# ON ATMOSPHERE EXTERNALITY AND CORRECTIVE TAXES

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**Abstract**

It has been argued that in the presence of an 'Atmosphere Externality' and competitive behavior by households, a uniform commodity tax on the externality - generating good attains the first best. It is demonstrated, however, that if income redistribution is desirable then personalized taxes are required for a second-best optimum. Each of these taxes is the sum of a uniform (across households) tax and a component, positive or negative, which depends on the household's income and demand elasticities. Second-best optimal indirect taxes and rules for investment in externality-reducing measures are also considered.

Kewords: atmosphere externality, optimal taxation,

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# 1 Introduction

It is commonly argued (e.g. Myles [1995], ch. 10 and the references therein) that in the presence of a negative '*Atmosphere Externality*' and competitive ("large-number") behavior by households, a uniform tax on an externality-generating good suffices to attain a first-best welfare optimum. This holds, however, only if income redistribution is not socially desirable. When income distribution is not optimal, the second best optimum entails non-uniform (personalized) taxes on the externality-generating good. It is shown that these taxes have an additive form: a *uniform* component that reflects the damage to the 'atmosphere' generated by each household's marginal consumption (the "*efficiency factor*") and a component that varies across households and reflects an income redistribution objective (the "*redistributive factor*"). When only a uniform commodity tax is feasible, then the optimal tax is the sum of the "*efficiency factor*" and a weighted combination of the "*redistribution factors*" for all households. We also examine the rule for optimal investment in externality reducing devices. Generally, the first-best rule for optimal investment does not hold when redistribution is desirable. However, under weak separability, the first-best rule carries-over to the second-best.

In the second-best optimum, we also examine the desirability of taxes on goods that do not generate externalities. It is shown that the purpose of such taxes is *solely* for redistribution and that they are not directly related to the level of the externality.

# 2 The Model

'The economy consists of $H$ households, denoted $h = 1, 2, ..., H$, who consume two goods: $X$, the externality-generating good, and $Y$ (the 'numeráire'). The utility of household $h$, $u^h$, is written

$$u^h = u^h(x_h, y_h, \phi) \qquad h = 1, 2, ..., H, \tag{1}$$

Where $\phi = \phi(x, e)$, $x = \sum_{h=1}^{H} x_h$ and $e$ is a scalar. The quantities $x_h$ and $y_h$ are the consumptions of $X$ and $Y$, respectively, by household $h$. It is assumed that $u^h$ is $C^2$ and strictly quasi-concave in ($x_h$, $y_h$). The function $\phi$, called '*Atmosphere Externality*', depends on the *aggregate consumption* of the externality-generating good, $X$, and on a resource input, $e$. For concreteness, we shall discuss *external diseconomies*, i.e. $u_3^h = \frac{\partial u^h}{\partial \phi} < 0$. (All conclusions carry-over, with obvious sign changes, to the case of external economies).

While atmosphere externality depends positively on aggregate consumption, $\phi_1 > 0$, it is possible to invest in $e$ to reduce the externality, $\phi_2 < 0$.

For simplicity, production of $X$, $Y$ and $e$ is assumed to be linear.[1] That is, unit costs of $X$, $p$, of $Y$, $1(= \textit{unity})$ and of $e$, $\pi$, are all fixed positive numbers. The level of total resources, $M$, is assumed to be given. Thus, feasible allocations satisfy

$$M - px - y - \pi e \geq 0. \tag{2}$$

Where $y = \sum_{h=1}^{H} y$ is the aggregate consumption of $Y$.

