

Cheap-Talk with Incomplete Information on Both Sides

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Abstract

We provide a characterization of the set of equilibria of two-person cheap-talk games with incomplete information on both sides. Each equilibrium generates a martingale with certain properties and one can obtain an equilibrium from each such martingale. Moreover, the characterization depends on the number of possible messages. It is shown that for every natural number n , there exist equilibrium payoffs that can be obtained only when the number of possible messages is at least n .

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1 Introduction

Cheap-Talk is a communication that costs nothing and occurs before the players choose their actions. The payoffs to the players depend only on their actions and not on the messages that were sent during the Cheap-Talk phase. Cheap-Talk with Incomplete Information is a three phase game. In the first phase the players receive their private information, in the second phase they talk (communicate) and in the last phase they choose actions and get payoffs. Again, the payoffs depend only on the actions. The players can use the Cheap-Talk to transfer information and to choose an equilibrium from a possible set of equilibria. The players can also ignore the Cheap-Talk and in this case the set of equilibria is the same as in the Incomplete Information game without Cheap-Talk. The Cheap-Talk enlarges the equilibrium set, and a question that rises in this setup is the structure of the new set.

A full characterization in the Two-Player One-Sided Information case was given by Aumann and Hart (1996). In that work they show that the concept of a bi-martingale, first used to characterize the equilibrium set of Repeated Games with Incomplete Information on One Side (Hart 1985, Aumann and Hart 1986), is applicable also in the Cheap-Talk games. They also show that the set of possible messages does not affect the equilibrium set, provided that it includes at least two different messages.

In this work we study the general (finite) two player case. In the Two-Sided Information case we show that bi-martingales are replaced by an appropriate class of “admissible martingales”. However, these do not have all the nice properties of bi-martingales. In particular, the size of the set of possible messages does affect the equilibrium set, and in a very strong sense, as an example will show.

In section 2 we discuss some general properties of games with incomplete information and in section 3 we define the notion of admissible martingales and state the main results. In section 4 an example is analyzed. The three sections are independent and can be read in any order.

2 Games with Incomplete Information

2.1 The model

We define two games - G and $\Gamma(M)$. G is a (finite) game of incomplete information, and $\Gamma(M)$ is its cheap-talk extension. In $\Gamma(M)$ the cheap-talk occurs after the players have received their private information.

- $G = (N, (C_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (u_n)_{n \in N})$ is defined by the following:
 1. N is a finite set of players. Without loss of generality (and some abuse of notation) we assume that $N = \{1, 2, 3, \dots, N\}$.
 2. C_n is a finite set of actions for player $n \in N$. Let $C := \prod_{n \in N} C_n$ and $C_{-n} := \prod_{m \in N \setminus \{n\}} C_m$.
 3. K_n is a finite set of types of player $n \in N$. Each player $n \in N$ knows his own type $k_n \in K_n$. Let $K := \prod_{n \in N} K_n$ and $K_{-n} := \prod_{m \in N \setminus \{n\}} K_m$.

4. $p_n : K_n \rightarrow \Delta(K_{-n})$ for ¹ all $n \in N$. The belief of type k_n of player n is $p_n(k_n)$. We will denote $p_n(k_n)$ by p_{k_n} , hence $p_{k_n}(k_{-n})$ is the probability assigned by type k_n of player n to the combination of types $k_{-n} \in K_{-n}$.
5. $u_n : C \times K \rightarrow \mathbb{R}$ is the payoff function of player n . That is, $u_n(c, k)$ is the payoff of player n for the profile of actions $c \in C$ and profile of types $k \in K$.
6. The game is played as follows: each player $n \in N$ chooses (simultaneously) an action $c_n \in C_n$. Let $c := (c_n)_{n \in N}$. The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is:

$$E_{k_n}(c) := \sum_{k_{-n} \in K_{-n}} u_n(c, (k_n, k_{-n})) p_{k_n}(k_{-n})$$

7. 1,2,3,4,5 and 6 are common knowledge.

- The game $\Gamma(M) = ((N, (C_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (u_n)_{n \in N}, M)$ is defined by 1,2,3,4,5 and in addition:

8. A finite set M , the set of possible messages in the talk phase. We assume that $|M| \geq 2$.
9. The game $\Gamma(M)$ has two phases:

The Talk Phase : This phase is divided into periods $t=1,2,3,\dots$. For each t and $n \in N$, player n chooses a message $m_t^n \in M$. The choices are made simultaneously.

The Action Phase : Each player $n \in N$ chooses (simultaneously) an action $c_n \in C_n$. Let $c := (c_n)_{n \in N}$. The subjective expected payoff of type $k_n \in K_n$ of player $n \in N$ is (see item 6 above): $E_{k_n}(c) := \sum_{k_{-n} \in K_{-n}} u_n(c, (k_n, k_{-n})) p_{k_n}(k_{-n})$.

