lambda_{h_t}(j) X^k_{(h_t,(i',j))} \geq \max_{i' \text{ s.t. } \sigma(k,h_t)(i')>0} \sum_{j\in J} \lambda_{h_t}(j) a^k_{(h_t,(i',j))}$$

$$= \max_{i' \text{ s.t. } \sigma(k,h_t)(i')>0} \sum_{j\in J} \lambda_{h_t}(j) c^k_{(h_t,(i',j))} \quad (3)$$

From equations (2) and (3) and the fact that $\sum_{i' \text{ s.t. } \sigma(h_t,k)(i')>0} \sigma(h_t, k)(i') = 1$ follows that for all i' such that $\sigma(h_t, k)(i') > 0$ there exists

$$c^k_{h_t} = \sum_{j\in J} \lambda_{h_t}(j) c^k_{(h_t,(i',j))}$$

In case 2 $\sigma(h_t, k)(i) > 0$. ∎

Corollary 4.3:
$c_{h_t} \in W^1_{q_{h_t}}$ and $d_{h_t} \in W^2_{p_{h_t}}$.

Proof: Again we will prove only for c_{h_t}. $X_{h_t} \in \underline{W}^1_{q_{h_t}}$ (immediate from the definition of $\underline{W}^1_{q_{h_t}}$). $c_{h_t} \geq X_{h_t} \in \underline{W}^1_{q_{h_t}}$ (lemma 4.2), hence $c_{h_t} \in \underline{W}^1_{q_{h_t}}$. The fact that $\underline{W}^1_{q_{h_t}} = W^1_{q_{h_t}}$ (see the discussion after definition 2.5) completes the proof.
∎

Corollary 4.4: *$\{(c_t, d_t, p_t, q_t)\}_{t=0}^{\infty}$ is a martingale with respect to the fields $\{\mathcal{H}_t\}_{t=0}^{\infty}$ and the probability $P{:=}P_{\sigma,\tau,p,q}$.*

Proof: We will prove the corollary only for c_t.

$$E(c_{t+1} \mid h_t) = \sum_{i\in I, j\in J} P((h_t,(i,j)) \mid h_t) c_{(h_t,(i,j))} = \sum_{i\in I}\sum_{j\in J} \mu_{h_t}(i)\lambda_{h_t}(j) c_{(h_t,(i,j))}$$

$$= \sum_{i\in I} \mu_{h_t}(i) \sum_{j\in J} \lambda_{h_t}(j) c_{(h_t,(i,j))} = \sum_{i\in I} \mu_{h_t}(i) c_{h_t} = c_{h_t} = E(c_t \mid h_t)$$

∎

For $h_\infty = ((i_1,j_1),(i_2,j_2),\ldots) \in H_\infty$ denote by $(h_\infty)^t \in H_t$, the history defined by the first t coordinates of h_∞, i.e, $(h_\infty)^t := ((i_1,j_1),(i_2,j_2),\ldots,(i_t,j_t))$. For $h_\infty \in H_\infty$ define $c_{h_\infty} := \lim_{t\to\infty} c_{(h_\infty)^t}$, $d_{h_\infty} := \lim_{t\to\infty} d_{(h_\infty)^t}$, $p_{h_\infty} := \lim_{t\to\infty} p_{(h_\infty)^t}$ and $q_{h_\infty} := \lim_{t\to\infty} q_{(h_\infty)^t}$.

Corollary 4.5: *$(c_\infty, d_\infty, p_\infty, q_\infty)$ exists P-a.s.*

Proof: Corollary 4.4 ∎

Lemma 4.6:
For P-almost every h_∞ there exists $P((h_\infty)^t) > 0$ for all $t \in \mathbb{N}$.

Proof: Let $t \in \mathbb{N}$. H_t is a finite set hence

$$P(\ \{h_\infty \in H_\infty \ s.t.\ P((h_\infty)^t) = 0\}\ ) = P(\ \{h_t \in H_t \ s.t.\ P(h_t) = 0\}\ ) = 0$$

$\{t \in \mathbb{N}\}$ is a countable set, hence

$$P(\ \bigcup_{t\in\mathbb{N}} \{h_\infty \in H_\infty \ s.t.\ P((h_\infty)^t) = 0\}\ ) = 0$$

∎

Lemma 4.7:
For P-almost every h_∞ there exist

1. *If $p_{h_\infty}(k) > 0$ then $c^k_{h_\infty} = \lim_{t\to\infty} a^k_{(h_\infty)^t}$.*
2. *If $q_{h_\infty}(l) > 0$ then $d^l_{h_\infty} = \lim_{t\to\infty} b^l_{(h_\infty)^t}$.*

Proof: Fix $k \in K$ and h_∞ such that $p_{h_\infty}(k) > 0$. We can assume that c_∞ exists (corollary 4.5). Denote $(h_\infty)^t$ by h_t. Using lemma 4.6 we can assume that $P(h_t) > 0$ for all t. $p_{h_\infty}(k) > 0$ implies that $p_{h_t}(k) > 0$ for all t. $P(h_t) > 0$ implies that $P(\mathbf{k} = k \mid h_t) = p_{h_t}(k) > 0$ hence $P^{k\cdot}(h_t) = \frac{p_{h_t}(k)P(h_t)}{p(k)} > 0$, hence (lemma 4.2) $c^k_{h_t} = a^k_{h_t}$ for all t and $c^k_{h_\infty} = \lim_{t\to\infty} a^k_{(h_\infty)^t}$. ∎

Lemma 4.8: $(c_\infty, d_\infty, p_\infty, q_\infty) \in IR$ *P-a.s.*

Proof: Fix h_∞ such that c_{h_∞}, d_{h_∞}, p_{h_∞} and q_{h_∞} exist (from corollary 4.5, this happens P-a.s.). From corollary 4.3 and the fact that W^1_q and W^2_p are upper semi continuous we have that $c_{h_\infty} \in W^1_{q_{h_\infty}}$ and $d_{h_\infty} \in W^2_{p_{h_\infty}}$. Hence, $(c_{h_\infty}, d_{h_\infty}, p_{h_\infty}, q_{h_\infty}) \in IR$. ∎

Lemma 4.9: *Let $\{X_n\}_{n=1}^\infty$ be a bounded sequence of real random variables, converging a.s as $n \to \infty$, and let $\{\mathcal{F}_n\}_{n=1}^\infty$ be a nondecreasing sequence of σ-fields such that X_n is measurable with respect to $\mathcal{F}_n$. Define $Y_n := \sup_{m\geq n}|X_m - X_n|$, then $E(Y_n \mid \mathcal{F}_n) \to 0$ a.s as $n \to \infty$.*