Let

$$(\underline{x}^*,\ \underline{y}^*,\ e^*) = \textit{argmax} \left\{ \sum_{h=1}^{H} u^h(x_h, y_y, \phi) \mid M - px - y - \pi e \geq 0 \right\} \tag{3}$$

be the *First-Best* allocation. Assuming that $(\underline{x}^*,\ \underline{y}^*,\ e^*) >> 0$, the following first-order conditions (FOC) are satisfied at $(\underline{x}^*,\ \underline{y}^*,\ e^*)$:

$$u_1^h + \left( \sum_{h=1}^{H} u_3^h \right) \phi_1 - \lambda p = 0, \qquad h = 1, 2, ..., H; \tag{4}$$

$$u_2^h - \lambda = 0, \qquad\qquad h = 1, 2, ..., H; \tag{5}$$

$$\left( \sum_{h=1}^{H} u_3^h \right) \phi_2 - \lambda \pi = 0 \tag{6}$$

[1] The qualitative results carry-over to more general production functions.

where $\lambda > 0$ is the shadow-price of the resource constraint (2) and subscripts denote partial derivatives.

Let

$$\tau^* = \frac{-\left(\sum_{h=1}^{H} u_3^h\right)\phi_1}{\lambda} > 0, \tag{7}$$

where the R.H.S. of (7) is evaluated at ($\underline{x}^*$, $\underline{y}^*$, $e^*$). It is obvious from (4), why it has often been argued (e.g. Myles [1995], ch. 10, Aoki [1971], Chipman [1970]) that a uniform-tax, $\tau^*$, imposed on $X$ leads to the first-best allocation. However, as we shall argue, this inference is not warranted whenever condition (5) is not satisfied, i.e. *whenever income redistribution is desirable.* Note also that the optimum investment rule for reduction of atmosphere externality, (6), equates marginal costs, $\pi$, with the value of marginal benefits. It is of interest to examine how this rule is modified in second-best situations.

# 3 Competitive Economy: Personal Taxes

Assume that the markets for $X$ and $Y$ are competitive, i.e. producer prices are $p$ and 1, respectively, and a unit-tax of $t_h$ is imposed on the purchase of the externality-generating good $X$ by household $h$. Households regard the atmosphere as given, disregarding the effect of their own consumption on $\phi$. This is a familiar "*large numbers*" assumption. Under this assumption, the FOC for household $h$, assuming an interior solution, are:

$$\begin{aligned} u_1^h - (p + t_h)u_2^h = 0 \\ m_h - (p + t_h)x_h - y_h = 0 \quad h = 1, 2, ..., H \end{aligned} \tag{8}$$

where $m_h$ is the exogenous income of household $h$. Let the *simultaneous* solution of equations (8) be $(\widehat{x}_h(\underline{t}, \underline{m}, e),\ \widehat{y}_h(\underline{t}, \underline{m}, e))$, $h = 1, 2, ..., H$, where $\underline{t} = (t_1, t_2, ..., t_H)$ and $\underline{m} = (m_1, m_2, ..., m_H)$.

In general, the dependence of $\widehat{x}_h$ and $\widehat{y}_h$ on $\underline{t}$, in particular the sign of $\dfrac{\partial \widehat{x}_h}{\partial t_h}$, cannot be established (Diamond [1973]). The reason is that the equilibrium solutions, $(\underline{\widehat{x}}, \underline{\widehat{y}})$ depend on conditions (8) for *all* households in the economy, a complex inter-dependence. In some special cases, though, this dependence can be simplified. For example, *weak separability* of $u_h$ in $\phi$ makes $\widehat{x}_h$ and $\widehat{y}_h$ depend only on $t_h$ (and not on $t_j$ $j \neq h$) and independent of $e$. Under the (standard) assumption of normality of $X$ we then have that $\dfrac{\partial \widehat{x}_h}{\partial t_h} < 0$. We shall discuss this case below.

Let

$$(\widehat{\underline{t}}, \widehat{e}) = \underset{t,e}{argmax} \left\{ \sum_{h=1}^{H} u_h(\widehat{x}_h, \widehat{y}_h, \widehat{\phi}) \mid \sum_{h=1}^{H} t_h \widehat{x}_h - \pi e \geq 0 \right\}, \tag{9}$$

where $\widehat{\phi} = \phi(\widehat{x}),\ \widehat{x} = \sum_{h=1}^{H} \widehat{x}_h(\underline{t}, \underline{m}, e)$.