10. All players have perfect recall.
11. 1,2,3,4,5,8,9,10 are common knowledge.

Definition 2.1: The beliefs $(p_n)_{n \in N}$ are said to be (*Harsanyi*) *consistent* if there exists a common prior distribution on K , $P \in \Delta(K)$, such that

$$p_{k_n}(k_{-n}) = \frac{P(k_n, k_{-n})}{P(k_n)} \text{ for all } (k_n, k_{-n}) \in K \text{ and } n \in N$$

(Note that $P(k_n) := \sum_{l_{-n} \in K_{-n}} P(k_n, l_{-n})$). In addition we say that the information is *independent* if $P = \prod_{n \in N} P^n$ for $P^n \in \Delta(K_n)$.

Note: For the results in this paper we do not assume consistency nor independence. These special cases will be useful in our proofs.

The players have perfect recall, hence m_t^n are functions of the history of messages of length $t-1$, $h_{t-1} := ((m_1^1, m_1^2, \dots, m_1^N), (m_2^1, m_2^2, \dots, m_2^N), \dots, (m_{t-1}^1, m_{t-1}^2, \dots, m_{t-1}^N))$. The actions, chosen by the players in the action phase, are functions of the infinite sequence defined in the talk phase, $h_\infty := ((m_1^1, m_1^2, \dots, m_1^N), (m_2^1, m_2^2, \dots, m_2^N), \dots, (m_t^1, m_t^2, \dots, m_t^N) \dots)$. Let $H_t = (M^N)^t$ be the set of histories of length t . Define $H_0 = \{\phi\}$. Let $H_\infty = \prod_{t=1}^{\infty} (M^N)$ be the set of infinite histories. On H_∞ , we define, for every t , a finite field \mathcal{H}_t as the field generated by the first t coordinates. That is,

¹for a finite set X , $\Delta(X)$ is the $|X| - 1$ dimensional simplex of probability vectors on X .

$h_\infty^1, h_\infty^2 \in H_\infty$ are in the same atom of \mathcal{H}_t if and only if for every $1 \leq u \leq t$ there ² exists $h_\infty^1(u) = h_\infty^2(u)$. Let \mathcal{H}_∞ be the σ -field generated by $\{\mathcal{H}_t\}_{t=1}^\infty$. Our basic measure space is $(\Omega, \mathcal{A}) = (K \times H_\infty \times C, 2^K \otimes \mathcal{H}_\infty \otimes 2^C)$. A point in Ω is a triple (k, h_∞, c) , where $k \in K$ is a profile of types, $h_\infty \in H_\infty$ is an history of the game (the communication that took place) and $c \in C$ is a profile of actions.

Since $\Gamma(M)$ is a game with perfect recall, we can restrict ourselves to behaviour strategies (see Aumann 1964). To shorten the writing, whenever we write 'strategy' we will mean a behaviour strategy.

Definition 2.2: A strategy α^n of player $n \in N$ in G is a function $\alpha^n : K_n \rightarrow \Delta(C_n)$.

Let Ψ^n be the set of strategies of player $n \in N$. Denote $\Psi := \prod_{n \in N} \Psi^n$ and $\Psi^{-n} := \prod_{m \in N \setminus \{n\}} \Psi^m$.

Definition 2.3: The subjective payoff of type k_n of player $n \in N$, given a profile of strategies $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^N)$ is

$$E_{k_n}^G(\alpha) := \sum_{k_{-n} \in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1, c_2, \dots, c_N) \in C} u_n(c, (k_n, k_{-n})) \prod_{m \in N} \alpha_{k_m}^m(c_m)$$

Definition 2.4:

A strategy σ^n of player $n \in N$ in $\Gamma(M)$ is $\sigma^n = (\{\sigma_t^n\}_{t \in \mathbb{N}}, \sigma_\infty^n)$ such that: ³

1. $\sigma_t^n : K_n \times H_{t-1} \rightarrow \Delta(M)$ for all $t \in \mathbb{N}$.
2. $\sigma_\infty^n : K_n \times H_\infty \rightarrow \Delta(C_n)$.
3. σ_∞^n is $2^K \otimes \mathcal{H}_\infty$ -measurable.

Note that in $\Gamma(M)$ the strategy has a talk component, $\{\sigma_t^n\}_{t \in \mathbb{N}}$, and an action component, σ_∞^n . Let Σ^n be the set of strategies of player n for $n \in N$. Let $\Sigma := \prod_{n \in N} \Sigma^n$.