Proof: Lemma 4.24 in (Hart 85) ∎

Lemma 4.10: $(c_\infty, d_\infty, p_\infty, q_\infty) \in F^+$ *P-a.s.*

Proof: Fix h_s such that $P_{\sigma,\tau,p,q}(h_s) > 0$. Let $\bar\sigma$ and $\bar\tau$ be the average non revealing strategies defined by

$$\bar\sigma(k, h_t) := \sum_{k'\in K} p_{h_t}(k')\sigma(k', h_t) \qquad \text{for all } k \in K \text{ and } h_t$$

and

$$\bar\tau(l, h_t) := \sum_{l'\in L} q_{h_t}(l')\tau(l', h_t) \qquad \text{for all } l \in L \text{ and } h_t$$

Now define the strategies $\tilde\sigma$ and $\tilde\tau$ of playing σ and τ (respectively), and switching to $\bar\sigma$ and $\bar\tau$ (respectively), after h_s, i.e,

$$\tilde\sigma(k, h_t) := \begin{cases} \bar\sigma(k, h_t) & h_t = (h_s, h_x) \\ \sigma(k, h_t) & \text{otherwise} \end{cases}$$

and

$$\tilde\tau(l, h_t) := \begin{cases} \bar\tau(l, h_t) & h_t = (h_s, h_x) \\ \tau(l, h_t) & \text{otherwise} \end{cases}$$

Denote $P := P_{\sigma,\tau,p,q}$, $\tilde P := P_{\tilde\sigma,\tilde\tau,p,q}$, $E := E_{\sigma,\tau,p,q}$, and $\tilde E := E_{\tilde\sigma,\tilde\tau,p,q}$. Denote $P_k := P^{k\cdot}_{\sigma,\tau,q}$, $\tilde P_k(\cdot) := P_{\tilde\sigma,\tilde\tau,p,q}(\cdot \mid \mathbf{k} = k)$, $E_k := E^{k\cdot}_{\sigma,\tau,q}$, and $\tilde E_k := P_{\tilde\sigma,\tilde\tau,p,q}(\cdot \mid \mathbf{k} = k)$.

$$P(\ (i_{t+1}, j_{t+1}) = (i,j) \mid h_t\ ) = \sum_{k\in K} p_{h_t}(k)\sigma(k,h_t)(i) \sum_{l\in L} q_{h_t}(l)\tau(l,h_t)(j) = \tilde P(\ (i_{t+1}, j_{t+1}) = (i,j) \mid h_t\ )$$

for every $i \in I, j \in J$ and $h_t \in H_t$ such that $P(h_t) > 0$, hence, $P(h_T \mid h_s) = \tilde P(h_T \mid h_s)$ for every $h_T \in H_T$. $\bar\sigma$ and $\bar\tau$ are non-revealing strategies hence for all $k \in K$, $l \in L$, and h_T such that $P(h_T \mid h_s) > 0$ there exist $\tilde P(\mathbf{k} = k \mid h_T) = p_{h_s}(k)$ and $\tilde P(\mathbf{l} = l \mid h_T) = q_{h_s}(l)$. Recall that for every $k \in K$ and $l \in L$:

$$a^k_{h_s} := \mathcal{L}[E_k(a_T \mid h_s)] \quad \text{and} \quad b^l_{h_s} := \mathcal{L}[E_l(b_T \mid h_s)]$$

Denote

$$\tilde{a}^k_{h_s} := \mathcal{L}[\tilde{E}_k(a_T \mid h_s)] \quad \text{and} \quad \tilde{b}^l_{h_s} := \mathcal{L}[\tilde{E}_l(b_T \mid h_s)]$$

Fix $k \in K$ and $T > s$. Let $A^{k,l}(h_T) := \frac{1}{T}\sum_{1\leq t\leq T} A^{k,l}(h_T(t))$ (note that $h_T(t) \in I \times J$). $A^{k,l}(h_T)$ is the average payoff to player 1 when h_T occurs and $(\mathbf{k},\mathbf{l}) = (k,l)$.

$$E_k(a_T \mid h_s) = \sum_{h_T \in H_T} P_k(h_T \mid h_s) \sum_{l\in L} q_{h_T}(l) A^{k,l}(h_T)$$

hence

$$p_{h_s}(k) E_k(a_T \mid h_s) = \sum_{h_T \in H_T} p_{h_s}(k) P_k(h_T \mid h_s) \sum_{l\in L} q_{h_T}(l) A^{k,l}(h_T)$$

Using Bayes' rule (recall that $P(h_s) > 0$) we have

$$p_{h_s}(k) E_k(a_T \mid h_s) = \sum_{h_T \in H_T} P(h_T \mid h_s) p_{h_T}(k) \sum_{l\in L} q_{h_T}(l) A^{k,l}(h_T) \tag{4}$$

similarly (using the facts that $P(h_T \mid h_s) = \tilde{P}(h_T \mid h_s)$, $\tilde{P}(\mathbf{k} = k \mid h_T) = p_{h_s}(k)$ and $\tilde{P}(\mathbf{l} = l \mid h_T) = q_{h_s}(l)$

$$p_{h_s}(k) \tilde{E}_k(a_T \mid h_s) = \sum_{h_T \in H_T} P(h_T \mid h_s) p_{h_s}(k) \sum_{l\in L} q_{h_s}(l) A^{k,l}(h_T) \tag{5}$$

Recall that k is fixed and denote $x_t(l) := p_t(k) q_t(l)$ for all $t \in \mathbb{N}$. From equations (4) and (5) we have

$$p_{h_s}(k)|E_k(a_T \mid h_s) - \tilde{E}_k(a_T \mid h_s)| = |\sum_{h_T\in H_T} P(h_T \mid h_s) \sum_{l\in L} A^{k,l}(h_T)(p_{h_T}(k) q_{h_T}(l) - p_{h_s}(k) q_{h_s}(l))|$$

$$\leq \sum_{h_T\in H_T} P(h_T \mid h_s) \sum_{l\in L} |A^{k,l}(h_T)||x_{h_T}(l) - x_{h_s}(l)| \leq Z \sum_{h_T\in H_T} P(h_T \mid h_s) \sum_{l\in L} |x_{h_T}(l) - x_{h_s}(l)|$$

$$= Z \cdot E(\sum_{l\in L} |x_T(l) - x_s(l)| \mid h_s) = Z \cdot \sum_{l\in L} E(|x_T(l) - x_s(l)| \mid h_s)$$

hence

$$p_{h_s}(k)|a^k_{h_s} - \tilde{a}^k_{h_s}| = p_{h_s}(k)|\mathcal{L}[E_k(a_T \mid h_s)] - \mathcal{L}[\tilde{E}_k(a_T \mid h_s)]|$$

$$\leq \sup_{T\geq s} p_{h_s}(k)|E_k(a_T \mid h_s) - \tilde{E}_k(a_T \mid h_s)| \leq Z \sup_{T\geq s} \sum_{l\in L} E(|x_T(l) - x_s(l)| \mid h_s)$$

$$\leq Z \sum_{l\in L} \sup_{T\geq s} E(|x_T(l) - x_s(l)| \mid h_s) \leq Z \sum_{l\in L} E(\sup_{T\geq s} |x_T(l) - x_s(l)| \mid h_s)$$

