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COMPLEXITY OF OPTIMAL LOBBYING IN THRESHOLD AGGREGATION

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Complexity of Optimal Lobbying in Threshold Aggregation

Monotone Desired Outcomes Sets

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Abstract. This work studies the computational complexity of Optimal Lobbying under Threshold Aggregation. Optimal Lobbying is the problem a lobbyist or a campaign manager faces in a voting scenario of a multi-issue referendum when trying to influence the result. The Lobby is faced with a profile that specifies for each voter and each issue whether the voter approves or rejects the issue, and seeks to find the smallest set of voters it can influence to change their vote, for a desired outcome to be obtained. This problem also describes problems arising in other scenarios of aggregation, such as principal-agents incentives scheme in a complex combinatorial problem, and bribery in Truth-Functional Judgement Aggregation. We study cases when the issues are aggregated by a threshold aggregator, that is, an anonymous monotone function, and the desired outcomes set is upward-closed. We analyze this problem with regard to two parameters: the minimal number of supporters needed to pass an issue, and the size of the maximal minterm of the desired set. For these parameters we separate tractable cases from untractable cases and in that generalize the *NP-complete* result of Christian et al. [8]. We show that for the extreme values of the parameters, the problem is solvable in polynomial time, and provide algorithms. On the other hand, we prove the problem is not solvable in polynomial time for the non-extremal values, which are common values for the parameters.

Keywords: Optimal Lobbying, Threshold Function, Computational Complexity

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1 Introduction

This paper studies the problem of Optimal Lobbying in multi-issue elections. In *Multi-issue Elections* n voters are voting on m issues. Each voter i declares his Boolean position on each issue j: x_i^j. The outcome on each issue j, o^j, is decided by aggregating the votes on the issue using some *aggregation function* φ: $o^j = \varphi\left(x_1^j, \ldots, x_n^j\right)$. The *Optimal Lobbying* problem formalizes the challenge that faces an outside entity (*the Lobby*) that desires to affect the outcome of the vote and can do so by changing the votes of some of the voters, but at a cost. Formally, given the *profile* $\left(x_i^j\right)$ stating for each voter his vote for each of the issues, the Lobby's goal is to find the minimal set K of voters such that changing the votes $\left(x_i^1 \cdots x_i^m\right)$ of voters $i \in K$ results in some desired outcome, where the *desired outcomes set* is captured by its indicator function $\psi\left(o^1, \ldots, o^m\right)$.

This model captures many voting scenarios, e.g., voting on a series of clauses of a bill in the parliament or elections using one ballot for several positions and decisions. The budget constraint on the number of voters ($|K|$) captures that the Lobby may need to compensate voters for the change or the need to invest time and money in personalized advertising. As we discuss in Section 2.1, Optimal Lobbying also models other problems in scenarios of aggregating complex opinions.

Clearly the difficulty of lobbying depends on both the aggregation function φ and on the desired outcomes function ψ. A natural example is where aggregation is done by simple majority vote and the desired outcome is defined by unanimity (i.e., the Lobby wants to achieve a majority on *all* issues). This scenario was studied by Christian et al. [8] who showed that the problem is *NP*-hard. This was generalized by Bredereck et al. [6] who showed this problem is *NP*-hard even under some (extreme) input constraints.[1] On the other hand, it is easy to verify that for some aggregation rules and desired outcomes sets the problem is easy. For example, if we use unanimity for aggregation and the desired outcomes set is also defined by unanimity, then it is easy to find (by a greedy algorithm) the minimal set of voters to influence. The problem is also easy when issues are aggregated using majority and the Lobby wants at least one issue to pass (i.e., the Lobby wants to achieve a majority on *any* issue). In many real-life situations one finds non-majority issue-aggregation functions (e.g., when approval of two-thirds of the voters is needed to rule against the status quo) or desired outcomes sets consisting of more than one outcome (e.g, when there is a trade-off in the eyes of the Lobby between several issues or issues combinations), hence there is a place to extend the study of Optimal Lobbying to these cases as well.

In this paper, we mainly study the computational complexity of Optimal Lobbying for the following natural families of aggregation functions and of desired outcomes sets: The aggregation function is an anonymous and monotone function, that is, an issue passes if at least t voters approve it, for a predefined

[1] When each voter approves at most three issues and the budget is $\left(\left\lceil\frac{n+1}{2}\right\rceil - 1\right)$, one less than the required majority threshold.

threshold t;[2] And the desired outcomes set is an upward-closed set, i.e., if x is a desired outcome, and all the issues that pass in x, also pass in y, then y is a desired outcome too.

In Section 5 (and more so in a subsequent work), we show a more general analysis for cases in which the assumptions do not hold. We present a systematic study of which combinations of an aggregation method and a desired outcomes set allow efficient lobbying, and which give rise to a computationally hard problem. This study generalizes all previously known computational complexity results in this setting. It turns out that the complexity of the lobbying problem hinges mainly on two parameters:

1. t - The *Threshold of Aggregation* – the minimal number of votes that is needed to pass an issue.
2. z - The *Maximal minterm size* of the desired outcomes set – that is, the maximal size of a minimal inclusion-wise desired issues set. Equivalently, the minterms are the terms in the minimal DNF form of the function ψ.[3]

We essentially show that the problem is tractable if and only if either of these two parameters is *bounded by a constant*. When both of them are at least *polynomial* in the input size (i.e., $(mn)^c$ for some $0 < c < 1$) then the problem becomes *NP*-hard. This is true for both the decision and search variants of the Optimal Lobbying problem.

We show two dichotomy theorems. One for the *unanimity case* in which the issues are aggregated using the unanimity function (that is, an issue passes if and only if all voters support it), and the second for the *non-unanimity* case. It is especially interesting to note the sharp threshold phenomenon between the unanimity case and the *almost-unanimity*. For instance, if the Lobby allows at most one issue to fail, if the issues are aggregated using unanimity we show the problem to be solvable in polynomial time, but if the issues are aggregated using almost-unanimity (even in the extreme case in which at most one voter is allowed to vote against without causing the issue to be rejected) we show the problem to be *NP-complete*.

Throughout this paper we use asymptotic notions for boundaries on the parameters t and z. E.g., we say they are *bounded* when there is a constant independent of n and m bounding them from above, *super-constant* when there is no such constant, and we say they are *polynomial* when there is a polynomial in n and m bounding them (either from above or below). In cases where the notion is not clear enough, we add a formal definition of the bounds.

Theorem 1 (For the full formal statement see Thm. 5, 11). *Let the aggregation function be unanimity and the desired outcomes be the outcomes in which at least z issues pass.*

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[2] In the Judgement Aggregation literature, such voting method is also called *Uniform Quota Rule* [10].

[3] For example, the minterms of the set represented by $\psi = \left(x^1 \wedge x^2\right) \vee \left(x^2 \wedge x^3 \wedge x^4\right)$ are 1100 and 0111, and the minterms of the set of majorities for $2k-1$ issues are all the issues sets of size k.

– *If z or $(m - z)$ is bounded,* Optimal Lobbying *can be solved in polynomial time.*
– *If both z and $(m - z)$ are polynomial (i.e., $z \in [m^\epsilon,\ m - m^\epsilon]$ for some $\epsilon > 0$), then* Optimal Lobbying *is NP-complete.*
– *There exists a constant $\alpha^\star > 1$ s.t. if both z and $(m - z)$ are poly-logarithmic of degree $\alpha^\star$ ($z \in \left[(\log m)^{\alpha^\star},\ m - (\log m)^{\alpha^\star}\right]$), then* Optimal Lobbying *is not in P, assuming ETH.*[4]

Theorem 2 (For the full formal statement see Thm. 3, 4, 6). *Let the aggregation function be the threshold function with threshold $t < n$ and let z be the maximal size of a minimal desired outcome (inclusion-wise) .*

– *If t or z is bounded,* Optimal Lobbying *can be solved in polynomial time.*
– *If both t and z are polynomial (i.e., $t \geqslant n^\epsilon$ and $z \geqslant m^\epsilon$ for some $\epsilon > 0$), then* Optimal Lobbying *is NP-complete.*
– *There exists a constant $\alpha^\star > 1$ s.t. if both t and z are poly-logarithmic of degree $\alpha^\star$ (i.e., $t \geqslant (\log n)^{\alpha^\star}$ and $z \geqslant (\log m)^{\alpha^\star}$), then* Optimal Lobbying *is not in P, assuming ETH.*[4]

2 The Optimal Lobbying Problem

The problem Optimal Lobbying models a society of n voters – $[n] = \{1, 2, \dots, n\}$ – that decides on m Boolean issues using a voting method, and a lobbyist that desires to influence the decision to be in a set desired by it. It consists of a profile, a voting method, and a desired outcomes set.

Profile: The profile defines the vote of each of the voters on each of the issues. We model it by a Boolean matrix $X \in \{0,1\}^{n \times m}$, where $\mathbf{n}$ is the number of *voters* and $\mathbf{m}$ is the number of *issues*. That is, an entry X_i^j denotes the vote of the $i^{\underline{th}}$ voter for the $j^{\underline{th}}$ issue and we consider 1 as an acceptance vote and 0 as a rejection vote. Throughout this paper, we use superscript notation when indexing issues and subscript notation when indexing voters.