Every pair $(\sigma, k) \in \Sigma \times K$ defines a probability measure $\pi_{\sigma, k}$ on $(H_\infty, \mathcal{H}_\infty)$, i.e., let $h_t := ((m_1^1, m_2^1, \dots, m_1^n), (m_1^2, m_2^2, \dots, m_2^n), \dots, (m_1^t, m_2^t, \dots, m_t^n))$, then $\pi_{\sigma, k}(h_t)$ is the probability that the first message sent by player 1 is m_1^1 , the first message sent by player 2 is m_2^2 , the second message sent by player 1 is m_1^2 , and so on, given that $k = (k_1, k_2, \dots, k_n)$ is the profile of types and player n plays according to σ^n ($\sigma = (\sigma^1, \sigma^2, \dots, \sigma^N)$) for all $n \in N$.

Definition 2.5: The subjective expected payoff of type k_n in $\Gamma(M)$, given a profile of strategies $\sigma = (\sigma^1, \sigma^2, \dots, \sigma^N)$, is:

$$E_{k_n}^{\Gamma(M)}(\sigma) := \sum_{k_{-n} \in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1, c_2, \dots, c_N) \in C} u_n(c, (k_n, k_{-n})) \int_{H_\infty} \prod_{m \in N} \sigma_\infty^m(k_m, h_\infty)(c_m) d\pi_{\sigma, k}(h_\infty)$$

Definition 2.6: $a \in \prod_{n \in N} \mathbb{R}^{K_n} = \mathbb{R}^{\cup_{n \in N} K_n}$ is an equilibrium payoff vector in G if there exists α , a profile of strategies, such that

1. $a^{k_n} = E_{k_n}^G(\alpha)$ for all $n \in N$ and $k_n \in K$.

² $h_\infty(u)$ is the vector of N messages sent by the players at period u , according to the infinite history h_∞ .

³ \mathbb{N} denotes the set of natural numbers $\{1, 2, 3, \dots\}$.

2. $a^{k_n} \geq E_{k_n}^G(\alpha'^n, \alpha^{-n})$ for all $n \in N$, $k_n \in K$ and $\alpha'^n \in \Psi^n$.

Let $\mathcal{E}^G := \{a \in \mathbb{R}^{\cup_{n \in N} K_n} \text{ s.t. } a \text{ is an equilibrium payoff vector in } G\}$.

Definition 2.7: $a \in \mathbb{R}^{\cup_{n \in N} K_n}$ is an *equilibrium payoff vector* in $\Gamma(M)$ if there exists $\sigma \in \Sigma$ (a profile of strategies) such that

1. $a^{k_n} = E_{k_n}^{\Gamma(M)}(\sigma)$ for all $n \in N$ and $k_n \in K$.
2. $a^{k_n} \geq E_{k_n}^{\Gamma(M)}(\sigma'^n, \sigma^{-n})$ for all $n \in N$, $k_n \in K$ and $\sigma'^n \in \Sigma^n$.

Let $\mathcal{E}^{\Gamma(M)} := \{a \in \mathbb{R}^{\cup_{n \in N} K_n} \text{ s.t. } a \text{ is an equilibrium vector payoff in } \Gamma(M)\}$.

2.2 On some properties of games with incomplete information

Definition 2.8: The games $G^1 = (N^1, (C_n^1)_{n \in N^1}, (K_n^1)_{n \in N^1}, (p_n^1)_{n \in N^1}, (u_n^1)_{n \in N^1})$ and $G^2 = (N^2, (C_n^2)_{n \in N^2}, (K_n^2)_{n \in N^2}, (p_n^2)_{n \in N^2}, (u_n^2)_{n \in N^2})$ are *equivalent* if the following holds:

1. $N^1 = N^2 = N$.
2. $C_n^1 = C_n^2 = C_n$ for all $n \in N$.
3. $K_n^1 = K_n^2 = K_n$ for all $n \in N$.
4. $E_{k_n}^{G^1}(\alpha) = E_{k_n}^{G^2}(\alpha)$ for all $n \in N$, $k_n \in K_n$ and $\alpha \in \Psi$ (note that 1,2 and 3 implies $\Psi^1 = \Psi^2 = \Psi$).

Remark: Instead of the last condition we could use:

5. For every $n \in N$ and $k_n \in K_n$ there exists $a_n > 0$ and b_n such that $E_{k_n}^{G^1}(\alpha) = a_n E_{k_n}^{G^2}(\alpha) + b_n$ for all $\alpha \in \Psi$.