From lemma 4.9 we have that for every $l \in L$

$$\lim_{s\to\infty} E(sup_{T\geq s}|x_T(l) - x_s(l)| \mid h_s) = 0 \quad P - a.s.$$

hence $\lim_{s\to\infty} p_s(k)|a^k_s - \tilde{a}^k_s| = 0$ P-a.s. Hence we have that for almost every h_∞ if $p_{h_\infty}(k) > 0$ then $\liminf_{s\to\infty} a^k_{(h_\infty)^s} = \liminf_{s\to\infty} \tilde{a}^k_{(h_\infty)^s}$ (note that $\lim_{s\to\infty} a^k_{(h_\infty)^s}$ might not exist). Similar arguments yield that for almost every h_∞ if $q_{h_\infty}(l) > 0$ then $\liminf_{s\to\infty} b^l_{(h_\infty)^s} = \liminf_{s\to\infty} \tilde{b}^l_{(h_\infty)^s}$. $P(h_t) > 0$ $P - a.s.$ (lemma 4.6). $p_{h_\infty}(k) > 0$ implies $p_{(h_\infty)^t}(k) > 0$ for all t, hence $p_{h_\infty}(k) > 0$ implies $a^k_{(h_\infty)^t} = c^k_{(h_\infty)^t}$ for all t $P - a.s$ (lemma 4.2), hence

$$p_{h_\infty}(k) > 0 \text{ implies } \lim_{t\to\infty} c_t^k = \liminf_{t\to\infty} a_t^k = \liminf_{t\to\infty} \tilde{a}_t^k \quad P\text{-a.s} \tag{6}$$

Similarly

$$q_{h_\infty}(l) > 0 \text{ implies } \lim_{t\to\infty} d_t^l = \liminf_{t\to\infty} b_t^l = \liminf_{t\to\infty} \tilde{b}_t^l \quad P\text{-a.s} \tag{7}$$

Using the fact that $P^{k,\cdot}_{\tilde{\sigma},\tau,q}(h_T) = P(h_T)$ we have

$$E^{k,\cdot}_{\tilde{\sigma},\tau,q}(a_T \mid h_s) = \sum_{h_T\in H_T} P(h_T \mid h_s) \sum_{l\in L} q_{h_T}(l) A^{k,l}(h_T)$$

similarly

$$\tilde{E}_k(a_T \mid h_s) = \sum_{h_T\in H_T} P(h_T \mid h_s) \sum_{l\in L} q_{h_s}(l) A^{k,l}(h_T) \tag{8}$$

hence

$$E^{k,\cdot}_{\tilde{\sigma},\tau,q}(a_T \mid h_s) - \tilde{E}_k(a_T \mid h_s) = \sum_{h_T\in H_T} P(h_T \mid h_s) \sum_{l\in L} (q_{h_T}(l) - q_{h_s}(l)) A^{k,l}(h_T)$$

Let $a'^k_{h_s} := \mathcal{L}[E^{k,\cdot}_{\tilde{\sigma},\tau,q}(a_T \mid h_s)]$. The same argument used to show that $\lim_{s\to\infty} p_s(k)|a_s^k - \tilde{a}_s^k| = 0$ shows that $\lim_{s\to\infty} |a'^k_s - \tilde{a}_s^k| = 0$ P-a.s., hence $\liminf_{s\to\infty} a'^k_s = \liminf_{s\to\infty} \tilde{a}_s^k$ P-a.s. $X^k_{h_s} \geq a'^k_{h_s}$ for all $k \in K$, therefore $c^k_{h_s} \geq a'^k_{h_s}$ for all $k \in K$ (lemma 4.2) and $c^k_\infty \geq \liminf_{s\to\infty} \tilde{a}^k_s$ for all $k \in K$ P-a.s. Similarly $d^l_\infty \geq \liminf_{s\to\infty} \tilde{b}^l_s$ for all $l \in L$. Combining this with equations (6) and (7) it is enough to prove that $(\ (\liminf_{s\to\infty} \tilde{a}^k_{(h_\infty)^s})_{k\in K}, (\liminf_{s\to\infty} \tilde{b}^l_{(h_\infty)^s})_{l\in L}, p_{h_\infty}, q_{h_\infty}) \in F$. F is closed thus it is sufficient to show that $(\tilde{a}_{(h_\infty)^s}, \tilde{b}_{(h_\infty)^s}, p_{(h_\infty)^s}, q_{(h_\infty)^s}) \in F$. Denote $(h_\infty)^s$ by h_s and let

$$\hat{a}^k_{h_s,h_T} := \sum_{l\in L} q_{h_s}(l) A^{k,l}(h_T) \quad \text{for all } k \in K$$

and

$$\hat{b}^l_{h_s,h_T} := \sum_{k\in K} p_{h_s}(k) B^{k,l}(h_T) \quad \text{for all } l \in L$$

$(\hat{a}_{h_s,h_T}, \hat{b}_{h_s,h_T}) \in conv(F_{p_{h_s},q_{h_s}})$. $conv(F_{p_{h_s},q_{h_s}})$ is convex hence

$$\sum_{h_T\in H_T} P(h_T \mid h_s)(\hat{a}_{h_s,h_T}, \hat{b}_{h_s,h_T}) \in conv(F_{p_{h_s},q_{h_s}})$$

hence

$$(\sum_{h_T\in H_T} P(h_T \mid h_s)\hat{a}_{h_s,h_T}, \sum_{h_T\in H_T} P(h_T \mid h_s)\hat{b}_{h_s,h_T}, p_{h_s}, q_{h_s}) \in F$$

hence (equation 8)

$$((\tilde{E}_k(a_T \mid h_s))_{k\in K}, (\tilde{E}_l(b_T \mid h_s)_{l\in L}, p_{h_s}, q_{h_s}) \in F$$

Using again the fact that F is closed we get $(\tilde{a}_{h_s}, \tilde{b}_{h_s}, p_{h_s}, q_{h_s}) \in F$. ∎

Lemma 4.11: *$(c_\infty, d_\infty, p_\infty, q_\infty) \in NR^+$ P-a.s.*

Proof: Lemmas 4.10, 4.8, and 2.8. ∎

Now we get that $\{(c_t, d_t, r_t, s_t)\}_{t=0}^\infty$ is an admissible martingale starting at (a, b, p, q): Condition 1 (of definition 3.3) follows from lemma 4.2 and condition 2 follows from corollary 4.4. Condition 3 follows from lemma 4.2 and the fact that player 2 actions have no influence on p_t and player 1 does not affect q_t. Condition 4 follows from lemma 4.11. This ends the proof of the first part of the theorem.