Voting Method: The voting method used to aggregate the votes for a specific issue into an aggregated accept/reject opinion. It is defined by a function $\varphi\colon \{0,1\}^n \to \{0,1\}$ (applied on each of the issues – the columns of the matrix X).

Desired Outcomes: We model the Lobby as having a *dichotomous preferences*, that is, each outcome is either desired or undesired. We model the preference using a Boolean function $\psi\colon \{0,1\}^m \to \{0,1\}$ returning for each outcome (a vector of length m), whether it is desired. We use propositional formulas over $x^1, \dots, x^m$ to describe ψ, e.g., the desired outcomes set defined by $\psi = x^1 \vee \left(x^2 \wedge x^4\right)$ is all the outcomes in which either the first issue passes or both the second and forth issues pass.

[4] Exponential Time Hypothesis (ETH) [21]: 3-SAT cannot be solved in time less than $2^{\delta n}$ for some $\delta > 0$.

In Section 5 we discuss the more general case of a (non-Boolean) utility function for the Lobby.

Formally, the problem the Lobby is facing, while knowing the voting method φ and the desired outcomes set ψ, is modeled by the following optimization problem:

Problem (OL(φ,ψ) – Optimal Lobbying with voting method φ and desired outcomes set ψ).

INSTANCE: A VOTING PROFILE $X \in \{0,1\}^{n\times m}$.

TASK: FIND A COALITION OF VOTERS C OF MINIMAL SIZE AND A VOTING PROFILE $Y \in \{0,1\}^{n\times m}$ THAT DIFFERS FROM X ONLY FOR THE ROWS (VOTERS) IN C S.T. $\psi\left(\varphi\left(Y^1\right),\ldots,\varphi\left(Y^m\right)\right) = 1$ (Y^j BEING THE $j^{\underline{th}}$ COLUMN OF Y).

We define the corresponding decision problem to be:

Problem (OLD(φ,ψ) – Optimal Lobbying Decision Problem).

INSTANCE: A VOTING PROFILE $X \in \{0,1\}^{n\times m}$.
A BUDGET $k \leqslant n$.

QUESTION: CAN THE LOBBY CHANGE THE VOTES OF AT MOST k VOTERS TO GET A VOTING PROFILE $Y \in \{0,1\}^{n\times m}$ S.T. $\psi\left(\varphi\left(Y^1\right),\ldots,\varphi\left(Y^m\right)\right) = 1$?

φ and ψ are not part of the input but are parameters of the problem, assuming oracle access to them. This way we circumvent the question of representation compactness of these parameters. For instance, in the characterization theorems and in the proofs, we refer to the DNF representation of ψ and in particular to the maximal minterm in it, while not assuming that ψ is given in DNF form, that this form is compact, or that it is easy to compute the maximal minterm. The common cases of Optimal Lobbying usually consists of functions that are polynomially computable and representable, so results of similar flavor to the ones we show are generated naturally from this paper. We discuss further below the reasons we think this is the right way to model and analyze the problem.

In this paper, we study the computational complexity of OLD(φ,ψ) when ψ is an upward-closed desired outcomes set and φ is an anonymous monotone function. A family of sets is upward-closed if for any two sets of issues $A \subseteq B$, if A is in the family then so is B. Note that upward-closedness of the desired outcomes set is equivalent to monotonicity of its indicator function. We note that an anonymous monotone function φ can be equivalently defined by the threshold t defined by: φ returns 1 if and only if at least t out of the n voters approved the issue.[5] Two special cases are the *unanimity* functions, Unan_n, which are

[5] As a referee commented, t might also be a subjective threshold of the Lobby and not the objective threshold of the voting method. E.g., in a scenario in which there are several competing lobbyists, the Lobby will want to put together a non-minimal coalition of voters so that it is tougher for the competing lobbyist to break this coalition. In such case, t would be the subjective threshold that includes also this safety margins.

the threshold aggregation functions with threshold that is equal to the number of voters ($t = n$), and the *majority* functions, Maj_n, which are the threshold aggregation functions with threshold that is equal to a majority $\left(t = \lceil \frac{n}{2} \rceil\right)$.

In this case, $\text{OLD}(\varphi, \psi)$ is equivalent to the following combinatorial problem.

Problem ($\textbf{OLD}(\textbf{Threshold}_t, \psi)$ – with threshold t and monotone ψ).

<u>INSTANCE</u>: Sets $X^1, \ldots, X^m \subseteq [n]$.
A positive integer $k \leqslant n$.

<u>QUESTION</u>:[6] Is there a set $B \subseteq [n]$ of size k s.t. $\psi\left(\left\{j \mid \left|X^j \cup B\right| \geqslant t\right\}\right) = 1$?

The two problems are equivalent because w.l.o.g. an influenced voter contributes the most when approving all issues.

Modeling φ and ψ It might seem more natural to model the aggregation method φ and the desired outcomes set ψ as part of the input. We claim that both for modeling reasons and for analysis reasons, having them as parameters of the problem while assuming an oracle access to them is a better way.

- Since the lobby knows the aggregation method and clearly it knows its goal, it should not be part of the input it gets online. E.g., it can pre-process them for faster access or process them once before encountering several profiles over time.
- We would like to study which are the hard cases for this problem, in a similar sense to FPT analysis of Parameterized Complexity (φ and ψ being the parameters). Clearly, in cases in which these parameters are compactly represented and partially known in advance (e.g., threshold functions), analyzing the worst-case becomes finding whether there are hard 'slices' of these two parameters. For example, when the two functions can be any function, the problem is *NP-complete* since Christian et al. [8] showed this problem is hard for φ being majority and requiring all issues to pass.
- In addition, that way we study the inherent complexity in Optimal Lobbying beyond the complexity that lies in dealing with φ and ψ. When either φ or ψ are undecidable functions, the problem of calculating the desirability of the input profile's outcome, which is deciding whether influencing is needed, is undecidable. Similar problem arises when these functions are hard to compute. In this paper we want to study complexity issues that are inherent to Optimal Lobbying.

2.1 Motivation

We find the motivation for analyzing $\text{OLD}(\varphi, \psi)$ in several fields.

<u>Optimal Lobbying</u>: As shown in the introduction, this is a generalization of the Optimal Lobbying problem defined by Christian et al. [8]. This problem models a variety of situations in which the opinions of several voters are

[6] By a slight abuse of notation, we regard the domain of φ as $P([n])$, the power set of $[n]$.

aggregated and we wish to analyze the complexity of the problem a lobbyist or a campaign manager faces. For the ease of the story, we have in mind a human Lobby, but it is reasonable to imagine in a complex voting procedure that the lobbying task is delegated to a computerized agent needing to find the best way to get a desired result while the budget is a cost of negotiating with the different agents/voters.

Incentivizing in a Complex Combinatorial Project: The OPTIMAL LOBBYING problem can also be interpreted as the problem of a principal to incentivize the minimal number of workers in a complex project. A principal is interested in the success of a meta-project that is composed of m independent projects. The success of each of the projects depends on the effort exerted by a group of workers; i.e., there is a known technology function φ that, given the set of workers who exerted effort, returns whether the project succeeds. In addition, there is a production function ψ that, given the projects that succeeded, returns whether the meta-project succeeds.

For each of the workers, the principal knows in which projects he will exert effort "naturally" (for example, the projects that are close to him, are easy for him, or in which his effort is monitored). The principal can choose to incentivize a worker to exert effort (in some or all projects) in a costly way (e.g., offer a monetary payment to the worker or use a device to monitor his effort). Therefore, the principal faces a trade-off between the success of the meta-project and the cost of incentivizing the workers.

Finding the minimal set of workers that will cause the meta-project to succeed by exerting effort is equivalent to $\mathrm{OL}(\varphi, \psi)$. The problem of finding an incentives scheme in simple projects was presented and studied by (among others) Holmstrom [20] and Babaioff et al. [1] so this work is a generalization of these works. In this day and age, when huge complex projects can be run on the network in a distributed manner, e.g., on Mechanical Turk, the coordination is done by an agent that seeks to find the best incentivization scheme in order to maximize the probability of success.

Bribery: Bribery problems in Judgement Aggregation framework deal with the problem of finding the best group of voters whom one should cause to change their vote towards a preferred outcome, given an aggregation method and a restriction on the votes, e.g., preference aggregation.[7] The time complexity of bribery has been studied extensively in the framework of voting (e.g., [11,13,14]), which is a variant of preference aggregation. OPTIMAL LOBBYING can be defined as Bribery for Truth-Functional Agendas[8] [23] where the preference of the briber is on one of the conclusions issues. Optimal Lobbying and Preference aggregation are the only two agenda families for which the bribery problem was defined

[7] For introduction to the field of Judgement Aggregation, one can read [22, 23, 24].