Making use of conditions (8), $(\widehat{\underline{t}}, \widehat{e})$ must satisfy the following FOC:

$$\sum_{h=1}^{H} \widehat{t}_h \frac{\partial \widehat{x}_h}{\partial t_i} = \left( \frac{u_2^i - \mu}{\mu} \right) \widehat{x}_i + \widehat{\tau} \frac{\partial \widehat{x}}{\partial t_i} \qquad i = 1, 2, ...H \tag{10}$$

$$\sum_{h=1}^{H} \widehat{t}_h \frac{\partial \widehat{x}_h}{\partial e} = \frac{\left( \sum_{h=1}^{H} u_3^h \right) \phi_2}{\mu} - \pi + \widehat{\tau} \frac{\partial \widehat{x}}{\partial e} \tag{11}$$

where $\widehat{\tau} = -\dfrac{\left( \sum_{h=1}^{H} u_3^h \right) \phi_1}{\mu} > 0$, $\mu > 0$ is the shadow-price of the budget constraint in (9).

Examine first the case when lump-sum income redistribution is feasible. That is, in the maximization problem (9), allow also maximization on $\underline{m} = (m_1, m_2, ..., m_H)$ subject to the constraint $\sum_{h=1}^{H} m_h - M = 0$. Additional conditions are then obtained,

$$u_2^h - \mu = 0, \qquad h = 1, 2, ..., H \tag{12}$$

That is, equalization of the marginal utility of income across households.

Comparison of (4)-(6) and (10) reveals that in this case the unique solution is the first-best allocation, i.e.

$$\widehat{t}_1 = \widehat{t}_2 = ... \qquad = \widehat{t}_H = \tau^*, \tag{13}$$

$$\frac{\left(\sum u_3^h\right)\phi_2}{\mu} - \pi = 0 \quad at\ (\underline{x}^*,\ \underline{y}^*,\ e^*), \tag{14}$$

and $\mu = \lambda$. Note that (14) is the first-best condition for optimal investment in $e$.[2] Thus, *uniform taxation of the externality-generating good yields the first-best allocation* $(\underline{x}^*,\ \underline{y}^*,\ e^*)$ *whenever income redistribution is not desirable, i.e. when condition (12) holds.*

In the general case, we solve (10) for $\widehat{\underline{t}}$. Using a well-known rule about the expansion of determinants (with equal columns),

$$\widehat{t}_h = \widehat{\tau} + \widehat{\tau}_h \quad where \quad \widehat{\tau}_h = \sum_{h=1}^{H}\left(\frac{u_2^i - \mu}{\mu}\right)\widehat{x}_i\frac{J_{ih}}{J}, \tag{15}$$

$J$ is the determinant of the matrix whose element in the $i$-th row and in the $j$-th column is $\frac{\partial \widehat{x}_i}{\partial t_j}$ and $J_{ih}$ is the co-factor of the element in the $i$-th row and $h$-th column of this matrix.

The optimal second-best differentiated tax vector is seen to be composed of *two additive parts: a uniform tax*, $\widehat{\tau}$, equal to the value of the damage that the marginal consumption of $X$ by each household imposes on the *atmosphere*, divided by the shadow-price of the government's budget. This may be termed the "*efficiency-factor*". The second-part, $\widehat{t}_h$, varies across households and reflects a mixture of the redistributive objective and deadweight-loss considerations.[3] It is termed the "*redistributive-factor*".

[2] Assume alternatively that all $u^h$ are *linearly* separable in $y_h$, and have the same marginal utility of $Y$, say unity. This is a case in which income redistribution is not strictly desirable. Indeed, $u_2^h = \mu = 1$, $h = 1, 2, ..., H$, and the solution is the first-best, (13).

[3] Sandmo [1975] has a similar decomposition for the case of public goods and identical households.