Definition 2.9: The games $\Gamma^1(M) = (N^1, (C_n^1)_{n \in N^1}, (K_n^1)_{n \in N^1}, (p_n^1)_{n \in N^1}, (u_n^1)_{n \in N^1}, M)$ and $\Gamma^2(M) = (N^2, (C_n^2)_{n \in N^2}, (K_n^2)_{n \in N^2}, (p_n^2)_{n \in N^2}, (u_n^2)_{n \in N^2}, M)$ are *equivalent* if the following holds:

1. $N^1 = N^2 = N$.
2. $C_n^1 = C_n^2 = C_n$ for all $n \in N$.
3. $K_n^1 = K_n^2 = K_n$ for all $n \in N$.
4. $E_{k_n}^{\Gamma^1(M)}(\sigma) = E_{k_n}^{\Gamma^2(M)}(\sigma)$ for all $n \in N$, $k_n \in K_n$ and $\sigma \in \Sigma$.

remark: similar definition can be defined for repeated games.

Following Myerson (1991) we can state the following theorem:

Theorem 2.10: *Every finite game with incomplete information is equivalent to a game with consistent and independent incomplete information.*

Proof:

Fix a game $G = (N, (C_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (u_n)_{n \in N})$. We define a game, equivalent to G , $\tilde{G} = (N, (C_n)_{n \in N}, (K_n)_{n \in N}, (\tilde{p}_n)_{n \in N}, (\tilde{u}_n)_{n \in N})$ by:

$$\tilde{p}_{k_n}(k_{-n}) := \frac{1}{|K_{-n}|} \quad \text{for all } n \in N, k_n \in K_n \text{ and } k_{-n} \in K_{-n}.$$

and

$$\begin{aligned} \tilde{u}_n(c, k) &:= |K_{-n}| p_{k_n}(k_{-n}) u_n(c, k) \quad \text{for all } n \in N, k_n \in K_n \text{ and } k_{-n} \in K_{-n}. \\ E_{k_n}^{\tilde{G}}(\alpha) &:= \sum_{k_{-n} \in K_{-n}} \tilde{p}_{k_n}(k_{-n}) \sum_{c=(c_1, c_2, \dots, c_N) \in C} \tilde{u}_n(c, (k_n, k_{-n})) \prod_{m \in N} \alpha_{k_m}^m(c_m) \\ &= \sum_{k_{-n} \in K_{-n}} \frac{1}{|K_{-n}|} \sum_{c=(c_1, c_2, \dots, c_N) \in C} |K_{-n}| p_{k_n}(k_{-n}) u_n(c, (k_n, k_{-n})) \prod_{m \in N} \alpha_{k_m}^m(c_m) \\ &= \sum_{k_{-n} \in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1, c_2, \dots, c_N) \in C} u_n(c, (k_n, k_{-n})) \prod_{m \in N} \alpha_{k_m}^m(c_m) = E_{k_n}^G(\alpha) \end{aligned}$$

Hence G and \tilde{G} are equivalent and \tilde{G} is a game with independent incomplete information. ■

In general theorem 2.10 is not correct for specific families of games with incomplete information. For example, zero-sum games with incomplete information are not always equivalent to zero-sum games with independent incomplete information (and indeed the characterization of zero-sum games with independent incomplete information is in general simpler than the characterization of the general (even the consistent) zero-sum games with incomplete information). The transformation in the proof of theorem 2.10 changes zero-sum games with incomplete information into games with independent incomplete information that are not necessarily zero-sum. On the other hand theorem 2.10 is still correct for cheap-talk games and for repeated games as the following theorem shows.

Theorem 2.11: *Every cheap-talk game with incomplete information is equivalent to a cheap-talk game with consistent and independent incomplete information.*

Proof:

Fix a game $\Gamma(M) = (N, (C_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (u_n)_{n \in N}, M)$. We define a game, $\tilde{\Gamma}(M) = (N, (C_n)_{n \in N}, (K_n)_{n \in N}, (\tilde{p}_n)_{n \in N}, (\tilde{u}_n)_{n \in N}, M)$, equivalent to $\Gamma(M)$, by $\tilde{p}_{k_n}(k_{-n}) := \frac{1}{|K_{-n}|}$ and $\tilde{u}_n(c, k) := |K_{-n}| p_{k_n}(k_{-n}) u_n(c, k)$ for all $n \in N, k_n \in K_n$ and $k_{-n} \in K_{-n}$ (this is the same transformation used in the proof of theorem 2.10).

$$\begin{aligned} E_{k_n, \sigma}^{\tilde{\Gamma}(M)} &= \sum_{k_{-n} \in K_{-n}} \tilde{p}_{k_n}(k_{-n}) \sum_{c=(c_1, c_2, \dots, c_N) \in C} \tilde{u}_n(c, (k_n, k_{-n})) \int_{H_\infty} \prod_{m \in N} \sigma_\infty^m(k_m, h_\infty)(c_m) d\pi_{\sigma, k}(h_\infty) \\ &= \sum_{k_{-n} \in K_{-n}} \frac{1}{|K_{-n}|} \sum_{c=(c_1, c_2, \dots, c_N) \in C} |K_{-n}| p_{k_n}(k_{-n}) u_n(c, (k_n, k_{-n})) \int_{H_\infty} \prod_{m \in N} \sigma_\infty^m(k_m, h_\infty)(c_m) d\pi_{\sigma, k}(h_\infty) \\ &= \sum_{k_{-n} \in K_{-n}} p_{k_n}(k_{-n}) \sum_{c=(c_1, c_2, \dots, c_N) \in C} u_n(c, (k_n, k_{-n})) \int_{H_\infty} \prod_{m \in N} \sigma_\infty^m(k_m, h_\infty)(c_m) d\pi_{\sigma, k}(h_\infty) = E_{k_n}^{\Gamma(M)}(\sigma) \end{aligned}$$