[8] In a *truth-functional agenda*, in addition to the unconstrained issues (the premises), there are *conclusion* issues. Each conclusion j is characterized by a Boolean function α_j over the premises and a vote is legitimate if the vote on a conclusion issue is consistent with applying the function α_j on the vote on the premise issues. I.e., $\{x \in \{0,1\}^m \mid x^j = \alpha_j(\text{premises}) \text{ for every conclusion issue } j\}$.

(Baumeister et al. [5] define a problem of bribery for judgement aggregation, but in their definition the agenda is part of the input and they don't show which agendas are hard to bribe). We see Optimal Lobbying as a step toward studying bribery in the more general framework of Judgement Aggregation.

Computational Complexity Theory: The definition of Optimal Lobbying seems to be a minor tweak of classic *NP-complete* problems like Hitting Set (Problem 3), Set Cover [18, Problem SP5], and Set Multi-Cover [27, p. 112], and vice versa. Yet, we did not find an embedding between Optimal Lobbying and neither of them (the complexity of $\text{OLD}\left(\text{Maj}_n, \bigwedge_{j=1}^{m} x^j\right)$ is not derived trivially, as far as we found, from the complexity of these problems). We think that the tweak of *majority constraints* (the constraints being to cover a majority of the items), albeit it looks small, changes the problem dramatically. To the best of our knowledge, the literature of computational complexity did not deal with such majoritized versions of the classic problems. Hence, we think that these results might be of independent interest, and that such majoritized variants of classic problems should be explored further, studying the impact of this change on the complexity. We hope it will add a new trait of interesting problems and contribute to the study of complexity theory.

Why Analyze Computational Complexity of Social Choice Problems? There is a long strand of works analyzing the computational hardness of problems in Social Choice, specifically of possible attacks (manipulation, bribery, control, etc.), and this work joins this strand. Yet, we thought there is place to detail the value we find in such works, both as a practical barrier for a possible attacker (the Lobby in our case) and as results showing what cannot be proved (barrier to the theoretician).

The literature on computation hardness as a **barrier for manipulation in elections** started in the late 80s and early 90s by Bartholdi, Tovey and Trick [3,4] and Bartholdi and Orlin [2]. They defined the property of a voting rule being *computationally resistant* as the *NP*-hardness of the problem a manipulator is facing. The intuition behind this definition is that a manipulator, when faced with this *NP*-hard problem, will prefer not to manipulate but to submit his true vote. This line of thought was continued for other forms of attacks (e.g., bribery and group manipulation) and voting scenarios (e.g., multi-winner elections); For a survey on these works, see [16] and [15]. This intuition is also supported by experiments like the ones done by Harrison et al. [19]. They showed that, in an actual survey, participants might give up strategic voting and answer survey questions truthfully while the questions are not incentive compatible, due to computational complexity to figure out the optimal strategy. In that they partially answer the common critique on this approach that it relies on *NP*-hardness as a measure of computational difficulty while it is a worst-case and asymptotic notion.

From the **researchers or designers point of view**, we would like to have a "nice" characterization of the profiles and of the desired outcomes sets that are vulnerable to bribery. While there is no clear formal definition of "nice" characterizations or of "useful" ones, it seems that a necessary condition for such a characterization should be that it can be transformed to a polynomial (or an almost polynomial) algorithm. Indeed, most of the characterizations in the literature satisfy this property. In that sense, proving that there is no polynomial (or sub exponential) algorithm for the bribery problem shows that there is no hope to find a nice characterization as well.

3 Results

We analyze the computational complexity of $\mathrm{OLD}(\varphi, \psi)$ for a threshold function φ and a monotone function ψ. We find that the two parameters that characterize (nearly-fully) the complexity of the problem are t (the threshold of φ) and z (the size of the maximal minterm of ψ). For all the cases in which we prove the decision problem is solvable in polynomial time, we provide also direct polynomial algorithms for the search problem as well, showing no discrepancy between the two.

For the non-unanimity case $t < n$, we characterize the complexity of the problem as a function of t and z. In this case, the problem is tractable if either the threshold of the issue-aggregation function is bounded by a constant (which does not grow with the input size), or if the size of all minterms of the desired outcomes set are bounded by a constant. On the other hand, we show that when these parameters are polynomially large, the problem is *NP-complete*, and under mild computation assumptions it is not tractable even for poly-logarithmic values, and by that we get as specific cases the results shown by Christian et al. [8] and Bredereck et al. [7, (Partial Lobbying)]. There remains a small gap between the two ranges we deal with, when the values are super-constant (i.e., not bounded from above by a constant) or poly-logarithmic with a small degree, and in that sense this is only an almost-full characterization.

On the other hand, for the cases involving issue-aggregation using the unanimity function ($t = n$), we show that the complexity cannot be characterized solely by the size of the maximal minterm. We show that the problem is tractable when the desired outcomes set is simple (can be described using a polynomial number of minterms). While we conjecture the number of minterms characterize the intractability results as well, we characterize the complexity of $\mathrm{OLD}(Unan, \psi)$ only for desired outcomes sets defined by a threshold, that is, the Lobby is indifferent between the issues but has a quota of desired issues to pass. In this case we show that, unlike the non-unanimity case, the behaviour is symmetric in the sense that the complexity is characterized by $z' = \min(z, m - z)$ - the minimum between the number of issues needed to pass in order to guarantee success of the Lobby, and the number of issues needed to fail in order to guarantee failure of the Lobby. Similar to the non-unanimity case, when this parameter z' is bounded by a constant, the problem is tractable, and when it is super-constant, we show that the problem is intractable. Specifically, we show that when it is polynomially large, the problem is *NP-complete*, and under mild computation assumptions it is not tractable even for poly-logarithmic values. In this case too, the intermediate small range between constant and poly-logarithmic remains open.

Tractability Results

When the parameters are very small or large, we find polynomial algorithms to find the optimization problem.

The first algorithm is a simple brute-force enumeration of all "cheap" coalitions and when the threshold is small enough, we get a polynomial algorithm.

Theorem 3. $\mathrm{OL}(\varphi, \psi)$ *can be solved in time* $O\left(m \cdot tn^t\right)$. *In particular, when the threshold* t *is bounded by a constant not dependent on* m *and* n*, then* $\mathrm{OL}(\varphi, \psi)$ *can be solved in polynomial time.*

Proof. We know that influencing any coalition of size t will satisfy ψ and hence the minimal such coalition is of size at most t. We find the minimal coalition by iterating over all coalitions of size at most t (the number of such coalitions is $\sum_{i=0}^{t} \binom{n}{i} \leqslant tn^t$) and checking whether influencing all their members (to approve all issues) satisfies ψ.

Our second algorithm uses the anonymity of threshold functions, and solves the problem by iterating over all "representative coalitions."[9]

Theorem 4. *If* ψ *can be written as the disjunction of* r *minterms, each of which is a conjunction of at most* z *variables, then* $\mathrm{OL}(\varphi, \psi)$ *can be solved in time* $O\left(r \cdot n^{2^z}\right)$. *In particular, when all minterms of* ψ *are bounded by a constant not dependent on* m *and* n*, then* $\mathrm{OL}(\varphi, \psi)$ *can be solved in polynomial time.*

Proof. For each of the r minterms we'll use the following procedure in order to find the smallest coalition the Lobby can influence in order to satisfy the minterm (and hence ψ).

For a minterm of size z, the input can be summarized by stating for each voter his vote for these z issues. Moreover, due to the anonymity of φ, it can be summarized by stating the number of voters supporting each vote combination out of the 2^z possible combinations. Similarly, a coalition of convinced voters is represented by how many of its members voted for each of these 2^z combinations.

Thus, finding the minimal coalition can be done by exhaustively iterating over all these representative coalitions and calculating for each of them whether convincing it results in a desired outcome. We can bound the number of supporters of a combination by n, and bound the number of representative coalitions by n^{2^z}. Hence, by checking for each representation whether such a coalition exists and whether influencing its members satisfy ψ, we can find the smallest coalition in time $O\left(z \cdot n^{2^z}\right) = O\left(n^{2^z}\right)$.

[9] Note that bounding all the minterms is equivalent to bounding the maximal minterm.

The third algorithm solves the case in which the issue-aggregation function φ is the unanimity function ($t = n$). In utilizes the fact that the only way for the Lobby to get a desired result is to choose a minterm $\vee_{j \in M} x^j$ and convince all the voters that reject an issue in M to change their vote.

Theorem 5. *If ψ can be written as the disjunction of r minterms, each of which is a conjunction of at most z variables, then* $\mathrm{OL}(Unan, \psi)$ *can be solved in time* $O(r \cdot zn)$. *In particular, when ψ can be written as the disjunction of* $\mathrm{poly}(m)$ *minterms, then* $\mathrm{OL}(Unan, \psi)$ *can be solved in polynomial time.*

Proof. For each of the r minterms we use the following procedure in order to find the smallest coalition the Lobby can influence in order to satisfy the minterm (and hence ψ); The minimal coalition of those r coalition is the optimal coalition. For a minterm of size z, the minimal coalition needed is the coalition of all voters that rejected at least one issue from these z issues. This coalition can be found in time $O(zn)$.