The nature of this last term can be seen clearly in the 2-good case. From (15), when $H = 2$,

$$\begin{aligned} \widehat{\tau}_1 &= \frac{1}{\Delta}\left[\left(\frac{u_2^1-\mu}{\mu}\right)\widehat{x}_1\frac{\partial \widehat{x}_2}{\partial t_2} - \left(\frac{u_2^2-\mu}{\mu}\right)\widehat{x}_2\frac{\partial \widehat{x}_2}{\partial t_1}\right] \\ \widehat{\tau}_2 &= \frac{1}{\Delta}\left[\left(\frac{u_2^2-\mu}{\mu}\right)x_2\frac{\partial \widehat{x}_1}{\partial t_1} - \left(\frac{u_2^1-\mu}{\mu}\right)x_1\frac{\partial \widehat{x}_1}{\partial t_2}\right] \end{aligned} \tag{16}$$

Where $\Delta = \dfrac{\partial \widehat{x}_1}{\partial t_1}\dfrac{\partial \widehat{x}_2}{\partial t_2} - \dfrac{\partial \widehat{x}_1}{\partial t_2}\dfrac{\partial \widehat{x}_2}{\partial t_1}$.

Assume further that $u_h$ is *weakly separable* in $\phi$.[3] In this case, $\widehat{x}_h$ (and $\widehat{y}_h$) depends only on $t_h$ (and $m_h$) and is independent of $t_j$, $j \neq h$ and $e$. Equation (16) then becomes, in the familiar elasticity-form,

$$\widehat{\tau}_1 = \left(\frac{u_2^1-\mu}{\mu}\right)t_1\frac{1}{\epsilon_{11}}, \qquad \widehat{\tau}_2 = \left(\frac{u_2^2-\mu}{\mu}\right)t_2\frac{1}{\epsilon_{22}} \tag{17}$$

where $\epsilon_{hh} = \dfrac{t_h}{\widehat{x}_h}\dfrac{\partial \widehat{x}_h}{\partial t_h}$, $h = 1, 2$ is the own tax elasticity of $x_h$. This is the well-known *inverse-elasticity* rule. The differential tax on individual $h$'s purchase of $X$ is seen to be positive if $u_2^h - \mu > 0$ ("high-income" person) and is inversely related, due to deadweight-loss consideration, to demand elasticity, $\epsilon_{hh} < 0$. For a "low-income" household, $u_2^h - \mu > 0$ and $\widehat{\tau}_h < 0$, i.e. a reduction from the uniform efficiency level, $\tau$, is called for.

It is interesting to note that, with weak-separability, the optimal second best investment rule for externality reduction by means of $e$, (11), is the same as the first-best rule, (6). Of course, since the other first-best conditions do not hold, it does not imply that the second-best optimal level of $e$, $\widehat{e}$, is equal to $e^*$. It is worth noting that fact, however, that the optimal investment rule does not require estimates of consumption propensities could be important in a decentralized decision-making context (see Lau, Sheshinski and Stiglitz [1978]).

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[3] $\mu^h$ is weakly-separable in $\phi$ if $u_1^h \;\; u_2^h$ is independent of $\phi$.

# 4 Uniform Taxation

Suppose that personalized taxes, which require identification of households, are not feasible. Consider, instead the imposition of a uniform commodity tax $t$ on $X$.

Demands $(\widehat{x}_h, \widehat{y}_h)$ now depend on $t$ and $e$, and the optimum levels of these instruments, denoted $(\widehat{t}, \widehat{e})$, one given by

$$(\widehat{t}, \widehat{e}) = \underset{t,e}{argmax} \left\{ \sum u^h(\widehat{x}_h, \widehat{y}_h, \phi) \mid t\widehat{x} - \pi e \geq 0 \right\}. \tag{18}$$