Hence $\Gamma(M)$ and $\tilde{\Gamma}(M)$ are equivalent and $\tilde{\Gamma}(M)$ is a game with independent incomplete information. ■

Remark: The analogous theorem is correct for repeated games (with and without discounting).

The transformation used in the proof of theorems 2.10 and 2.11 may fail to preserve a known-own-payoffs property (private value assumption, i.e. $u_n(c, k) = u_n(c, k_n)$ for all $n \in N$, $c \in C$ and $k = (k_1, k_2, \dots, k_N) \in K$). This problem can be solved using the next definition and theorem.

Definition 2.12: G^1 and G^2 are *semi-equivalent* if $\mathcal{E}^{G^1} = \mathcal{E}^{G^2}$. $\Gamma^1(M)$ and $\Gamma^2(M)$ are *semi-equivalent* if $\mathcal{E}^{\Gamma^1(M)} = \mathcal{E}^{\Gamma^2(M)}$.

Theorem 2.13: *every game with incomplete information is semi-equivalent to a game with known own payoffs and with the same information structure (i.e., N , $(K_n)_{n \in N}$ and $(p_n)_{n \in N}$ are not changed).*

Proof: Fix a game $G = (N, (C_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (u_n)_{n \in N})$. Let Z be the upper bound of the absolute value of the possible payoffs, that is $Z := \max_{n \in N, c \in C, k \in K} \{|u_n(c, k)|\}$. We define a game, semi-equivalent to G , $\tilde{G} = (N, (\tilde{C}_n)_{n \in N}, (K_n)_{n \in N}, (p_n)_{n \in N}, (\tilde{u}_n)_{n \in N})$ by:

1. For all $n \in N$ define $\tilde{C}_n := \cup_{k_n \in K_n} \tilde{C}^{k_n}$, where \tilde{C}^{k_n} is a set isomorphic to C_n with the isomorphism $s_{k_n} : \tilde{C}^{k_n} \rightarrow C_n$. Denote $s_n = \cup_{k_n \in K_n} s_{k_n}$ (i.e., $s_n : \tilde{C}_n \rightarrow C_n$). Define a function $k^n : \tilde{C}_n \rightarrow K_n$ by $k^n(\tilde{c}_n) = k_n$ if $\tilde{c}_n \in \tilde{C}^{k_n}$.
2. For all $n \in N$, $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_N) \in \prod_{m \in N} \tilde{C}_m$ and $k = (k_1, k_2, \dots, k_N) \in K$ define:

$$\tilde{u}_n(\tilde{c}, k) := \begin{cases} u_n((s_1(\tilde{c}_1), \dots, s_N(\tilde{c}_N)), (k^1(\tilde{c}_1), \dots, k^{n-1}(\tilde{c}_{n-1}), k_n, k^{n+1}(\tilde{c}_{n+1}), \dots, k^N(\tilde{c}_N))) & \tilde{c}_n \in \tilde{C}^{k_n} \\ -(Z + 1) & \text{otherwise} \end{cases}$$

\tilde{G} is a game with known own payoff property. We will prove now that G and \tilde{G} are semi-equivalent. Fix $a \in \mathcal{E}^G$. We will show that $a \in \mathcal{E}^{\tilde{G}}$. There exists $\alpha \in \Psi$ satisfying items 1 and 2 of definition 2.6. Define $\tilde{\alpha} \in \tilde{\Psi}$ by:

$$\tilde{\alpha}_{k_n}^n(\tilde{c}_n) := \begin{cases} \alpha_{k_n}^n(s_{k_n}(\tilde{c}_n)) & \tilde{c}_n \in \tilde{C}^{k_n} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

for all $n \in N$, $k_n \in K_n$ and $\tilde{c}_n \in \tilde{C}_n$. Despite the abuse of notation we will denote by c_n the strategy that assigns probability 1 to the action c_n . Now $E_{k_n}^{\tilde{G}}(\tilde{\alpha}) = a_{k_n}$ for all $n \in N$ and $k_n \in K_n$ and $E_{k_n}^{\tilde{G}}(\tilde{\alpha}^{-n}, \tilde{c}'_n) \leq E_{k_n}^G(\alpha^{-n}, s_n(\tilde{c}'_n)) \leq a_{k_n}$ for all $n \in N$, $k_n \in K_n$ and $\tilde{c}'_n \in \tilde{C}_n$, hence $a \in \mathcal{E}^{\tilde{G}}$. Thus we proved that $\mathcal{E}^G \subset \mathcal{E}^{\tilde{G}}$. to prove that $\mathcal{E}^G \supset \mathcal{E}^{\tilde{G}}$ fix $a \in \mathcal{E}^{\tilde{G}}$. There exists $\tilde{\alpha}$ such that