Intractability Results

We prove separately and using different reductions intractability results for the cases where "the threshold is smaller than n (non-unanimity)" and the cases where "$t = n$ (unanimity)."

Non-Unanimity issue-aggregation $t < n$

For the case $t < n$ we prove the following intractability result.

Theorem 6. *Let φ be a threshold function with threshold $t < n$, and ψ a monotone function with maximal minterm of size z.*

1. *If t is polynomial ($\exists \epsilon > 0 \quad t \geqslant n^\epsilon$), and z is polynomial ($\exists \epsilon > 0 \quad z \geqslant m^\epsilon$), then* $\mathrm{OLD}(\varphi, \psi)$ *is NP-complete.*
2. *For any problem* $\mathrm{A} \in NP$ *and any $\beta \in (0,1)$, there exists $\alpha > 1$ s.t. if* $\mathrm{OLD}(\varphi, \psi)$ *is solvable in polynomial time and if t and z are poly-logarithmic of degree α ($t \geqslant (\log n)^\alpha$, $z \geqslant (\log m)^\alpha$), then* A *is solvable in time* $2^{O\left(n^\beta\right)}$.

As a corollary we get the following,

Corollary 7.

1. *(Christian et al. [8])* $\mathrm{OLD}(\varphi, \psi)$ *is NP-complete for φ the simple majority function and $\psi = \wedge_{j=1}^m x^j$, i.e., requiring all issues to pass.*
2. *Assuming ETH, there exists $\alpha^\star > 1$ s.t. if t and z are poly-logarithmic of degree $\alpha^\star$ ($t \geqslant (\log n)^{\alpha^\star}$, $z \geqslant (\log m)^{\alpha^\star}$), then* $\mathrm{OLD}(\varphi, \psi)$ *is not in P.*
3. *Assuming NP $\nsubseteq$ SUBEXP,*[10] *if t and z are super-poly-logarithmic (there is no $\alpha > 1$ s.t. $t \geqslant (\log n)^\alpha$ and $z \geqslant (\log m)^\alpha$ for all large enough n and m), then* $\mathrm{OLD}(\varphi, \psi)$ *is not in P.*

[10] $SUBEXP = \cap_{\epsilon > 0} DTIME\left(2^{s^\epsilon}\right)$ is the class of problems solvable in sub-exponential time. The assumption $NP \nsubseteq SUBEXP$ means assuming that there exists at least one NP problem that cannot be solved in time less than $2^{n^{\Omega(1)}}$.

We prove Theorem 6 by constructing the three reductions described below. The first two (Lemmas 8 and 9) are reductions from HITTING SET and VERTEX COVER, respectively, to $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ and the third reduction (Lemma 10) finishes the proof by reducing $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ to the general case $\text{OLD}(\varphi, \psi)$. The first two reductions are different and aim at different ranges of the threshold t. Although any reduction from HITTING SET is also a reduction from VERTEX COVER, we prefer constructing a reduction from the former to get a stronger result (wider range of the parameters).

Problem (Hitting Set).

<u>`INSTANCE`</u>: COLLECTION $C = \{C^j\}$ OF $|C|$ SUBSETS OF A UNIVERSE S. A POSITIVE INTEGER $k \leqslant |S|$.

<u>`QUESTION`</u>: IS THERE A HITTING SET $H \subseteq S$ OF SIZE k, I.E., A SET $H \subseteq S$ S.T. $\forall j \; C^j \cap H \neq \emptyset$?

REFERENCE: THE PROBLEM IS *NP-complete* – [18, PROBLEM SP8].

Lemma 8. *Given a family of threshold functions φ over n voters with threshold t that satisfies "t and $(n-t)$ diverge to infinity," there exists a reduction that runs in time linear in the output size;*[11] *and given an instance of* HITTING SET – $\big(S, C = \{C^j\}, k\big)$, *it produces an instance of* $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ – $\Big(X \in \{0,1\}^{n \times m}, k'\Big)$ – *s.t. the following is satisfied:*

- $\min(t, n-t) = |S|$,
- $m = n + |C|$,
- $k' = k$,
- *and* (S, C, k) *is satisfiable if and only if* (X, k') *is satisfiable.*
- *In addition the reduction's output satisfies that each issue is supported by either* $t-1$ *or* $t-k$ *supporters.*

<u>**Proof Sketch of Lemma 8**</u>

Given an instance of HITTING SET – $\big(S, C = \{C^j\}, k\big)$, we construct the instance of $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ – (X, k) by using the same threshold k and defining the profile X to be:

Voters \ Issues	$\|C\|$ issues	$m - \|C\|$ issues
$\|S\|$ voters	A	0
$n - \|S\|$ voters	In the $j^{\underline{th}}$ column there are $t - 1 - \big(\|S\| - \|C^j\|\big)$ ones	- In each column there are $t-k$ ones - No all zeroes line
Total (out of n voters)	$t-1$	$t-k$

when entries of the matrix $A \in \{0,1\}^{|S| \times |C|}$, $A_{i,j}$, are 1 if $i \notin C^j$ and 0 otherwise.

[11] In cases in which the output size is polynomial in the input size, which are the ones in which we apply this reduction, we get that the running time is polynomial in the regular sense, i.e., in the input size.

Due to the way we defined the bottom-right sub-profile, the Lobby can convince at most k voters in order to pass all issues, if and only if it can do so by convincing k voters from the top $|S|$ voters. Due to the definition of the top-left sub-profile, it can achieve that only by finding a hitting set of size k. □

The reduction from Vertex Cover is similar to the above but differs in the gadgets used for embedding of the incidence matrix in the profile. The full reductions can be found in the appendix.

Problem (Vertex Cover).

<u>INSTANCE</u>: An undirected graph $G = (V, E)$.
A positive integer $k \leqslant |V|$.
<u>QUESTION</u>: Is there a cover $C \subseteq V$ of size k in G, i.e., a set $C \subseteq V$ s.t. for each edge $e = \{u, v\}$, u or v belongs to C?
Reference: The problem is *NP-complete* – [18, Problem GT1].

Lemma 9. *Given a family of threshold functions φ over n voters with threshold t that satisfies "$t \leqslant n-1$ and $\frac{t}{n}$ converges to one," there exists a reduction that runs in time linear in the output size; and given an instance of* Vertex Cover *– $(G = (V, E), k)$, it produces an instance of $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m} x^j\right)$ – $\left(X \in \{0,1\}^{n \times m}, k'\right)$ – s.t. the following is satisfied:*

- $\frac{n}{n-t} = |V|^2$,
- $m = |E|$,
- $k' = k$,
- *and (G, k) is satisfiable if and only if (X, k') is satisfiable.*

The third reduction is from $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m} x^j\right)$ to the general case $\mathrm{OLD}(\varphi, \psi)$.

Lemma 10. *Given a family of threshold functions φ over n voters with threshold t and a family of monotone functions ψ over m variables (issues) with a minterm of size z that diverges to infinity with m, there exists a reduction that runs in time linear in the output size; and given an instance of $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m'} x^j\right)$ – $\left(Y \in \{0,1\}^{n' \times m'}, k'\right)$, it produces an instance of $\mathrm{OLD}(\varphi, \psi)$ $\left(X \in \{0,1\}^{n \times m}, k\right)$ s.t. the following is satisfied:*

- $n = n'$,
- *$z = m'$ (and m is set according to z),*
- $k = k'$,
- *and (Y, k') is satisfiable if and only if (X, k) is satisfiable.*

In addition, each issue in X either corresponds to one of the issues in Y or is not approved by any of the voters. That is, if the maximal minterm is $\wedge_{j=1}^{z} x^j$, then $\forall j \leqslant z \quad X^j = Y^j$ and $\forall j > z \quad X^j = \overline{0}$ (the all zeroes vector).

<u>Proof Sketch of Lemma 10</u>

W.l.o.g., we assume that $\wedge_{j=1}^{z} x^j$ is a minterm of ψ. Given an instance of $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m'} x^j\right)$ – $\left(Y \in \{0,1\}^{n' \times m'}, k'\right)$, we construct the instance of $\mathrm{OLD}(\varphi, \psi)$ – $\left(X \in \{0,1\}^{n \times m}, k'\right)$ by defining the threshold $k' = k$ and defining the profile X to be:

Voters \ Issues	first z issues	$(m - z)$ issues
n voters	Y	0

for $n = n'$, $z = m'$, and $k = k'$.

Due to the way we define the profile, the only way for the lobby to satisfy ψ is by satisfying the first z issues and this can be done only by solving the original problem $\left(Y \in \{0,1\}^{n' \times m'}, k'\right)$. □

We prove the intractability theorem using the above reductions.