5 From admissible martingales to equilibria

Now let $\Gamma_\infty(p,q)$ be a tight game and assume that $\{(c_t,d_t,r_t,s_t)\}_{t=0}^{\infty}$ is an admissible martingale starting at (a,b,p,q) and converging to NR^+. We will build σ and τ, equilibrium strategies in $\Gamma_\infty(p,q)$, such that a will be the expected vector payoff for player 1, and b for player 2.

Lemma 5.1:
If there exists an admissible martingale starting at (a,b,p,q), then there exists an exact admissible martingale starting at (a,b,p,q)

Proof: See lemma 3.27 in (Amitai 1996). ▮

Using lemma 5.1 we will assume that the martingale is exact. Let $x_t := (c_t,d_t,r_t,s_t)$. Let $f^t \in \mathcal{F}_t$. There is an exact split of $E(X_t \mid f^t)$, $S = (\{E(x_{t+1} \mid f^t_{u,v})\}_{1\le u\le|I|,1\le v\le|J|}, \mu_{f^t}, \lambda_{f^t})$. $\sum_{1\le u\le|I|,1\le v\le|J|} E(f^t_{u,v} \mid f^t) = 1$, therefore if $E(f^{t+1} \mid f^t) > 0$ then $f^{t+1} \in \{ f^t_{u,v} \mid 1 \le u \le |I|, 1 \le v \le |J| \}$. $\mathcal{F}_{t+1} \supset \mathcal{F}_t$, hence for all $f^{t+1} \in \mathcal{F}_{t+1}$ such that $P(f^{t+1}) > 0$ there exists a unique $f^t \in \mathcal{F}_t$ such that $E(f^{t+1} \mid f^t) > 0$, and therefore $f^{t+1} = f^t_{u,v}$ for some $(u,v) \in \{1,2,...,|I|\} \times \{1,2,...,|J|\}$. From the last two facts we can conclude that to every $f^t \in \mathcal{F}_t$, such that $P(f^t) > 0$, there corresponds a unique sequence from $(I \times J)^t$. This map is one-to-one, since the martingale is exact. Hence to every $f^t \in \mathcal{F}_t$, such that $p(f^t) > 0$ there corresponds an history $h_t \in H_t$. Denote by f_{h_t} the $f^t \in \mathcal{F}_t$ corresponding to h_t. We will write h_t instead of f_{h_t}. We will write $\mu_{h_t}, \lambda_{h_t}, c_{h_t}, d_{h_t}, r_{h_t}$ and s_{h_t} instead of $\mu_{f_{h_t}}, \lambda_{f_{h_t}}, c_{f_{h_t}}, d_{f_{h_t}}, r_{f_{h_t}}$ and $s_{f_{h_t}}$ respectively.

For all $t \in \mathbb{N}$ and $l \in L$ define $\alpha_t := Z \cdot \sum_{l\in L} E(|s_\infty(l) - s_t(l)| \,|f_t)$.

Lemma 5.2: $\lim_{t\to\infty} \alpha_t = 0$ *P-a.s.*

Proof: $s_t(l)$ converges P-a.s. to $s_\infty(l)$ for all $k \in K$. Hence $|s_\infty(l) - s_t(l)|$ converges P-a.s. to 0 and $\lim_{t\to\infty} \alpha_t = 0$ P-a.s. ▮

Lemma 5.3: $(c_t,d_t,r_t,s_t) \in IR$

Proof: $(c_\infty,d_\infty,r_\infty,s_\infty) \in IR$ a.s. (definition 3.3 and lemma 2.8), IR is convex (lemma 2.7), and $\{(c_t,d_t,r_t,s_t)\}_{t\in\mathbb{N}}$ is an exact martingale (i.e, $P(h_t) > 0$ for all h_t). ▮

Corollary 5.4: *For all t there exists $\tilde\sigma_t \in \Sigma^1$ such that for all $\tau \in \Sigma^2$ and $l \in L$ there exists:*

$$\limsup_{T\to\infty} E^{\cdot l}_{\tilde\sigma_t,\tau,r_t}(b_T) \le b^l_t$$

For all t there exists $\tilde\tau_t \in \Sigma^2$ such that for all $\sigma \in \Sigma^1$ and $k \in K$ there exists:

$$\limsup_{T\to\infty} E^{k\cdot}_{\sigma,\tilde\tau_t,s_t}(a_T) \le a^k_t$$

Corollary 5.5: $w^{r_t,s_t}_{c_t,d_t}$ *(see definition 2.9) always exists.*

$(c_\infty,d_\infty,r_\infty,s_\infty) \in F^+$ a.s. (definition 3.3 and lemma 2.8), hence there exists a random variable $w_\infty \in \Delta(I \times J)$, measurable with respect to P, such that P-a.s there exist: $\sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty) \le c^k_\infty$, $\sum_{k\in K} r_\infty(k) B^{k,l}(w_\infty) \le d^l_\infty$, $r_\infty(k) > 0$ implies $\sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty) = c^k_\infty$ and $s_\infty(l) > 0$

implies $\sum_{k\in K} r_\infty(k)B^{k,l}(w_\infty) = d^l_\infty$. Define $w_t := E(w_\infty \mid \mathcal{F}_t)$. $\{w_t\}_{t=0}^\infty$ is a bounded martingale, therefore it converges a.s. to w_∞. Define $z_t := ||w_t - w^{r_t,s_t}_{c_t,d_t}||_1 = \sum_{i\in I, j\in J} |w_t(i,j) - w^{r_t,s_t}_{c_t,d_t}(i,j)|$ and

$$\tilde{w}_t := \begin{cases} w^{r_t,s_t}_{c_t,d_t} & \alpha_t \geq \frac{1}{t} \\ w_t & \text{otherwise} \end{cases}$$

For every $w \in \Delta(I\times J)$ and $n \in \mathbb{N}$, we choose $\{\beta^n_w(u)\}_{u\in I\times J}$ such that:

1. $\beta^n_w(u)$ is a non-negative integer for all $u \in I \times J$.
2. $\sum_{u\in I\times J} \beta^n_w(u) = n$.
3. $|w(u) - \frac{\beta^n_w(u)}{n}| \leq \frac{1}{n}$ for all $u \in I\times J$.