1. $E_{k_n}^{\tilde{G}}(\tilde{\alpha}) = a_{k_n}$ for all $n \in N$ and $k_n \in K_n$.
2. $E_{k_n}^{\tilde{G}}(\tilde{\alpha}^{-n}, \tilde{c}'_n) \leq a_{k_n}$ for all $n \in N$, $k_n \in K_n$ and $\tilde{c}'_n \in \tilde{C}_n$.

The second condition implies that $\tilde{\alpha}_{k_n}(\tilde{c}_n) = \mathbf{0}$ for all $\tilde{c}_n \notin \tilde{C}^{k_n}$ (i.e., $\tilde{\alpha}_{k_n}(\tilde{C}^{k_n}) = 1$). Define $\alpha \in \Psi$ by $\alpha_{k_n}(c_{k_n}) := \tilde{\alpha}(s_{k_n}(\tilde{c}_{k_n}))$. $E_{k_n}^G(\alpha) = E_{k_n}^{\tilde{G}}(\tilde{\alpha}) = a$ and $E_{k_n}^G(\alpha^{-n}, c'_n) = E_{k_n}^{\tilde{G}}(\tilde{\alpha}^{-n}, s_{k_n}^{-1}(c'_n)) \leq a_{k_n}$ for all $n \in N$, $k_n \in K_n$ and $c'_n \in C_n$. Hence $a \in \mathcal{E}^G$. ■

Remarks:

1. Theorem 2.13 is also correct for cheap-talk games. One can prove it using the same transformation defined in the proof of theorem 2.13.
2. Theorem 2.13 is correct for repeated games with discounting. Here the transformation should be slightly changed, putting $\frac{-(2Z+1)}{\lambda}$ in the definition of \tilde{u}_n instead of $-(Z+1)$, for λ -discounted games.
3. This will not work in the case of repeated games (where the payoff is defined to be the limit of means), as the expected payoff of a player is not affected by the payoff from a finite number of periods. And indeed the theorem is not true for repeated games, as every equilibrium in repeated games with known own payoffs can be achieved by complete revelation of all the information in the first period of the game (Shalev (1994), Koren (1988)). This is not true in general for repeated games.
4. Theorem 2.13 is correct for stochastic games, and for repeated games with absorbing states. Here the $-(Z+1)$ should be replaced by an absorption with probability 1 and payoff $-(Z+1)$. (in discounted games $\frac{-(2Z+1)}{\lambda}$).
5. The finiteness condition is not essential. We only need u_n to be bounded for every $n \in N$.

3 Characterization of the set of equilibria

We characterize the set of equilibria for two player cheap-talk games. Using theorem 2.11 we can assume that the game is of independent incomplete information (using theorem 2.13 we can even assume that the game is with known own payoffs, but this assumption does not make the characterization simpler). In section 3.1 we define the model and in section 3.2 we introduce the concept of admissible-martingales. The geometrical properties of admissible-martingales are discussed in section 3.3 and in section 3.4 we give the main result.

3.1 The model

We repeat the definitions, given in the previous section, because here we deal with two-player games with consistent and independent information, which enable us to simplify the notations. In addition we insert the information structure (p, q) explicitly into the notations of the games. We define two games - $G(p, q)$ and $\Gamma(p, q, M)$. $G(p, q)$ is a game of incomplete information on both sides, and $\Gamma(p, q, M)$ is its cheap-talk extension. In $\Gamma(p, q, M)$ the cheap-talk occurs after the players have received their private information.

- $G(p, q)$ is defined by the following:
 1. Two players: player 1 and player 2.
 2. A finite set of actions I for player 1, and a finite set of actions J for player 2.
 3. Two finite sets, K and L , such that to each pair $(k \in K, l \in L)$ there corresponds a pair of $I \times J$ matrices $(A^{k,l}, B^{k,l})$. $A^{k,l} = (A^{k,l}(i, j))_{i \in I, j \in J}$, $B^{k,l} = (B^{k,l}(i, j))_{i \in I, j \in J}$.