Proof Sketch of Theorem 6

Combining the reductions of Lemmas 8 and 10, we get a reduction from Hitting Set to $\mathrm{OLD}(\varphi, \psi)$. This reduction satisfies the desired properties:

- If $z \geqslant m^\epsilon$ and $t, (n-t) \geqslant n^\epsilon$ (for some $\epsilon > 0$), we get a polynomial reduction from the *NP-complete* problem Hitting Set to $\mathrm{OLD}(\varphi, \psi)$, and by that proving the *NP*-hardness of $\mathrm{OLD}(\varphi, \psi)$.
- Similarly, if $z \geqslant (\log m)^\alpha$ and $t, (n-t) \geqslant (\log n)^\alpha$ (for some $\alpha > 1$), we get a reduction from Hitting Set with a blowup smaller than $2^{x^{1/\alpha}}$. Hence, since Hitting Set is *NP-complete*, we get that for any problem $\mathrm{A} \in NP$ if $z \geqslant (\log m)^\alpha$ and $t, (n-t) \geqslant (\log n)^\alpha$, there is a reduction from A to $\mathrm{OLD}(\varphi, \psi)$ with a blowup smaller than $2^{x^{C/\alpha}}$, for a constant C that depends on A. Given that, for any $\beta \in (0,1)$, there exists a large enough $\alpha > 1$ s.t. if $\mathrm{OLD}(\varphi, \psi)$ is solvable in polynomial time, $z \geqslant (\log m)^\alpha$, and $t, (n-t) \geqslant (\log n)^\alpha$, then A is solvable in time $2^{O\left(n^\beta\right)}$.

We prove the intractability results for the complementary domain $t \in [n - n^\epsilon, n-1]$, using the reduction from Vertex Cover (Lemma 9) in a similar way.

Unanimity issue-aggregation $t = n$

For the case $t = n$, we prove the following intractability result.

Theorem 11. *Let ψ be a threshold function with threshold z, i.e., the Lobby would like at least z issues to pass.*

1. *If z and $(m-z)$ are polynomial ($\exists \epsilon > 0 \quad z \in [m^\epsilon, m - m^\epsilon]$), then $\mathrm{OLD}(Unan, \psi)$ is NP-complete (under Turing reductions; See Footnote 12).*
2. *For any problem $\mathrm{A} \in NP$ and any $\beta \in (0,1)$, there exists $\alpha > 1$ s.t. if $\mathrm{OLD}(Unan, \psi)$ is solvable in polynomial time and if z and $(m-z)$ are poly-logarithmic of degree α ($z \in [(\log m)^\alpha, m - (\log m)^\alpha]$), then A is solvable in time $2^{O\left(n^\beta\right)}$.*

3. *Assuming ETH, there exists* $\alpha^\star > 1$ *s.t. if* z *and* $(m - z)$ *are poly-logarithmic of degree* $\alpha^\star$ ($z \in \left[(\log m)^{\alpha^\star}, m - (\log m)^{\alpha^\star}\right]$)*, then* $\mathrm{OLD}(Unan, \psi)$ *is not in P.*
4. *Assuming NP* $\nsubseteq$ *SUBEXP, if* z *and* $(m - z)$ *are super-poly-logarithmic (there is no* $\alpha > 1$ *s.t.* $z \geqslant (\log m)^\alpha$ *and* $(m - z) \geqslant (\log m)^\alpha$ *for all large enough* m*), then* $\mathrm{OLD}(Unan, \psi)$ *is not in P.*

We prove this theorem by constructing a reduction from the following problem:

Problem (EBNCD – Exact Balanced Node Cardinality Decision problem).

INSTANCE:	A BIPARTITE GRAPH $G = (L, R, E)$. A POSITIVE INTEGER $k \leqslant \min(\lvert L\rvert, \lvert R\rvert)$.
QUESTION:	DOES THERE EXIST A BICLIQUE OF SIZE (k, k) IN G, I.E., TWO SETS $A \subseteq L$ AND $B \subseteq R$, BOTH OF SIZE k, S.T. $\forall l \in A, r \in B : (l, r) \in E$?
REFERENCE:	THE PROBLEM IS *NP-complete* (UNDER TURING REDUCTIONS) – [9].[12]

Lemma 12. *Given a family of threshold functions* ψ *over* m *issues with threshold* z *that satisfies "*$(m - z)$ *and* z *diverges to infinity," there exists a reduction that runs in time linear in the output size; and given an instance of* EBNCD – $(G = (L, R, E), k)$*, it produces an instance of* $\mathrm{OLD}(Unan, \psi)$ – $\left(X \in \{0,1\}^{n \times m}, k'\right)$ – *s.t. the following is satisfied:*

- $n = \lvert L\rvert$,
- $\min(z, m - z) = \max(k, \lvert R\rvert - k)$,
- *and* $k' = n - k$*, and* (G, k) *is satisfiable if and only if* (X, k') *is satisfiable.*

Proof Sketch of Lemma 12

Given an instance of EBNCD – $(G = (L, R, E), k)$, represented by an incidence matrix $A \in \{0,1\}^{\lvert L\rvert \times \lvert R\rvert}$ and a function ψ defined by a threshold k, we construct the instance of $\mathrm{OLD}(Unan, \psi)$ – (X, k') by defining the threshold $k' = \lvert L\rvert - k$ and defining the profile X to be:

Voters \ Issues	$\lvert R\rvert$ issues	$z - k$ issues	$m - \lvert R\rvert - (z - k)$ issues
$\lvert L\rvert$ voters	A	1	strictly more than k' zeroes in each column

[12] The reduction presented by Dawande et al. [9] is a Turing reduction from an *NP-complete* problem. Hence, they show that EBNCD is an *NP-complete* problem in a weaker sense, and the same weakness is shared by our result regarding $\mathrm{OLD}(Unan, \psi)$. Notice that nevertheless, this weaker notion still proves that a polynomial algorithm to $\mathrm{OLD}(Unan, \psi)$ implies that *NP=P*.

Due to the way we define the right sub-profile, the only way for the lobby to satisfy ψ is by satisfying at least k of the left issues. This can be done only by finding a (k,k)-biclique in G; that is, k issues, all supported by k voters; and convincing the voters not corresponding to L-vertices of the clique to support the issues corresponding to the R-vertices of the clique. $\square$

4 Related Work

The Optimal Lobbying problem was first addressed by Christian et al. [8] who (essentially) showed that $\text{OLD}\left(\textsc{Maj}_n, \bigwedge_{j=1}^{m} x^j\right)$ is *NP-complete* and $W[2]$-*complete* with respect to the budget k.[13]

This problem was studied further by Bredereck et al. [7] in terms of its parameterized computational complexity with regard to other parameters – number of voters and issues, maximal number of ones and zeroes per row, and the maximal gap over all issues, where the gap of an issue is the number of voters the Lobby needs to influence in order to get a majority in this issue. This work included both intractability results (e.g., $W[2]$-hardness w.r.t. the maximal gap as a corollary from the work of Christian et al. [8] and NP-hardness for any positive constant constraint for the parameter $\left(\left\lceil \frac{n+1}{2} \right\rceil - k\right)$) and tractability results (e.g., an FPT algorithm w.r.t. m). In addition they presented two generalizations of this problem: Restricted Lobbying, in which the input includes (in addition to the profile and the budget) a parameter o' of the number of issue-votes an influenced voter is allowed to change; and Partial Lobbying, in which the desired outcomes are the outcomes in which at least a certain number of issues (r) pass instead of all issues (in our terms this is $\text{OLD}\left(\textsc{Maj}_n, \bigvee_{M \in \binom{[m]}{r}} \bigwedge_{j \in M} x^j\right)$, when r is part of the input and $\binom{[m]}{r}$ denotes the set of all the subsets of size r of $\{1,2,\ldots,m\}$). Compared to their model, we deal with these desired outcomes sets but we analyze the problem when r is exogenous as well as for more general issue-aggregation functions, and we analyze fully which are the hard values of r.

Another extension of the model of Christian et al. [8] was presented by Baumeister et al. [5]. They define several variants of related problems – Bribery and Manipulation of Premise Based Agendas. The question of bribery they describe is equivalent to the Optimal Lobbying. They prove hardness of the problem when $\varphi = \text{Maj}_n$ and ψ is part of the input and consisting of a single minterm, i.e., a unique desired outcome for a partial list of the issues. This result is obtained as a corollary of the work by Christian et al. [8] that showed hardness for a specific ψ, $\psi = \wedge_{j=1}^{m} x^j$, and it can be also be proved as a corollary of more general results that we prove here. The contribution of our work over this result is that in our work the function ψ is not part of the input but exogenous to the problem and hence we characterize the hard ψ-slices of the problem (the functions ψ for which the problem is tractable and those for which it is not).

[13] For background on the theory of parameterized complexity and in particular for the definition of $W[2]$, see Downey and Fellows [12], Niedermeier [26], and Flum and Grohe [17].

5 Generalizations

So far, in the main part of this paper, we characterized the computational complexity of Optimal Lobbying when **(a)** all issues are Boolean, **(b)** all issues are aggregated using the same monotone anonymous function φ, and **(c)** the desired outcomes set ψ is upward-closed. In this section we show how one can relax these constraints and lift the impossibility results to get impossibility results in more general frameworks as well.