It can be done by fixing an order on $I\times J$ and choosing

$$\beta^n_w(u) := \begin{cases} \lfloor n\cdot w(u)\rfloor & \text{if } \frac{1}{n}\sum_{u'<u}\beta^n_w(u') \geq \sum_{u'<u} w(u') \\ \lfloor n\cdot w(u)\rfloor + 1 & \text{otherwise} \end{cases}$$

For all $t\in\mathbb{N}$ define $w'_t, \tilde{w}'_t \in \Delta(I\times J)$ by $w'_t(u) := \frac{\beta^t_{w_t}(u)}{t}$ and $\tilde{w}'_t(u) := \frac{\beta^t_{\tilde{w}_t}(u)}{t}$. Denote by χ_D the characteristic function of the set D (i.e, $\chi(x) = 1$ if $x\in D$ and $\chi(x) = 0$ otherwise).

$$|A^{k,l}(w_t) - A^{k,l}(w'_t)| \leq \sum_{u\in I\times J} |w_t(u) - w'_t(u)|\cdot|A^{k,l}(u)| < \frac{Z|I\times J|}{t} \tag{9}$$

$$|A^{k,l}(w_t) - A^{k,l}(\tilde{w}'_t)| \leq \sum_{u\in I\times J} |w_t(u) - \tilde{w}'_t(u)|\cdot|A^{k,l}(u)|$$

$$\leq \sum_{u\in I\times J} |w_t(u) - \tilde{w}_t(u)|\cdot|A^{k,l}(u)| + \sum_{u\in I\times J} |\tilde{w}_t(u) - \tilde{w}'_t(u)|\cdot|A^{k,l}(u)|$$

$$< 2Z\cdot\chi_{\{\alpha_t\geq\frac{1}{t}\}} + \frac{Z|I|\cdot|J|}{t}$$

Hence

$$|A^{k,l}(w_t) - A^{k,l}(\tilde{w}'_t)| < \frac{Z|I|\cdot|J|}{t} + 2Z\cdot\chi_{\{\alpha_t\geq\frac{1}{t}\}} \tag{10}$$

For all $w\in\Delta(I\times J)$ and $n\in\mathbb{N}$ fix a function $\gamma_{w,n} : [n] \to I\times J$ such that $|\gamma^{-1}_{w,n}(u)| = \beta^n_w(u)$.

In order to define σ and τ, we define communication periods (in which the players play according to the martingale) and payoff periods. Between the n^{th} and the $(n+1)^{th}$ communication periods will be $n!$ payoff periods (hence the payoffs in the communication periods has no influence on the limit of means of the payoffs of the game). The communication takes place in the periods $1, 3, 6, 13, 38, ..., n + \sum_{n'<n} n'!, ...$. Denote $g(n) := n + \sum_{n'<n} n'!$. Thus $g(n)$ is the period in which the n^{th} communication period takes place. Let $COMMUN$ be the set of communication periods, that is $COMMUN := \{1, 3, 6, 13, 38, ..., n + \sum_{n'<n} n'!, ...\}$. For every $t\in\mathbb{N}$ let $COM(t)$ be the number of communication periods not exceeding t, i.e,

$$COM(t) := \min\{u\in\mathbb{N} \mid g(u) \leq t\}$$

let $PAY(t) := t - g(COM(t))$. For each history h_t, define $h'_t \in H_{COM(t)}$, the history reduced to the communication periods.

$$h'_t(u) := h_t(g(u)) \text{ for all } 1 \leq u \leq COM(t)$$

Now we can define the strategies σ and τ. As long as no deviation has been detected let

$$\sigma(h_t,k)(i) := \begin{cases} \mu_{h'_t}(i)\frac{r_{(h'_t,(i,j))}(k)}{r_{h'_t}(k)} & t+1 \in COMMUN \text{ and } r_{h'_t}(k) > 0 \\ \frac{1}{|I|} & t+1 \in COMMUN \text{ and } r_{h'_t}(k) = 0 \\ \gamma_{\tilde{w}_{h'_t},COM(t)}(PAY(t) \bmod COM(t))(i) & t+1 \notin COMUUN \\ & \text{and no deviation has been detected} \end{cases}$$

If a deviation has first been detected after the history h_t then player 1 switches to playing the strategy $\tilde{\sigma}_{COM(t)}$ (defined in corollary 5.4). $\sigma(h_t,k)(i)$ is well defined because $r_{(h'_t,(i,j))}$ is constant $\forall j \in J$. As long as no deviation has been detected let

$$\tau(h_t,l)(j) := \begin{cases} \lambda_{h'_t}(j)\frac{s_{(h'_t,(i,j))}(l)}{s_{h'_t}(l)} & t+1 \in COMMUN \text{ and } s_{h'_t}(l) > 0 \\ \frac{1}{|J|} & t+1 \in COMMUN \text{ and } s_{h'_t}(l) = 0 \\ \gamma_{\tilde{w}_{h'_t},COM(t)}(PAY(t) \bmod COM(t))(j) & t+1 \notin COMMUN \\ & \text{and no deviation has been detected} \end{cases}$$

If a deviation has first been detected after the history h_t then player 2 switches to playing the strategy $\tilde{\tau}_{COM(t)}$.

Denote p_{h_t} and q_{h_t} by $p_{h_t}(k) := P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h_t)$ and $q_{h_t}(l) := P_{\sigma,\tau,p,q}(\mathbf{l} = l \mid h_t)$. We will prove that σ and τ are equilibrium strategies with vector payoffs a and b. We need a few technical lemmas. recall that h'_t is the partial history $((i_1,j_1),(i_3,j_3),...,(i_{g(COM(t))},j_{g(COM(t)})))$. For every h_t consistent with σ and τ define $p_{h'_t}$ and $q_{h'_t}$ by $p_{h'_t}(k) := P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h'_t)$ and $q_{h'_t}(l) := P_{\sigma,\tau,p,q}(\mathbf{l} = l \mid h'_t)$.

Lemma 5.6:

1. *$P_{\sigma,\tau,p,q}(h'_t) = P(f_{h'_t}) > 0$ for all $h'_t \in H_{COM(t)}$*
2. *$r_{h'_t} = p_{h'_t}$ for all $h'_t \in H_{COM(t)}$.*
3. *$s_{h'_t} = q_{h'_t}$ for all $h'_t \in H_{COM(t)}$.*

Note that if h_t is consistent with h'_t then $p_{h_t} = p_{h'_t}$ and $q_{h_t} = q_{h'_t}$.