4. Two probability vectors: $p \in \Delta(K)$, $p = (p(k))_{k \in K}$ and $q \in \Delta(L)$, $q = (q(l))_{l \in L}$.
5. The game $G(p, q)$ has two phases: ⁴

The Information Phase : Nature chooses $\mathbf{k} \in K$ according to p and $\mathbf{l} \in L$ according to q . The choices are made independently, i.e, $Prob(\mathbf{k} = k \text{ and } \mathbf{l} = l) = p(k)q(l)$. \mathbf{k} is told to player 1 and \mathbf{l} is told to player 2.

The Action Phase : Player 1 chooses $i \in I$ and player 2 chooses $j \in J$. The choices are made simultaneously. The payoff to player 1 is $A^{\mathbf{k}, \mathbf{l}}(i, j)$ and the payoff to player 2 is $B^{\mathbf{k}, \mathbf{l}}(i, j)$.

6. 1,2,3,4,5 are common knowledge to both players.

- The game $\Gamma(p, q, M)$, a cheap-talk extension of $G(p, q)$, is defined by 1,2,3,4 and:

7. A finite set M , the set of possible messages in the Talk phase. We assume that $|M| \geq 2$.
8. The game $\Gamma(p, q, M)$ has three phases:

The Information Phase is the same as in $G(p, q)$.

The Talk Phase : This phase is divided into periods $t=1,2,3,\dots$. For each t , player 1 chooses a message $m_t^1 \in M$ and player 2 chooses a message $m_t^2 \in M$. The choices are made simultaneously.

The Action Phase : Player 1 chooses an action $i \in I$ and player 2 chooses an action $j \in J$. The choices are made simultaneously. The payoff to player 1 is $A^{\mathbf{k}, \mathbf{l}}(i, j)$ and the payoff to player 2 is $B^{\mathbf{k}, \mathbf{l}}(i, j)$.

9. The players have perfect recall.

10. 1,2,3,4,7,8,9 are common knowledge to both players.

The players have perfect recall, so m_t^1 and m_t^2 are functions of the history of length $t - 1$, $h_{t-1} := ((m_1^1, m_1^2), (m_2^1, m_2^2), \dots, (m_{t-1}^1, m_{t-1}^2))$. The actions, chosen by the players in the action phase, are functions of $h_\infty := ((m_1^1, m_1^2), (m_2^1, m_2^2), \dots, (m_t^1, m_t^2), \dots)$, the infinite sequence defined in the talk phase. Let $H_t = (M \times M)^t$ be the set of histories of length t . Define $H_0 = \{\phi\}$. Let $H_\infty = \prod_{t=1}^{\infty} (M \times M)$ be the set of infinite histories. On H_∞ , we define for every t , a finite field \mathcal{H}_t . $h_\infty^1, h_\infty^2 \in H_\infty$ are in the same atom of \mathcal{H}_t if and only if for every $1 \leq u \leq t$ there ⁵ exists $h_\infty^1(u) = h_\infty^2(u)$. Let \mathcal{H}_∞ be the σ -field generated by $\{\mathcal{H}_t\}_{t=1}^{\infty}$. Our basic probability space is $(\Omega, \mathcal{A}) = (K \times L \times H_\infty \times I \times J, 2^K \otimes 2^L \otimes \mathcal{H}_\infty \otimes 2^I \otimes 2^J)$. A point in Ω is a five-tuple (k, l, h_∞, i, j) , where (k, l) is a possible state of nature, $h_\infty \in H_\infty$ is an history of the game (the communication that took place), i is an action of player 1 and j is an action of player 2. Defining sequences of random variables, we will use the following notation: a_t, b_t, c_t, \dots will usually be random variables measurable with respect to (H_t, \mathcal{H}_t) , and $a_{h_t}, b_{h_t}, c_{h_t}, \dots$ will denote $a_t(h_t), b_t(h_t), c_t(h_t), \dots$. For $x \in \Delta(I)$ and $y \in \Delta(J)$ we will write $A^{k,l}(x, y)$ instead of $\sum_{i \in I, j \in J} x(i)y(j)A^{k,l}(i, j)$ and $B^{k,l}(x, y)$ instead of $\sum_{i \in I, j \in J} x(i)y(j)B^{k,l}(i, j)$.

Since $\Gamma(p, q, M)$ is a game with perfect recall, we can restrict ourselves to behaviour strategies (see Aumann 1964). To shorten the writing, whenever we write 'strategy' we will mean a behaviour strategy.

⁴This is an equivalent model to the model described in chapter 2. \mathbf{k} and \mathbf{l} are the types of the players.

⁵ $h_\infty(u)$ is the two messages sent by the players at period u , according to the infinite history h_∞ .