Whenever one can embed (the hard cases of) Optimal Lobbying into another problem, hardness results follow immediately. E.g., we get hardness results, to a variant when the threshold is included as part of the input, which is a more natural modeling of the problem (Notice that it can be done without changing the size of the input significantly). Similarly, we show hardness for the more general cases when there are several thresholds (e.g., , e.g., different subjective "safety margins" of the Lobby) or general issue-aggregation functions (as part of the input in a compact way). A similar natural lift can be done for the uniform pricing scheme to get hardness results for more complex pricing schemes when each voter asks for a different price, or a price that is monotone in the change.

Different kind of extensions, still using the above embedding idea, is modeling the Lobby as having a utility instead of a dichotomous preferences., for which the hard problem is whether the Lobby can achieve a utility higher than a given $u^\star$.

Last, we notice that although all our theorems assumed monotonicity of the issue-aggregation function and of the desired outcomes set, it is easy to show equivalence between these cases and cases of downward monotone issue-aggregation function or downward-closed desired outcomes set. These cases are equivalent to scenarios we solved. The equivalence is attained by negating φ, ψ, and the input matrix. For instance, assume the Lobby is faced with a profile X, a voting method of aggregating using a threshold t, and it desires that at most k issues would pass. That is, it desires that at least $(m - k + 1)$ issues would fail, when an issue fails if at least $(n - t + 1)$ voters reject it. Hence, this problem is equivalent to the problem defined by the profile $(1 - X)$ (i.e., negating all votes), a voting method of aggregating using a threshold $(n - t + 1)$, and monotone desired outcomes defined by a threshold $(m - k + 1)$. Using a similar technique (negating φ and X) we can also deal with downward monotone φ. Hence, all the results shown in this paper (tractability and intractability ones) can be lifted to any case that both φ and ψ are monotone, either upward monotone or downward monotone.

6 Conclusions

In this paper we defined the Optimal Lobbying decision and optimization problems (in their general forms) and characterized their computational complexity when **(a)** all issues are Boolean, **(b)** all issues are aggregated using the same monotone anonymous function φ, and **(c)** the desired outcomes set ψ is upward-closed.We also showed that actually we can relax these constraints to get characterization results for a larger set of Optimal Lobbying problems.

There are still some gaps in our characterization that we intend to fill. First, our characterization of the case of aggregating using unanimity characterizes mostly the case of desired outcomes set defined by a threshold. We conjecture that the real parameter that controls the complexity is the number of minterms (when the number of minterms is large, ψ is "close enough" to a threshold function).

Conjecture. *Let the aggregation function be unanimity and let the desired outcomes set be described using r minterms.*

- *If r is polynomial in m,* Optimal Lobbying *can be solved in polynomial time (We already proved this part in Thm. 5).*
- *If r is exponential in m ($r \geqslant 2^{m^{\alpha}}$ for some $\alpha > 0$), then* Optimal Lobbying *is NP-complete.*

A smaller gap in our characterization is the gap between bounded values, for which we proved tractability, to poly-logarithmic values, for which we proved intractability. We conjecture that the tractability results can be extended by smarter algorithms.

In cases where we proved the problem to be tractable, a natural question is whether the tractability still holds for extensions like we described in the previous section, like more complex price schemes, and more complex aggregation methods.

In a subsequent work, we extend this paper to studying the *parameterized* computational complexity of Optimal Lobbying. We describe the parameters that we think capture the complexity of an instance and by finer analysis of the reductions, analyze the parameterized complexity with regard to them. A natural extension of both works would be extension of the analysis to other issue-aggregating functions (φ) and other desired outcomes families (ψ), both monotone and non-monotone.

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A The Reductions

Lemma (Lemma 8).

Given a family of threshold functions φ over n voters with threshold t that satisfies "t and $(n-t)$ diverge to infinity," there exists a reduction that runs in time linear in the output size;[14] *and given an instance of* HITTING SET – $\left(S,\, C=\left\{C^j\right\},\, k\right)$, *it produces an instance of* $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m} x^j\right)$ – $\left(X \in \{0,1\}^{n\times m}, k'\right)$ *– s.t. the following is satisfied:*

- $\min(t, n-t) = |S|$,
- $m = n + |C|$,
- $k' = k$,
- *and* (S, C, k) *is satisfiable if and only if* (X, k') *is satisfiable.*
- *In addition the reduction's output satisfies that each issue is supported by either* $t-1$ *or* $t-k$ *supporters.*

Proof.

Given an instance of HITTING SET – $\left(S,\, C=\left\{C^j\right\},\, k\right)$, we construct the instance (X, k') of $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m} x^j\right)$ in the following way:

Let φ be an issue-aggregation function that satisfies $\min(t, n-t) = |S|$ and let $m = n + |C|$.

The cases "$\max_j \left|C^j\right| \leqslant 1$", "$k \leqslant 1$," and "$k = |S|$" can be easily solved and hence the reduction for these inputs is trivial. Hence we define (X, k') for the case "$\max_j \left|C^j\right| > 1$ and $2 \leqslant k \leqslant |S|-1$" as follows:

- The set of the n voters (matrix rows) is divided into
  - $|S|$ "*v voters*" that correspond to the elements of S: $\left\{v_1, \ldots, v_{|S|}\right\}$
  - $n-|S|$ "*u voters*" $\left\{u_1, \ldots, u_{n-|S|}\right\}$.
- The set of the m issues (matrix columns) is divided into
  - $|C|$ "*A issues*" of size $t-1$ that correspond to the sets in C s.t. $A^j = \left\{v_i \;\middle|\; i \notin C^j\right\} \cup \left\{u_1, \ldots, u_{t-1-(|S|-|C^j|)}\right\}$
  - n "*B issues*" of size $t-k$ s.t. each of the "v voters" supports none of these issues and each of the "u voters" supports at least one of these issues. E.g., we can define $B^j = \{u_i \mid (t-k)\cdot(j-1)+1 \leqslant i \leqslant (t-k)\cdot j \;\; [mod\ (n-|S|)]\}$

  Schematically, the matrix X is

Voters \ Issues	$\left\{A^j\right\}_{j=1}^{\lvert C\rvert}$	$B^j (n = m - \lvert C\rvert$ issues)
$\{v_i\}_{i\in S}$	$\begin{cases} 0 & i \in C^j \\ 1 & \text{otherwise} \end{cases}$	0
u_i ($n-\lvert S\rvert$ voters)	In each column there are $t-1-\left(\lvert S\rvert-\left\lvert C^j\right\rvert\right)$ ones	- In each column there are $t-k$ ones - No all zeroes line
Total(out of n voters)	$t-1$	$t-k$

[14] In cases in which the output size is polynomial in the input size, which are the ones in which we apply this reduction, we get that the running time is polynomial in the regular sense, i.e., in the input size.

Claim. *The reduction is feasible.*

Proof.

- The number of "B issues" is strictly positive since $m > |C|$.
  - The number of "u voters" is strictly positive:
    - Since $n - |S| \geqslant 2\min(t, n-t) - |S| = |S| > 0$.
  - The number of supporters in A^i among the "u voters" is non-negative but smaller than the total number of "u voters":
    - $t \geqslant |S|$ and $\left|C^j\right| \geqslant 1$ and hence $t - 1 - \left(|S| - \left|C^j\right|\right) \geqslant 0$
    - $n - t \geqslant |S| \geqslant \left|C^j\right|$ and hence $t - 1 - \left(|S| - \left|C^j\right|\right) < n - |S|$
  - The number of supporters in B^i among the "u voters" is strictly positive but smaller than the total number of "u voters":
    - $t \geqslant |S| > k$ and hence $t - k > 0$
    - $n - t \geqslant |S|$ and hence $t - k \leqslant n - |S| - k \leqslant n - |S|$
  - It is possible to tile the sub-matrix of "u voters" and "B issues" with $t - k$ in each column and with no zero line.
    - This holds since $\frac{n-|S|}{t-k} \leqslant n =$the number of "$B$ issues."

Claim. *The reduction can be generated in time and space linear in the output size. I.e. in $O(mn)$ time and space.*

Claim. *If there is a hitting set of size k, then the Lobby can influence k voters in order to get a desired outcome.*

Proof. Let $A \subseteq S$ be a hitting set of size k. Influencing the k voters $\{v_i \mid i \in A\}$ to approve all the issues adds at least one to each of the A^j issues and exactly k to each of the B^j issues. Hence, all the issues pass and we get the requested result.

Claim. *If the Lobby can influence k voters in order to get a desired outcome, then there exists a hitting set of size at most k.*

Proof. Let A be the set of influenced voters. We divide into two cases:

<u>Case 1</u>: There is a "u voter" u_i in A.

So, there is an issue B^j for which u_i initially voted one and hence did not add his vote due to influencing. This issue has at most $t-k+(k-1)$ votes which does not pass the threshold and we get a contradiction.

<u>Case 2</u>: All the voters in A are "v voters."

Each issue A^j now passes the threshold and hence at least one "v voter" $v_i \in A$ changed his vote on this issue from zero to one. Hence C^j is hit by i so we get that A is also a hitting set.