Proof: Let $h_t \in H_t$ be the strategy consistent with h'_t (there exists only one such strategy). We will prove the lemma by induction. For $t = 0 : h'_0 = \phi$ and $P_{\sigma,\tau,p,q}(h'_0) = P(f_{h'_0}) = 1$. $r_{h'_0} = p = p_{h'_0}$ and $s_{h'_0} = q = q_{h'_0}$. Now we assume that 1, 2 and 3 are satisfied for h'_t and prove them for $(h'_t,(i,j))$. The proof of 3 is similar to the proof of 2, thus we will just prove 1 and 2. Without loss of generality assume that $t+1 \in COMMUN$

1.

$$P_{\sigma,\tau,p,q}(h'_t,(i,j)) = P_{\sigma,\tau,p,q}(h'_t)\sum_{k\in K} p_{h'_t}(k)\sigma(k,h_t)(i)\sum_{l\in L} q_{h'_t}(l)\tau(l,h_t)(j)$$

$$= P(f_{h'_t}) \sum_{k\in K} r_{h'_t}(k)\sigma(k,h_t)(i) \sum_{l\in L} s_{h'_t}(l)\tau(l,h_t)(j)$$

and from the definition of σ and τ

$$= P(f_{h'_t}) \sum_{k\in K} \mu_{h'_t}(i) r_{h'_t,(i,j)}(k) \sum_{l\in L} \lambda_{h'_t}(j) s_{h'_t,(i,j)}(l)$$

$$= P(f_{h_t})\mu_{h'_t}(i)\lambda_{h'_t}(j) = P(f_{h'_t,(i,j)})$$

$\mu_{h'_t}(i) > 0$, $\lambda_{h'_t}(j) > 0$ (the martingale is exact) and $P(f_{h'_t}) > 0$ (the induction hypothesis), thus we have $P(f_{h'_t,(i,j)}) > 0$.

2.

$$p_{h'_t,(i,j)}(k) = \frac{p_{h'_t}(k)\sigma(k,h_t)(i)}{\sum_{k'\in K} p_{h'_t}(k')\sigma(k',h_t)(i)} = \frac{r_{h'_t}(k)\sigma(k,h_t)(i)}{\sum_{k'\in K} r_{h'_t}(k')\sigma(k',h_t)(i)}$$

$$= \frac{\mu_{h'_t}(i) r_{h'_t,(i,j)}(k)}{\sum_{k'\in K} \mu_{h'_t}(i) r_{h'_t,(i,j)}(k')} = r_{h'_t,(i,j)}(k)$$

∎

Corollary 5.7:

1. *$\mu_{h'_{g(t)}}(i) = P_{\sigma,\tau,p,q}(i_{g(t+1)} = i \mid h'_{g(t)})$ for all $h'_{g(t)}$ and $i \in I$.*

2. *$\lambda_{h'_{g(t)}}(j) = P_{\sigma,\tau,p,q}(j_{g(t+1)} = j \mid h'_{g(t)})$ for all $h'_{g(t)}$ and $j \in J$.*

Proof: We will prove only the first part as the proof of the second part is similar. Let $h_{g(t+1)-1}$ be the history of length $g(t+1)-1$ that is consistent with $h'_{g(t)}$ (i.e, such that $h'_{g(t+1)-1} = h'_{g(t)}$ and such that no deviation has occurred during $h_{g(t+1)-1}$).

$$P_{\sigma,\tau,p,q}(i_{g(t+1)} = i \mid h_{g(t+1)-1}) = \sum_{k\in K} p_{h'_{g(t)}}(k)\sigma(k,h_{g(t+1)-1})(i) = \sum_{k\in K} r_{h'_{g(t)}}(k)\sigma(k,h_{g(t+1)-1})(i)$$

$$= \sum_{k\in K} \mu_{h'_{g(t)}}(i) r_{(h'_{g(t)},(i,j))}(k) = \mu_{h'_{g(t)}}(i) \sum_{k\in K} r_{(h'_{g(t)},(i,j))}(k) = \mu_{h'_{g(t)}}(i)$$

∎

Corollary 5.8: $\lim_{t\to\infty} E(\{\alpha_t > \frac{1}{t}\}) = 0$ *and* $\lim_{t\to\infty} E^{k\cdot}(\{\alpha_t > \frac{1}{t}\}) = 0$.

Proof: Lemma 5.2 and the fact that $P^{k\cdot}$ in absolutely continuous with respect to P. ∎

Lemma 5.9: *$\{r_t(k)c_t^k\}_{t=0}^{\infty}$ is a martingale w.r.t. P, converging a.s. to $r_\infty(k)c_\infty^k$ for every $k \in K$.*

Proof:

$$E(r_{t+1}(k)c_{t+1}^k \mid h_t) = \sum_{i\in I}\sum_{j\in J} P((h_t,(i,j)) \mid h_t) r_{(h_t,(i,j))}(k) c^k_{(h_t,(i,j))}$$

$$= \sum_{i\in I} \mu_{h_t}(i) \sum_{j\in J} \lambda_{h_t}(j) r_{(h_t,(i,j))}(k) c^k_{(h_t,(i,j))}$$

Fix $j' \in J$. Using the fact that $r_{(h_t,(i,j))}$ is independent of j we get

$$E(r_{t+1}(k)c^k_{t+1} \mid h_t) = \sum_{i\in I} \mu_{h_t}(i) r_{(h_t,(i,j'))}(k) \sum_{j\in J} \lambda_{h_t}(j) c^k_{(h_t,(i,j))} = c^k_{h_t} \sum_{i\in I} \mu_{h_t}(i) r_{(h_t,(i,j'))}(k)$$

$$= c^k_{h_t} r_{h_t}(k) = E(r_t(k) c^k_t \mid h_t)$$

r_t converges to r_∞ a.s. and c_t converges to c_∞ a.s. hence $r_t c_t$ converges to $r_\infty c_\infty$ a.s. ∎

Lemma 5.10: *Fix $k \in K$ such that $p(k) > 0$ (recall that we assume that $p(k) > 0$ for all $k \in K$). If v_t is a random variable measurable with respect to $(H_t, \mathcal{H}_t)$, then $E^{k\cdot}(v_t) = \frac{1}{p(k)} E_{\sigma,\tau,p,q}(p_t(k) v_t)$.*

Proof: From Bayes' rule we have

$$P_{\sigma,\tau,p,q}(h_t \mid \mathbf{k} = k) = \frac{P_{\sigma,\tau,p,q}(\mathbf{k} = k \mid h_t) P_{\sigma,\tau,p,q}(h_t)}{P_{\sigma,\tau,p,q}(\mathbf{k} = k)} = \frac{p_{h_t}(k) P_{\sigma,\tau,p,q}(h_t)}{p(k)}$$

hence

$$E^{k\cdot}(v_t) = E_{\sigma,\tau,p,q}(v_t \mid \mathbf{k} = k) = \sum_{h_t\in H_t} P_{\sigma,\tau,p,q}(h_t \mid \mathbf{k} = k) v_t(h_t) = \sum_{h_t\in H_t} \frac{p_{h_t}(k) P_{\sigma,\tau,p,q}(h_t)}{p(k)} v_t(h_t)$$

$$= \frac{1}{p(k)} \sum_{h_t\in H_t} P_{\sigma,\tau,p,q}(h_t) p_{h_t}(k) v_t(h_t) = \frac{1}{p(k)} E_{\sigma,\tau,p,q}(p_t(k) v_t)$$

∎

Lemma 5.11: $\lim_{t\to\infty} E(r_t(k) \sum_{l\in L} s_t(l) A^{k,l}(w_t)) = p(k) a^k$ *for all* $k \in K$.