It can be shown that $(\widehat{t}, \widehat{e})$ satisfy

$$\widehat{t} = \widehat{\tau} + \sum_{h=1}^{H} \widehat{\tau}_h \beta_h \tag{19}$$

$$(\widehat{t} - \widehat{\tau}) \frac{\partial \widehat{x}}{\partial e} + \frac{\left( \sum_{h=1}^{H} u_3^h \right) \phi_2}{\mu} - \pi = 0 \tag{20}$$

where

$$\widehat{\tau} = -\frac{\left( \sum_{h=1}^{H} u_3^h \right) \phi_1}{\mu}, \ \widehat{\tau}_h = \left( \frac{u_2^h - \mu}{\mu} \right) \widehat{x}_h \ / \ \frac{\partial \widehat{x}_h}{\partial t} \ and \ \beta_h = \frac{\partial \widehat{x}_h}{\partial t} \ / \ \frac{\partial \widehat{x}}{\partial t}, \ h = 1, 2, ..., H.$$

In (19), $\widehat{\tau}_h$ are the individual optimal taxes in the previous (weak-separability) case, (15). Since $\sum \beta_h = 1$, it is tempting, in view of (19), to regard the optimal uniform tax as a *weighed-average* of these personal taxes. Though sensible, this is warranted only if all $\beta_h$ are non-negative, which requires additional assumptions (see Diamond [1973]).

# 5 Indirect Taxes

In second-best situations it may be desirable to impose taxes also on goods whose consumption does not generate externalities (Green-Sheshinski [1976]). What is the optimal level of such an 'indirect' commodity tax (or subsidy)? Specifically, how much of the tax is related to improving the externality problem. In particular, with a uniform tax on the externality-generating good, if it is optimal to impose tax (subsidies) on goods whose consumption is positively (negatively) correlated with the consumption of this good, how much of these taxes (subsidies) is related to income redistribution?

We discuss this issue for the case of a uniform tax. *A-priori*, compared with the case of personalized taxes, this case should call for reliance on additional taxes to improve resource allocation.[5]

Thus, let us add a third good, $Z$, whose consumption by household $h$ is denoted by $z_h$ and its unit cost is also constant. Consider the imposition of uniform unit tax of $s$ on $Z$, in addition to the uniform tax, $t$, on the externality-generating good $X$.

The optimal taxes $(\widehat{t},\ \widehat{s})$ solve

$$(\widehat{t}, \widehat{s}) = \underset{t,s}{argmax} \left\{ \sum u^h(\widehat{x}_u, \widehat{z}_u, \widehat{y}_h, \phi) \mid t\widehat{x} + s\widehat{z} - \pi e \geq 0 \right\} \tag{21}$$

where the demands $\widehat{x}$ and $\widehat{z} = \sum_{h=1}^{H} \widehat{z}_h$ depend on $t$ and $s$. The FOC for $\widehat{t}$ and $\widehat{s}$ can be written:

$$\widehat{t} = \widehat{\tau} + \sum_{h=1}^{h} \left( \frac{u_3^h - \mu}{\mu} \right) \left( \widehat{x}_h \frac{\partial \widehat{z}}{\partial s} - \widehat{z}_h \frac{\partial \widehat{z}}{\partial t} \right) \frac{1}{\Delta} \tag{22}$$

$$\widehat{s} = \sum_{h=1}^{H} \left( \frac{u_3^h - \mu}{\mu} \right) \left( \widehat{z}_h \frac{\partial \widehat{x}}{\partial t} - \widehat{x}_h \frac{\partial \widehat{x}}{\partial s} \right) \frac{1}{\Delta} \tag{23}$$

where $\Delta = \frac{\partial \widehat{x}}{\partial t} \frac{\partial \widehat{z}}{\partial s} - \frac{\partial \widehat{x}}{\partial s} \frac{\partial \widehat{z}}{\partial t}$ (*assumed* $\neq 0$).

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[5] The conclusions below carry-over to the case of personalized taxes.

The interesting feature of (23) is that, *in the presence of atmosphere externalities, taxes on goods which do not generate externality, are desirable only for distributional purposes.*

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