Lemma (Lemma 9).

Given a family of threshold functions φ over n voters with threshold t that satisfies "$t \leqslant n-1$ and $\frac{t}{n}$ converges to one," there exists a reduction that runs in time linear in the output size; and given an instance of Vertex Cover *–* $(G = (V, E)\,,\, k)$, *it produces an instance of* $\mathrm{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ – $\Big(X \in \{0,1\}^{n\times m}\,, k'\Big)$ *– s.t. the following is satisfied:*

- $\frac{n}{n-t} = |V|^2$,
- $m = |E|$,
- $k' = k$,
- *and* (G, k) *is satisfiable if and only if* (X, k') *is satisfiable.*

Proof.

Given an instance of Vertex Cover – $(G = (V, E)\,,\, k)$, we construct the instance (X, k') of $\mathrm{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ in the following way:

Let φ be an issue-aggregation function that satisfies $\frac{n-t}{n} = \frac{1}{|V|^2}$ and let $m = |E|$.

First, we deal with few extreme cases that are easy instances of Vertex Cover.

If $k \geqslant |V| - 1$ or $E = \emptyset$, define X to be the all ones $n \times m$ matrix.

If $E \neq \emptyset$ and $k = 0$, define X to be the all zeroes $n \times m$ matrix.

It is easy to see that in the above cases (X, k) has a solution iff (G, k) has a solution.

Hence, henceforth we'll assume $k \in [1, |V| - 2]$ and $E \neq \emptyset$.

In this case we notate $r = n - t - 1 \geqslant 0$ and construct the instance as follows:

We number the edges arbitrarily as $e_1, e_2, \ldots, e_{|E|}$

- The set of the n voters (matrix rows) is divided into
  - $|V|$ "*v voters*" $\big\{v_1, \ldots, v_{|V|}\big\}$
  - $r\,|E|$ "*e voters*" $\{e_{z,l} \mid z = 1, \ldots, |E|\,,\, l = 1, \ldots, r\}$
  - $n - |V| - r\,|E|$ "*u voters*" $\big\{u_1, \ldots, u_{n-|V|-r|E|}\big\}$
- The set of the m issues (matrix columns) is
  - $|E|$ "*A issues*" of size $t - 1$ s.t.
    $A^j = \{v_i \mid \text{The edge } j \text{ is not connected to the node } i\} \cup \{e_{z,l} \mid z \neq j\}$

Schematically, the matrix X is

<table>
<tr><td>Issues / Voters</td><td>$\big\{A^j\big\}_{j=1}^{|E|}$</td></tr>
<tr><td>$\{v_i\}_{i\in V}$</td><td>$\begin{cases} 0 & \text{The edge } j \text{ is connected to the node } i \\ 1 & \text{otherwise} \end{cases}$</td></tr>
<tr><td>$\big\{e_{z,l}\big\}_{z\in E, l\in\{1,\ldots r\}}$</td><td>$\begin{cases} 0 & z = j \\ 1 & z \neq j \end{cases}$</td></tr>
<tr><td>$\{u_i\}_{i=1}^{n-|V|-r|E|}$</td><td>1</td></tr>
<tr><td>Total(out of n voters)</td><td>$(|V|-2) + r\cdot(|E|-1) + (n-|V|-r\,|E|) = n - r - 2 = t - 1$</td></tr>
</table>

Claim. *The reduction is feasible.*

Proof. The number of "u voters" is non-negative since $n = (n-t)|V|^2 = |V|^2 + (n-t-1)|V|^2 \geqslant |V| + r|E|$

Claim. *The reduction can be generated in time and space linear in the output size. I.e. in $O(mn)$ time and space.*

Claim. *If there is a vertex cover of size k, then the Lobby can influence k voters in order to get a desired outcome.*

Proof. Let $C \subseteq V$ be a cover of size k. Convincing the k voters $\{v_i \mid i \in C\}$ to approve all issues, adds at least one to the support of each of the "A issues" and hence all issues pass and we get the desired outcome.

Claim. *If the Lobby can influence k voters in order to get the desired outcome, then there is a vertex cover of size k.*

Proof. Let I be the set of influenced voters. We choose an arbitrary orientation of the graph and define $C \subseteq V$ to be $\{i \in V \mid v_i \in I\} \cup \{i \in V \mid e_{z,l} \in I \text{ for an edge } z \text{ outgoing from } i\}$. Clearly $|C| \leqslant |I| = k$

We know that influencing I results in a desired outcome so all issues passed the threshold.

Let j be an edge. Then, the issue A^j passes the threshold due to a change in the vote of one of its non-supporters. If this influenced voter is a "v voter" – v_i, then j is covered by $i \in C$. Otherwise, this voter is an "e voter" – $e_{j,l}$, and j is covered by its source which is in C. Hence, C is a vertex cover of size at most k.

Lemma (Lemma 10).

Given a family of threshold functions φ over n voters with threshold t and a family of monotone functions ψ over m variables (issues) with a minterm of size z that diverges to infinity with m, there exists a reduction that runs in time linear in the output size; and given an instance of $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m'} x^j\right)$ – $\left(Y \in \{0,1\}^{n' \times m'}, k'\right)$*, it produces an instance of* $\mathrm{OLD}(\varphi, \psi)$ $\left(X \in \{0,1\}^{n \times m}, k\right)$ *s.t. the following is satisfied:*

- $n = n'$,
- $z = m'$ *(and m is set according to z),*
- $k = k'$,
- *and (Y, k') is satisfiable if and only if (X, k) is satisfiable.*

In addition, each issue in X either corresponds to one of the issues in Y or is not approved by any of the voters. That is, if the maximal minterm is $\wedge_{j=1}^{z} x^j$, then $\forall j \leqslant z \quad X^j = Y^j$ and $\forall j > z \quad X^j = \overline{0}$ (the all zeroes vector).

Proof.

Given an instance of $\mathrm{OLD}\left(\varphi, \wedge_{j=1}^{m} x^j\right)$ – $\left(Y \in \{0,1\}^{n' \times m'}, k'\right)$, we construct the instance (X, k) of $\mathrm{OLD}(\varphi, \psi)$ in the following way:

Let ψ be a desired outcomes set function that satisfies $z = m'$ and let $n = n'$. ψ includes a minterm of size z. W.l.o.g. $\wedge_{j=1}^{z} x^j$. We define the instance (X, k) as follows:

- $k = k'$
- The set of the n voters (matrix rows) is the same set as in Y.
- The set of the m issues (matrix columns) is divided into
  - $z = m'$ "*A issues*" that are defined to be identical to the issues of Y.
  - $(m - z)$ "*C issues*" of size 0. I.e., empty support.

Schematically, the matrix X is

Issues / Voters	$\left\{A^j\right\}_{j=1}^{z}$	$C^i((m-z)$ issues$)$
v_i (n voters)	Y	0

Claim. *The reduction can be generated in time and space linear in the output size. I.e. in $O(mn)$ time and space.*

Claim. *For a coalition $V \subset [n]$, if influencing the members of V to approve all the issues in the profile Y results in all issues passing, then influencing them to approve all the issues in the profile X results in a desired outcome (according to ψ).*

Proof. By convincing the voters of V the Lobby gets that all the "*A issues*" pass. Since we assume that these z issues correspond to a minterm of ψ we get a desired outcome.

Claim. *For any coalition $V \subset [n]$, if influencing the members of V to approve all the issues in the profile X results in a desired outcome, then influencing them to approve all the issues in the profile Y results in all issues passing.*

Proof. If a "C issue" pass due to the members of V approving it, then from φ's monotonicity also all the "A issues" pass. Otherwise, none of the "C issues" passes and since the result is a desired outcome all the "A issues" pass. We get that all the "A issues" pass and hence all the issues of Y pass when V's members are influenced.

Lemma (Lemma 12).

Given a family of threshold functions ψ over m issues with threshold z that satisfies "$(m-z)$ and z diverges to infinity," there exists a reduction that runs in time linear in the output size; and given an instance of EBNCD – $(G=(L,R,E),k)$, *it produces an instance of* $\mathrm{OLD}(Unan,\psi)$ – $\left(X\in\{0,1\}^{n\times m},k'\right)$ – *s.t. the following is satisfied:*

- $n=|L|$,
- $\min(z,m-z)=\max(k,|R|-k)$,
- *and* $k'=n-k$, *and* (G,k) *is satisfiable if and only if* (X,k') *is satisfiable.*

Proof.

Given an instance of EBNCD – $(G=(L,R,E),k)$, we construct the instance (X,k') of $\mathrm{OLD}(Unan,\psi)$ in the following way:

Let ψ be a desired outcomes set function that satisfies $\min(z,m-z)=\max(k,|R|-k)$ and let $n=|L|$. We define the instance as follows:

- The set of the n voters (matrix rows)
  - $|L|$ "v voters" $\left\{v_1,\ldots,v_{|L|}\right\}$ that correspond to the vertices of L.
- The set of the m issues (matrix columns) is divided into
  - $|R|$ "A issues" s.t. $A^j=\{v_i\mid(i,j)\in E\}$
  - $z-k$ "B issues" of size n. I.e., full support
  - $m-(z-k+|R|)$ "C issues" of size at most $k-1$. E.g., $C^j=\{v_1,\ldots,v_{k-1}\}$.
- The budget is $k'=|L|-k$.