Definition 3.1:

A strategy σ of player 1 in $\Gamma(p, q, M)$ is $\sigma = (\{\sigma_t\}_{t \in \mathbb{N}}, \sigma_\infty)$ such that:⁶

1. $\sigma_t : K \times H_{t-1} \rightarrow \Delta(M)$ for all $t \in \mathbb{N}$.
2. $\sigma_\infty : K \times H_\infty \rightarrow \Delta(I)$.
3. σ_∞ is $2^K \otimes \mathcal{H}_\infty$ -measurable.

A strategy τ of player 2 in $\Gamma(p, q, M)$ is $\tau = (\{\tau_t\}_{t \in \mathbb{N}}, \tau_\infty)$ such that:

1. $\tau_t : L \times H_{t-1} \rightarrow \Delta(M)$ for all $t \in \mathbb{N}$.
2. $\tau_\infty : L \times H_\infty \rightarrow \Delta(J)$.
3. τ_∞ is $2^L \otimes \mathcal{H}_\infty$ -measurable.

Let Σ^i be the set of strategies for player i for $i = 1, 2$. Denote:

$$Z := \max_{k \in K, l \in L, i \in I, j \in J} \{|A^{k,l}(i, j)|, |B^{k,l}(i, j)|\}.$$

That is, Z is the upper bound of the absolute value of the possible payoffs.

Now we define the equilibrium in the game $G(p, q)$. Later, we will use the equilibrium in $G(p, q)$ in the characterization of the equilibrium in the cheap-talk extension $\Gamma(p, q, M)$.

Definition 3.2:

$a \in [0, Z]^K$ and $b \in [0, Z]^L$ are *equilibrium vector payoffs* in $G(p, q)$ if there exist $\alpha \in (\Delta(I))^K$ and $\beta \in (\Delta(J))^L$ such that:

- 3.2.1 $a^k = \sum_{l \in L} q(l) A^{k,l}(\alpha^k, \beta^l)$ for all $k \in K$ such that $p(k) > 0$.
- 3.2.2 $b^l = \sum_{k \in K} p(k) B^{k,l}(\alpha^k, \beta^l)$ for all $l \in L$ such that $q(l) > 0$.
- 3.2.3 $a^k \geq \sum_{l \in L} q(l) A^{k,l}(\gamma, \beta^l)$ for all $k \in K$ and $\gamma \in \Delta(I)$.
- 3.2.4 $b^l \geq \sum_{k \in K} p(k) B^{k,l}(\alpha^k, \delta)$ for all $l \in L$ and $\delta \in \Delta(J)$.

Note the difference between definition 3.2 and definition 2.6. In definition 3.2 we allow a^k to be greater than the payoff of type k when $p(k) = 0$.

Going back to the Cheap-Talk extension, we need a few definitions. Every 4-tuple $(\sigma, \tau, k, l) \in \Sigma^1 \times \Sigma^2 \times K \times L$ defines a *probability* measure $\pi_{\sigma, \tau, k, l}$ on $(H_\infty, \mathcal{H}_\infty)$, i.e, for an history $h_t := ((m_1^1, m_1^2), (m_2^1, m_2^2), \dots, (m_t^1, m_t^2))$, $\pi_{\sigma, \tau, k, l}(h_t)$ is the probability that the first message sent by player 1 is m_1^1 , the first message sent by player 2 is m_1^2 , the second message sent by player 1 is m_2^1 , and so on, given that $\mathbf{k} = k, \mathbf{l} = l$, player 1 plays according to σ and player 2 plays according to τ . We derive from $\pi_{\sigma, \tau, k, l}$ another probability measure on $(H_\infty \times K \times L, \mathcal{H}_\infty \otimes 2^K \otimes 2^L)$

$$P_{\sigma, \tau, p, q}(h_t, k, l) := p(k)q(l)\pi_{\sigma, \tau, k, l}(h_t)$$

Note that $P_{\sigma, \tau, p, q}(\mathbf{k} = k, \mathbf{l} = l) = \sum_{h_t \in H_t} P_{\sigma, \tau, p, q}(h_t, k, l) = p(k)q(l)$. Denote by $E_{\sigma, \tau, p, q}$ the expectation with respect to $P_{\sigma, \tau, p, q}$. We will denote $P_{\sigma, \tau, p, q}(\cdot | \mathbf{k} = k)$, $E_{\sigma, \tau, p, q}(\cdot | \mathbf{k} = k)$, $P_{\sigma, \tau, p, q}(\cdot | \mathbf{l} = l)$ and $E_{\sigma, \tau, p, q}(\cdot | \mathbf{l} = l)$ by $P^k(\cdot)$, $E^k(\cdot)$, $P^l(\cdot)$ and $E^l(\cdot)$ respectively and $P_{\sigma, \tau, p, q}$ by P . Denote by \mathbf{a} and \mathbf{b} the (random) payoff of player 1 and player 2 respectively.

⁶ \mathbb{N} denotes the set of natural numbers $\{1, 2, 3, \dots\}$.