Proof:
Fix $k \in K$. $r_\infty(k) \sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty) = r_\infty(k) c^k_\infty$ P-a.s. (see the definition of w_∞). Hence (lemma 5.9)

$$\lim_{t\to\infty} E(r_\infty(k) \sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty)) = \lim_{t\to\infty} E(r_\infty(k) c^k_\infty) = r_0(k) c^k_0 \tag{11}$$

On the other hand

$$E(\, |r_t(k) \sum_{l\in L} s_t(l) A^{k,l}(w_t) - r_\infty(k) \sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty)|\,)$$

$$\leq \sum_{l\in L} E(\, |r_t(k) - r_\infty(k)| \cdot |s_t(l) A^{k,l}(w_t)| + |s_t(l) - s_\infty(l)| \cdot |r_\infty(k) A^{k,l}(w_t)| + |A^{k,l}(w_t) - A^{k,l}(w_\infty)| \cdot |r_\infty(k) s_\infty(l)|\,)$$

$$\leq \sum_{l\in L} Z \cdot E|r_t(k) - r_\infty(k)| + Z \cdot E|s_t(l) - s_\infty(l)| + Z \cdot E(\| w_t - w_\infty \|_1)$$

therefore

$$\lim_{t\to\infty} |E(\, r_t(k) \sum_{l\in L} s_t(l) A^{k,l}(w_t) - r_\infty(k) \sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty)\,)| \leq$$

$$\lim_{t\to\infty} E(\ |r_t(k)\sum_{l\in L} s_t(l)A^{k,l}(w_t) - r_\infty(k)\sum_{l\in L} s_\infty(l)A^{k,l}(w_\infty)|\ )$$

$$\leq \sum_{l\in L} Z\cdot \lim_{t\to\infty}(\ E|r_t(k)-r_\infty(k)| + E|s_t(l)-s_\infty(l)| + E(\| w_t - w_\infty \|_1)\ ) = 0$$

hence

$$\lim_{t\to\infty} E(r_t(k)\sum_{l\in L} s_t(l)A^{k,l}(w_t)) = \lim_{t\to\infty} E(r_\infty(k)\sum_{l\in L} s_\infty(l)A^{k,l}(w_\infty))$$

because the right hand side exists (equation (11)) and (again from (11))

$$= r_0(k)c_0^k = p(k)a^k$$

∎

Lemma 5.12:

1. $\lim_{T\to\infty} E^{k\cdot}(a_T) = a^k$ *for every* $k \in K$.
2. $\lim_{T\to\infty} E^{\cdot l}(b_T) = b^l$ *for every* $l \in L$.

Proof: $P_{\sigma,\tau,p,q}(h_t) = P(h'_t)$ (P is the probability with respect to which the martingale is defined), $p_{h_t} = r_{h'_t}$ and $q_{h_t} = s_{h'_t}$ (see lemma 5.6). Let $T = g(n)-1$ ($n > 1$) and choose $h_T \in H_T$ consistent with h'_T. $E^{k\cdot}(a_T \mid h_T) = \frac{1}{T}\sum_{l\in L} q_{h_T}(l)\sum_{t=1}^{T} A^{k,l}(h_T(t)) = \frac{1}{T}\sum_{l\in L} q_{h'_T}(l)\sum_{t=1}^{T} A^{k,l}(h_T(t))$, hence

$$|E^{k\cdot}(a_T \mid h_T) - \sum_{l\in L} q'_{h_T}(l)A^{k,l}(w_{h'_T})| = |\sum_{l\in L} q_{h_T}(l)\frac{1}{T}\sum_{t=1}^{T} A^{k,l}(h_T(t)) - \sum_{l\in L} q_{h_T}(l)A^{k,l}(w_{h'_T})|$$

and from equation (10) we have

$$\leq \frac{1}{T}(\ g(n-1)2Z + (T-g(n-1)Z(\frac{|I|\cdot|J|}{n} + 2\cdot\chi_{\{\alpha_n \geq \frac{1}{n}\}}))\ )$$

$$\leq Z(\frac{2g(n-1)}{g(n)-1} + \frac{|I|\cdot|J|}{n} + 2\cdot\chi_{\{\alpha_n\geq\frac{1}{n}\}})$$

hence

$$|E^{k\cdot}(a_T) - E^{k\cdot}(\sum_{l\in L} q'_{h_T}(l)A^{k,l}(w_{h'_T})| \leq Z(\frac{2g(n-1)}{g(n)-1} + \frac{|I|\cdot|J|}{n} + 2E^{k\cdot}(\{\alpha_n \geq \frac{1}{n}\}))$$

$Z(\frac{2g(n-1)}{g(n)-1} + \frac{|I|\cdot|J|}{n} + 2E^{k\cdot}(\{\alpha_n \geq \frac{1}{n}\}))$ converges to 0 (corollary 5.8), $w_{h'_T}$ converge to w_∞ (also with respect to $P^{k\cdot}$ because $P^{k\cdot}$ is absolutely continuous with respect to $P_{\sigma,\tau,p,q}$), and $q_{h_t} = s_{h_t}$. Therefore (recall that $T := g(n)-1$)

$$lim_{n\to\infty}E^{k\cdot}(a_T) = lim_{n\to\infty}E^{k\cdot}(\sum_{l\in L} s_{h'_T}(l)A^{k,l}(w_{h'_T}))$$

$$= \frac{1}{p(k)}\lim_{t\to\infty} E_{\sigma,\tau,p,q}(\ p_{h'_T}(k)\sum_{l\in L} s_{h'_T}(l)A^{k,l}(w_{h'_T})\ ) \quad \text{(lemma 5.10)}$$

$$= \frac{1}{p(k)} \lim_{t\to\infty} E_{\sigma,\tau,p,q}(\ r_{h'_T}(k) \sum_{l\in L} s_{h'_T}(l) A^{k,l}(w_{h'_T})\ ) = a^k \qquad \text{(lemma 5.6 and lemma 5.11)}$$

n and $(n-1)!$ are negligible with respect to $n!$, therefore $\lim_{T\to\infty} E^{k\cdot}(a_t^k)$ exists and equals a^k. The proof for b is similar. ∎

Remark: by similar proof we have $\lim_{T\to\infty} E^{k\cdot}(a_T \mid h_s) = c^k_{h_s}$ for every $k \in K$ and h_s. Now we have to show that for every $\sigma' \in \Sigma^1$ and $k \in K$ there exists $\limsup_{T\to\infty} E_{\sigma',\tau,k,q}(a_T) \le a^k$ (the proof for player 2 is similar).