Schematically, the matrix X is

Voters \ Issues	$\left\{A^j\right\}_{j\in R}$	B^j	C^j
$\{v_i\}_{i\in L}$	The adjacency matrix of G	1	strictly more than $\|L\|-k$ zeroes in each column

Claim. *The reduction is feasible.*

Proof. We chose ψ such that $\min(z,m-z)=\max(k,|R|-k)$ and hence the number of "B issues" $z-k$ is non-negative and the number of "C issues" $m-z-(|R|-k)$ is non-negative

Claim. *The reduction can be generated in time and space linear in the output size. I.e. in $O(mn)$ time and space.*

Claim. *If there is a biclique of size (k,k) in G, then the Lobby can influence $(n-k)$ voters in order to get a desired outcome.*

Proof. Let $A\subseteq L$ and $B\subseteq R$ be a biclique of size (k,k). We know that all the voters in A approved all the "A issues" in B. Hence by convincing the $n-k$ voters $\{v_i\mid i\notin A\}$ to approve all issues, the Lobby changes the outcome so all the "A issues" in B and all the "B issues" pass. So at least z issues pass and hence this is a desired outcome.

Claim. *If the Lobby can influence $(n-k)$ voters in order to get a desired outcome, then there is a biclique of size (k,k).*

Proof. Let $I \subseteq L$ be the set of influenced voters. So $|I| \leqslant n-k$ and $|L \setminus I| \geqslant k$. From our construction, none of the "C issues" pass and hence at least k of the "A issues" passed. We'll notate the issues that passed by $Y \subseteq R$. Consequently, $L \setminus I \subseteq L$ and $Y \subseteq R$ form a biclique of size at least (k,k) in G.

B Proofs – Intractability Results

Theorem (Theorem 6.1(a)).

Let φ be a threshold function with threshold t. If t is not $\text{poly}(n)$*-small or* $\text{poly}(n)$*-large ($\exists\epsilon > 0 \quad t \in [n^{\epsilon},\, n - n^{\epsilon}]$), then* $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ *is NP-complete.*

Proof.

It is trivial that the problem belongs to *NP*. In order to prove hardness we construct the following polynomial reduction from Hitting Set. It's given that $t \in [n^{\epsilon},\, n - n^{\epsilon}]$ and hence $n - t, t \geqslant n^{\epsilon}$ and both diverge to infinity. Based on Lemma 8, we get that there exists a reduction that runs in time linear in the output size and given an instance of Hitting Set – $\big(S,\, C = \big\{C^j\big\}\,,\, k\big)$, produces an instance of $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ – $\Big(X \in \{0,1\}^{n \times m}\,, k'\Big)$ – s.t. $n^{\epsilon} \leqslant \min(t, n - t) = |S|$, $m = n + |C|$, and $k' = k$.

Hence, we get that both m and n are polynomial in the size of the input so the reduction is polynomial. We get a polynomial reduction from the *NP-complete* problem Hitting Set to $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ and hence $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ is *NP-complete*.

Theorem (Theorem 6.1 (b)).

Let φ be a threshold function with threshold t. If t is $\text{poly}(n)$*-large ($\exists\epsilon > 0 \quad t \in [n - n^{\epsilon},\, n - 1]$), then* $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ *is NP-complete.*

Proof.

It is trivial that the problem belongs to *NP*. In order to prove hardness we construct the following polynomial reduction from Vertex Cover. It's given that $t \in [n - n^{\epsilon}, n - 1]$ and hence $\frac{t}{n}$ converges to 1. Based on Lemma 9, we get that there exists a reduction that runs in time linear in the output size and given an instance of Vertex Cover – $(G = (V, E)\,,\, k)$, produces an instance of $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ – $\Big(X \in \{0,1\}^{n \times m}\,, k'\Big)$ – s.t. $\frac{1}{|V|^2} = \frac{n-t}{n} \leqslant \frac{n^{\epsilon}}{n} = \frac{1}{n^{1-\epsilon}}$, $n \leqslant |V|^{\frac{2}{1-\epsilon}} m = |E|$, and $k' = k$. Hence, we get that both m and n are polynomial in the size of the input so the reduction is polynomial. We get a polynomial reduction from the *NP-complete* problem Vertex Cover to $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ and hence $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ is *NP-complete*.

Theorem (Theorem 6.2).

Let φ be a threshold function with threshold t. If t is not polylog(n)-small or polylog(n)-large ($\forall\alpha > 1 \quad t \in \left[(\log n)^{\alpha}\,,\, n - (\log n)^{\alpha}\right]$ for large enough n), then $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ *is not in P (assuming NP $\nsubseteq$ SUBEXP).*

Proof.

In order to prove hardness we construct the following sub-exponential reduction from Hitting Set. It's given that $t \in \left[(\log n)^{\alpha}\,,\, n - (\log n)^{\alpha}\right]$ and hence $n, n - t \geqslant (\log n)^{\alpha}$ and both converge to infinity. Based on lemma 8, we get that there exists a reduction that runs in time linear in the output size and given an instance of HS – $\big(S,\, C = \big\{C^j\big\}\,,\, k\big)$, produces an instance of $\text{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$

– $\left(X \in \{0,1\}^{n\times m}, k'\right)$ – s.t. $\log^\alpha n \leqslant \min(t, n-t) = |S|$, $m = n + |C|$, and $k' = k$.

Hence, we get that both n and m are smaller than $2^{(|S||C|)^{1/\alpha}}$ so the reduction is sub-exponential in the input size. We get a sub-exponential reduction from the *NP-complete* problem HS to $\mathrm{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$. Hence, if we can solve $\mathrm{OLD}\big(\varphi, \wedge_{j=1}^{m} x^j\big)$ in polynomial time we can solve any *NP* problem in sub-exponential time.

Theorem (Theorem 11.1).

If ψ is a threshold function with threshold z and z is not $\text{poly}(m)$*-small or* $\text{poly}(m)$*-large* ($\exists \epsilon > 0 \quad z \in [m^{\epsilon}, \, m - m^{\epsilon}]$)*, then* $\text{OLD}(Unan, \psi)$ *is NP-complete.*

Proof.

It is trivial that the problem belongs to *NP*. In order to prove hardness we construct the following polynomial reduction from EBNCD. It's given that $z \in [m^{\epsilon}, \, m - m^{\epsilon}]$ and hence $m - z, z \geqslant m^{\epsilon}$ and both diverge to infinity. Based on Lemma 12, we get that there exists a reduction that runs in time linear in the output size and given an instance of EBNCD – $(G = (L, R, E), k)$, produces an instance of $\text{OLD}(Unan, \psi)$ – $\left(X \in \{0,1\}^{n \times m}, k'\right)$ – s.t. $n = |L|$, $m^{\epsilon} \leqslant \min(z, m - z) = \max(k, |R| - k)$, and $k' = n - k$. Hence, we get that both m and n are polynomial in the size of the input so the reduction is polynomial. We get a polynomial reduction from the *NP-complete* problem EBNCD to $\text{OLD}(Unan, \psi)$ and hence $\text{OLD}(Unan, \psi)$ is *NP-complete*.

Theorem (Theorem 11.4).

If ψ is a threshold function with threshold z and z is not $\text{polylog}(m)$*-small or* $\text{polylog}(m)$*-large* ($\forall \alpha > 1 \quad z \in [(\log m)^{\alpha}, \, m - (\log m)^{\alpha}]$ *for large enough m), then* $\text{OLD}(Unan, \psi)$ *is not in P (assuming NP $\not\subseteq$ SUBEXP).*

Proof.

In order to prove hardness we construct the following sub-exponential reduction from EBNCD. It's given that $z \in [(\log m)^{\alpha}, \, m - (\log m)^{\alpha}]$ and hence $m - z, z \geqslant (\log m)^{\alpha}$ and both diverge to infinity. Based on Lemma 12, we get that there exists a reduction that runs in time linear in the output size and given an instance of EBNCD – $(G = (L, R, E), k)$, produces an instance of $\text{OLD}(Unan, \psi)$ – $\left(X \in \{0,1\}^{n \times m}, k'\right)$ – s.t. $n = |L|$, $(\log m)^{\alpha} \leqslant \min(z, m - z) = \max(k, |R| - k)$, and $k' = n - k$. Hence, we get that m is smaller than $2^{|R|^{1/\alpha}}$ and n is linear in $|L|$ so the reduction is sub-exponential in the input size. We get a sub-exponential reduction from the *NP-complete* problem EBNCD to $\text{OLD}(Unan, \psi)$. Hence, if we can solve $\text{OLD}(Unan, \psi)$ in polynomial time we can solve any *NP* problem in sub-exponential time.