Lemma 5.13: $c_t^k \ge \sum_{l\in L} s_t(l) A^{k,l}(w_t) - \alpha_t$ *for all* $k \in K$ *and* $t \in \mathbb{N}$.

Proof: $c_t^k \ge E(\sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty) \mid h_t)$ (see the definition of w_∞), hence,

$$\sum_{l\in L} s_t(l) A^{k,l}(w_t) - c_t^k \le E(\sum_{l\in L} s_t(l) A^{k,l}(w_t) - \sum_{l\in L} s_\infty(l) A^{k,l}(w_\infty) \mid h_t)$$

$$= E(\sum_{l\in L} s_t(l) A^{k,l}(w_t - w_\infty) \mid h_t) + E(\sum_{l\in L}(s_t(l) - s_\infty(l)) A^{k,l}(w_\infty) \mid h_t)$$

$$\le 0 + Z \cdot E(\sum_{l\in L} |s_t(l) - s_\infty(l)| \mid h_t) \le \alpha_t$$

∎

Corollary 5.14: $c_t^k \ge \sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}'_t) - \frac{1}{t} - Z\frac{|I||J|}{t})$ *for all* $k \in K$ *and* $t \in \mathbb{N}$.

Proof: If $\alpha_t \ge \frac{1}{t}$ then $\sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}_t) = \sum_{l\in L} s_t(l) A^{k,l}(w^{r_t,s_t}_{c_t,d_t}) \le c_t^k$. and

$$c_t^k - \sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}'_t) = (c_t^k - \sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}_t)) + (\sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}_t) - \sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}'_t))$$

and from equation (9) we have

$$\ge -Z\frac{|I||J|}{t}$$

If $\alpha_t < \frac{1}{t}$ then $\tilde{w}_t = w_t$ and

$$c_t^k - \sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}'_t) = (c_t^k - \sum_{l\in L} s_t(l) A^{k,l}(w_t)) + (\sum_{l\in L} s_t(l) A^{k,l}(w_t) - \sum_{l\in L} s_t(l) A^{k,l}(w'_t))$$

using lemma 5.13 and equation (9) we have

$$c_t^k - \sum_{l\in L} s_t(l) A^{k,l}(\tilde{w}'_t) \ge -\alpha_t - Z\frac{|I||J|}{t} \ge -\frac{1}{t} - Z\frac{|I||J|}{t}$$

∎

Fix $\sigma' \in \Sigma^1$ with no detectable deviation with respect to σ (i.e, σ' defers from σ only in communication periods). Let $P' := P_{\sigma',\tau,p,q}$ and $E' := E_{\sigma',\tau,p,q}$. Let $P'^k(\cdot) := P'(\cdot \mid \mathbf{k} = k)$ and $E'^k(\cdot) := E'(\cdot \mid \mathbf{k} = k)$. Let $p'_t(k) := P'(\mathbf{k} = k \mid h_t)$. Let $\tilde{c}_{h_t} := c_{h'_t}$ and let $H'_t := \bigcup_{h_t \in H_t} h'_t$.

Lemma 5.15: $\{\tilde{c}^k_{g(t)}\}_{t\in\mathbb{N}}$ *is a* P'^k*-martingale with respect to the fields* $\{H'_{g(t)}\}_{t\in\mathbb{N}}$ *(which are isomorphic to* $\{H_t\}_{t\in\mathbb{N}}$*).*

Proof:

$$E'^k(\tilde{c}^k_{g(t+1)} \mid h'_t) = \sum_{i\in I}\sum_{j\in J} P'^k(i_{g(t+1)} = i, j_{g(t+1)} = j \mid h'_t)c^k_{(h'_t,(i,j))}$$
$$= \sum_{i\in I} P'^k(i_{g(t+1)} = i \mid h'_t) \sum_{j\in J} \lambda_{h'_t}(j)c^k_{(h'_t,(i,j))}$$

and from lemma 4.2

$$= \sum_{i\in I} P'^k(i_{g(t+1)} = i)c^k_{h'_t} = c^k_{h'_t} = \tilde{c}_{h_t} = E'^k(\tilde{c}^k_t \mid h'_t)$$

∎

Lemma 5.16: $\limsup_{T\to\infty} E'^k(a_T) \le a^k$ *for all* $k \in K$*. (see lemma* 5.12*)*

Recall that σ' has no detectable deviation.
Proof: Let $T = g(n) - 1$ $(n > 2)$ and let $T' = g(n-1)$.

$$E'^k(a_T \mid h_{T'}) \le \frac{1}{T}(T'Z + (T - T')\sum_{l\in L} s_{h'_{T'}}(l)A^{k,l}(\tilde{w}'_{h'_{T'}})$$

and from corollary 5.14 we have

$$E'^k(a_T \mid h_{T'}) \le \frac{T'}{T}Z + \frac{T-T'}{T}(c^k_{h'_{T'}} + \frac{1}{t} + Z\frac{|I||J|}{t})$$

hence

$$E'^k(a_T) \le \frac{T'}{T}Z + \frac{T-T'}{T}(E'^k(c^k_{h'_{T'}}) + \frac{1}{t} + Z\frac{|I||J|}{t})$$
$$= \frac{T'}{T}Z + \frac{T-T'}{T}(a^k + \frac{1}{t} + Z\frac{|I||J|}{t})$$

(lemma 5.15 and lemma 4.2 (b)) Therefore ($\frac{T'}{T} \to 0$)

$$\limsup_{n\to\infty} E'^k(a^k_{g(n)-1}) \le a^k$$

n and $(n-1)!$ are negligible with respect to $n!$, therefore

$$\limsup_{n\to\infty} E'^k(a^k_n) \le a^k$$

∎

Now choose $\sigma'' \in \Sigma^1$. Denote $P''^k := P^{k,\cdot}_{\sigma'',\tau,q}$ and $E''^k := E^{k,\cdot}_{\sigma'',\tau,q}$.

Lemma 5.17: *There exists* $\sigma' \in \Sigma^1$ *with no detectable deviation (with respect to* σ*) such that* $\limsup_{T\to\infty} E^{k,\cdot}_{\sigma',\tau,q}(a_T) \ge \limsup_{T\to\infty} E''^k(a_T)$.